Lecture Notes on Perfect Incompressible Fluids

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Abstract

This is my study notes on the English language translation of Perfect Incompressible Fluids by Jean-Yves Chemin [1] when I was the first year PhD student at University of Göttingen, 2019-2020. I rewrite all the details with my understanding, including a section-by-section aim and summary of each chapter. I have also remarked on those things that I found confusing, or that I still don't clearly understand.

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5.1 Yudovich's theorem

In this chapter, we are going to study the following formulation of the Euler's equation in $\mathbb{R}^2 \times \mathbb{R}$:

$$(E) \begin{cases} \partial_t v + \operatorname{div} v \otimes v = -\nabla p \\ \operatorname{div} v = 0 \\ v_{|t=0} = v_0 \end{cases}$$

We want to prove an existence and uniqueness theorem by energy method when the initial vector field has it curl bounded and compactly supported. Let us now state the main theorem as following:

Theorem 5.1 *Let m be a real number and v_0 a divergence-free vector field belonging to the space E_m. Assume, in addition, that ω_0 belongs to $L^\infty \cap L^a$ with $1 < a < +\infty$. Then, there exists a unique solution (v, p) of (E) belonging to the space $C(\mathbb{R}; E_m) \times L^\infty_{loc}(\mathbb{R}; L^2)$ and such that the vorticity ω of the vector field v is in $L^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}; L^a(\mathbb{R}^2))$.*

Moreover, this vector field v has a flow. More precisely, there exists a unique mapping ψ, continuous from $\mathbb{R} \times \mathbb{R}^2$ to $\mathbb{R}^2$, such that

$$\psi(t, x) = x + \int_0^t v(s, \psi(s, x)) ds$$

In addition, there exists a constant C such that

$$\psi(t) - \mathrm{Id} \in C^{\exp\left(-Ct\|\omega_0\|_{L^\infty \cap L^a}\right)}$$

Remark 5.1 *For the uniqueness of the solutions (v, p). According to Proposition 1.3.3 and (1.15), the pressure p is uniquely determined by the vector field v since it is square integrable.*

We first prove the uniqueness in the following sense: In the space L^2 and at the time t, the distance between two solutions in terms of the distance, again in L^2, between the initial data, and using only a control on the $L^\infty \cap L^a$ norm of the vorticity. The following lemma implies the uniqueness in an obvious manner.

Lemma 5.1 *Given a real number $a > 1$, there exists a constant C verifying the following property:*

Let (v_1, p_1) and (v_2, p_2) be two solutions of the incompressible Euler system (E). Assume that they both belong to the same space $L^\infty_{loc}(\mathbb{R}; E_m) \times L^\infty_{loc}(\mathbb{R}; L^2)$ and that ω_i belongs to $L^\infty \cap L^a$. Define

$$\alpha(t) = \left(C \max_i \|v_i(0) - \sigma\|_{L^2} e^{t\|\nabla\sigma\|_{L^\infty}} + \max \|\omega_i\|_{L^\infty \cap L^a} + 1\right)^{\frac{2}{a}} \quad \text{and}$$
$$\beta(t) = e \int_0^t \alpha(s) ds.$$

Then we have the following relation:

$$\|v_1(0) - v_2(0)\|^2_{L^2} \le e^{-a(\exp\beta(t) - 1)} \Rightarrow \|v_1(t) - v_2(t)\|^2_{L^2} \le \|v_1(0) - v_2(0)\|^{2\exp(-\beta(t))}_{L^2} e^{a(1 - \exp(-\beta(t)))}.$$

Remark 5.2

- *For any stationary vector field σ whose total vorticity is m, $v_i(0) - \sigma \in L^2(\mathbb{R}^2)$. So we can choose an arbitrary such σ and fixing it.*

- *As for $\|\nabla\sigma\|_{L^\infty}$, we can bound it using a calculation like the one we used to verify the expression for $\sigma \cdot \nabla\sigma$ on p. 11 (link missing). Therefore, the definition in Lemma 5.1 is well-defined.*

Proof: First, for all $b \geq a$ and $v = v_1 - v_2$, by Theorem 3.1 and the fact that $f \in L^p \cap L^q \Rightarrow f \in L^b$ for all $1 \leq p \leq b \leq q \leq \infty$, we have $\nabla v = \nabla v_1 - \nabla v_2 \in L^b$. In the following, we can get each $v_i \in L^\infty$, so we have $v \cdot \nabla v$ belongs to L^b It doesn't seem possible to get around $v \in L^\infty$ here! Maybe this can be assured by some hidden hypothesis but I don't know.

For the pressure we have know

$$\begin{aligned} p &= \Delta^{-1}\Delta p = \Delta^{-1}\left(\partial_t \operatorname{div} v + \Delta p\right) \\ &= \Delta^{-1}\left(\partial_t \sum_j \partial_j v^j + \sum_j \partial_j^2 p\right) \\ &= \sum_j \Delta^{-1}\left(\partial_j \partial_t v^j + \partial_j \partial_j p\right) \\ &= -\sum_j \partial_j \Delta^{-1}\left(v \cdot \nabla v^j\right) \\ &= \sum_j \partial_j \Delta^{-1}\left(\partial_t v^j + \partial_j p\right) \\ &= -\sum_{j,k} \partial_j \Delta^{-1}\left(v^k \partial_k v^j\right). \end{aligned} \tag{5.1}$$

Applying ∇ to both sides gives

$$\nabla p = -\nabla \sum_{j,k} \partial_j \Delta^{-1}\left(v^k \partial_k v^j\right). \tag{5.2}$$

Hence, the fact that (v_i, p_i) are solutions of (E) ensures that

$$\forall b \geq a, \quad \partial_t v_i = -\nabla p_i - (v_i \cdot \nabla v_i) \in L^b. \tag{5.3}$$

Now, define the function

$$I(t) \stackrel{\text{def}}{=} \|(v_1 - v_2)(t)\|_{L^2}^2$$

and

$$I_\epsilon(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \chi(\epsilon x) \left|(v_1 - v_2)(t,x)\right|^2 dx$$

for a strictly positive real number ϵ.

Let ψ_1 be the flow for v_1 and change variables, using the fact that $\psi_1(t)$ is measure preserving, we have

$$\begin{aligned} I_\epsilon(t) &= \int_{\mathbb{R}^2} \chi(\epsilon x) \left|(v_1 - v_2)(t,x)\right|^2 dx \\ &= \int_{\mathbb{R}^2} \chi\left(\epsilon \psi_1(t,x)\right) \left|(v_1 - v_2)(t, \psi_1(t,x))\right|^2 dx. \end{aligned}$$

(Here, χ is presumably the function of Chapter 2, which is smooth and supported on a ball centered at the origin.) Then we calculate,

$$\begin{aligned} &\frac{d}{dt}\left|(v_1 - v_2)(t, \psi_1(t,x))\right|^2 \\ &= \sum_j \partial_j \left|(v_1 - v_2)(t, \psi_1(t,x))\right|^2 \partial_t \psi_1^j(t,x) + \partial_t \left|(v_1 - v_2)(t, \psi_1(t,x))\right|^2 \\ &= \sum_j \partial_j \left|(v_1 - v_2)(t, \psi_1(t,x))\right|^2 v_1^j\left(t, \psi_1(t,x)\right) \\ &\quad + 2\left(v_1 - v_2\right)(t, \psi_1(t,x)) \cdot \partial_t \left(v_1 - v_2\right)(t, \psi_1(t,x)) \end{aligned}$$

where we used $\partial_t \psi_1(t,x) = v_1(t, \psi_1(t,x))$.

Making essentially the same calculation that we made to verify that $\partial_t(v(t,\psi(t,x))) = (\partial_t v + v\cdot\nabla v)(t,\psi(t,x))$, we have

$$\partial_t (v_i(t,\psi_1(t,x))) = (\partial_t v_i + v_1\cdot\nabla v_i)(t,\psi_1(t,x))$$

so

$$\begin{aligned}
&\partial_t (v_1 - v_2)(t,\psi_1(t,x)) \\
&= (\partial_t v_1 - \partial_t v_2 + v_1\cdot\nabla v_1 - v_1\cdot\nabla v_2)(t,\psi_1(t,x)) \\
&= (-\nabla p_1 - v_1\cdot\nabla v_1 + \nabla p_2 + v_2\cdot\nabla v_2 + v_1\cdot\nabla v_1 - v_1\cdot\nabla v_2)(t,\psi_1(t,x)) \\
&= (-\nabla p - (v_1 - v_2)\cdot\nabla v_2)(t,\psi_1(t,x)).
\end{aligned}$$

Putting this all together, and changing variables from $\psi_1(t,x)$ back to x at opportune times, we get

$$\begin{aligned}
I'_\epsilon(t) &= \int_{\mathbb{R}^2} \chi(\epsilon x)\frac{d}{dt}\left|(v_1 - v_2)(t,x)\right|^2 dx \\
&= \int_{\mathbb{R}^2} \chi(\epsilon\psi_1(t,x))\frac{d}{dt}\left|(v_1 - v_2)(t,\psi_1(t,x))\right|^2 dx \\
&= \int_{\mathbb{R}^2} \chi(\epsilon\psi_1(t,x))\begin{pmatrix} \sum_j \partial_j |(v_1 - v_2)(t,\psi_1(t,x))^2 v_1^j(t,\psi_1(t,x)) \\ +2(v_1 - v_2)(t,\psi_1(t,x))\cdot\partial_t (v_1 - v_2)\cdot(t,\psi_1(t,x)) \end{pmatrix} dx \\
&= \int_{\mathbb{R}^2} \chi(\epsilon\psi_1(t,x))\begin{pmatrix} \sum_j \partial_j |(v_1 - v_2)(t,x)|^2 v_1^j(t,x) \\ +2(v_1 - v_2)(t,x)\cdot\partial_t (v_1 - v_2)\cdot(t,x) \end{pmatrix} dx \\
&= \sum_j \int_{\mathbb{R}^2} \chi(\epsilon x)\partial_j \left|(v_1 - v_2)(t,x)\right|^2 v_1^j(t,x) dx \\
&\quad - 2\int_{\mathbb{R}^2} \chi(\epsilon x)(v_1 - v_2)(t,x)\cdot\nabla p dx \\
&\quad - 2\int_{\mathbb{R}^2} \chi(\epsilon x)(v_1 - v_2)(t,x)\cdot\left[(v_1 - v_2)(t,x)\cdot\nabla v_2(t,x)\right] dx.
\end{aligned}$$

Note that we can justify the following calculation:

$$\partial_t \left|(v_1 - v_2)(t,\psi_1(t,x))\right|^2 = 2(v_1 - v_2)(t,\psi_1(t,x))\cdot\partial_t (v_1 - v_2)(t,\psi_1(t,x))$$

is a distribution. Then we can use the integral by parts to get

$$I'_\epsilon(t) \le 2\int_{\mathbb{R}^2} \chi(\epsilon x)\left|(v_1 - v_2)(t,x)\right|^2 \left|\nabla v_2(t,x)\right| dx + R_\epsilon(t)$$

where

$$R_\varepsilon(t) = \epsilon\sum_j \int_{\mathbb{R}^2} \left|(v_1 - v_2)(t,x)\right|^2 v_1^j(t,x)(\partial_j\chi)(\epsilon x) dx + \epsilon\sum_i \int_{\mathbb{R}^2} (v_1 - v_2)^i(t,x)(\partial_i\chi)(\epsilon x) p(t,x) dx.$$

The vector field $v_1 - v_2$ belongs to the space $L^\infty_{loc}(\mathbb{R}; L^2\cap L^\infty)$:

- The L^∞_{loc} part comes from the assumption that $v_1, v_2 \in L^\infty_{loc}(\mathbb{R}; E_m)$.
- The L^2 part comes from the definition of E_m.
- The L^∞ part comes from the fact that $v_i \in L^\infty$ in the next,

and the pressure to the space $L^\infty_{loc}(\mathbb{R}; L^2)$. We have,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} |(v_1 - v_2)(t,x)|^2 v_1^j(t,x)(\partial_j\chi)(\epsilon x) dx\right| \\
&\quad \le \|(v_1 - v_2)(t,\cdot)\|_{L^\infty} \|(v_1 - v_2)(t,\cdot)\|_{L^2} \|v_1^j(t,x)(\partial_j\chi)(\epsilon\cdot)\|_{L^2} \\
&< \infty
\end{aligned}$$

where we used the fact that χ (and so $\partial_j\chi$) is compactly supported and the fact that v_1^j is a sum of a function in L^2 and a function that is bounded near the origin to conclude that the last norm above is finite.

We also have

$$\left|\int_{\mathbb{R}^2} (v_1 - v_2)^i (t,x) (\partial_i\chi)(\epsilon x) dx\right| \leq \|(\partial_i\chi)(\epsilon x)\|_{L^\infty} \|(v_1 - v_2)(t,\cdot)\|_{L^2} \|p(t,\cdot)\|_{L^2} < \infty.$$

Therefore, we infer that

$$R_\varepsilon(t) \leq C(t)\epsilon \quad \text{with} \quad C \in L^\infty_{loc}(\mathbb{R}).$$

Remark 5.3 *If we make this same calculation using a vector $v = \sigma + w \in E_m$ in place of v_1 and σ in place of v_2, we obtain,*

$$\frac{d}{dt}\|v(t) - \sigma\|_{L^2}^2 \leq 2\|\nabla\sigma\|_{L^\infty}\|v(t) - \sigma\|_{L^2}^2 + R_\epsilon(t)$$

As above, $R_\epsilon(t) \to 0$ as $\epsilon \to 0$. Then

$$\frac{d}{dt}\|v(t) - \sigma\|_{L^2}^2 \leq 2\|\nabla\sigma\|_{L^\infty}\|v(t) - \sigma\|_{L^2}^2.$$

Therefore we have equation (4.32) and from (4.33) we have

$$\|v(t) - \sigma\|_{L^2}^2 \leq \|v_0 - \sigma\|_{L^2}^2 \, e^{t\|\nabla\sigma\|_{L^\infty}}.$$

We will use this later.

Hölder's inequality implies that, for all $b \geq a$, we have

$$I'_\epsilon(t) \leq 2\left(\int_{\mathbb{R}^2} \chi(\epsilon x) |v_1(t,x) - v_2(t,x)|^{\frac{2b}{b-1}} dx\right)^{1-\frac{1}{b}} \left(\int_{\mathbb{R}^2} |\nabla v_2(t,x)|^b dx\right)^{\frac{1}{b}} + R_\epsilon(t),$$

where

$$\begin{aligned}
&\left(\int_{\mathbb{R}^2} \chi(\epsilon x) |v_1(t,x) - v_2(t,x)|^{\frac{2b}{b-1}} dx\right)^{1-\frac{1}{b}} \\
&= \left(\int_{\mathbb{R}^2} \chi(\epsilon x) |v_1(t,x) - v_2(t,x)|^2 |v_1(t,x) - v_2(t,x)|^{\frac{2}{b-1}} dx\right)^{1-\frac{1}{b}} \\
&\leq \|v_1(t,\cdot) - v_2(t,\cdot)\|_{L^\infty}^{\frac{2}{b-1}(1-\frac{1}{b})} \left(\int_{\mathbb{R}^2} \chi(\epsilon x) |v_1(t,x) - v_2(t,x)|^2\right)^{1-\frac{1}{b}} \\
&= \|v_1(t,\cdot) - v_2(t,\cdot)\|_{L^\infty}^{\frac{2}{b}} (I_\epsilon(t))^{1-\frac{1}{b}}.
\end{aligned}$$

Furthermore, for all $b \geq a$

$$I'_\epsilon(t) \leq 2\|v_1(t) - v_2(t)\|_{L^\infty}^{\frac{2}{b}} I_\epsilon(t)^{1-\frac{1}{b}} \|\nabla v_2(t)\|_{L^b} + R_\varepsilon(t).$$

According to Biot-Savart's law, Theorem 3.1 .1 and the conservation of the vorticity along the flow lines, it follows that for all $b \geq a$

$$\|\nabla v_2(t)\|_{L^b} \leq \frac{b^2}{b-1}\|\omega_2(0)\|_{L^b} \leq Cb\|\omega_2(0)\|_{L^a\cap L^\infty}.$$

Moreover, dropping the i subscript on v_i for convenience, we have

$$\begin{aligned}
\|v\|_{L^\infty} &= \|v - \sigma + \sigma\|_{L^\infty} = \|\chi(D)(v - \sigma) + \sigma + (\mathrm{Id} - \chi(D))(v - \sigma)\|_{L^\infty} \\
&\leq \|\chi(D)(v - \sigma)\|_{L^\infty} + \|\sigma\|_{L^\infty} + \|(\mathrm{Id} - \chi(D))(v - \sigma)\|_{L^\infty}.
\end{aligned}$$

Note that

$$\begin{aligned}
\|(\mathrm{Id} - \chi(D))\sigma\|_{L^\infty} &\leq \|\sigma\|_{L^\infty} + \|\chi(D)\sigma\|_{L^\infty} \\
&= \|\sigma\|_{L^\infty} + \|S_0\sigma\|_{L^\infty} \leq (1 + C)\|\sigma\|_{L^\infty} < \infty,
\end{aligned}$$

and

$$\begin{aligned}\|\chi(D)(v-\sigma)\|_{L^\infty} &\le C\|\chi(D)(v-\sigma)\|_{L^2} = C\|\chi(v-\sigma)\|_{L^2}\\ &\le C\|\chi(D)\|_{L^\infty}\|(v-\sigma)\|_{L^2} \le C\|(v-\sigma)\|_{L^2}\\ &= C\|(v(t)-\sigma)\|_{L^2}\end{aligned}$$

where we used Bernstein's lemma (Lemma 2.1.1) for the first inequality and the fact that $\chi \in L^\infty$ to absorb its norm into the constant, C.
Note also

$$\|(\mathrm{Id}-\chi(D))v)\|_{L^\infty} = \left\|\sum_{j=0}^{\infty}\Delta_j v\right\|_{L^\infty} \le \sum_{j=0}^{\infty}\left\|\Delta_j v\right\|_{L^\infty}.$$

By Bernstein's inequality, Lemma 2.1.1, applied twice,

$$\begin{aligned}\left\|\Delta_j v^i\right\|_{L^\infty} &\le C2^{-j}\left\|\partial_k\Delta_j v^i\right\|_{L^\infty} = C2^{-j}\left\|\Delta_j\partial_k v^i\right\|_{L^\infty}\\ &\le C2^{-j}\left\|\Delta_j\nabla v\right\|_{L^\infty} = C2^{-j}\left\|\nabla\Delta_j v\right\|_{L^\infty}\\ &\le C2^{-j}\left(2^j\right)^{2/a}\left\|\nabla\Delta_j v\right\|_{L^a} \le C2^{j(2/a-1)}a\left\|\omega\left(\Delta_j v\right)\right\|_{L^a}\\ &= C2^{j(2/a-1)}a\left\|\Delta_j\omega\right\|_{L^a}\end{aligned}$$

where the last inequality follows by Theorem 3.1.1. We then have

$$\begin{aligned}\|\Delta_j\omega\|_{L^a} &= \left\|\mathcal{F}^{-1}\left(\varphi\left(2^{-j}\cdot\right)\widehat{\omega}\right)\right\|_{L^a}\\ &= \left\|\mathcal{F}^{-1}\left(\mathcal{F}\left(\mathcal{F}^{-1}\left(\varphi\left(2^{-j}\cdot\right)\right)\right)\widehat{\omega}\right)\right\|_{L^a}\\ &= \left\|\mathcal{F}^{-1}\left(\mathcal{F}\left(\mathcal{F}^{-1}\left(\varphi\left(2^{-j}\cdot\right)\right)\right)*\omega\right)\right\|_{L^a}\\ &= \left\|\mathcal{F}^{-1}\left(\varphi\left(2^{-j}\cdot\right)\right)*\omega\right\|_{L^a} \le \left\|\mathcal{F}^{-1}\left(\varphi\left(2^{-j}\cdot\right)\right)\right\|_{L^1}\|\omega\|_{L^a}\\ &= \left\|\mathcal{F}^{-1}(\varphi)\right\|_{L^1}\|\omega\|_{L^a} = C\|\omega\|_{L^a}\end{aligned}$$

That $\left\|\mathcal{F}^{-1}\left(\varphi\left(2^{-j}.\right)\right)\right\|_{L^1} = \left\|\mathcal{F}^{-1}(\varphi)\right\|_{L^1}$ follows by a change of variables. Thus,

$$\|(\mathrm{Id}-\chi(D))v\|_{L^\infty} \le \sum_{j=0}^{\infty} C2^{j(2/a-1)}a\|\omega\|_{L^a} \le Ca\|\omega\|_{L^a}.$$

Therefore we know that

$$\|v_i\|_{L^\infty} \le C\left(\|v(0)-\sigma\|_{L^2}\exp\left(t\|\nabla\sigma\|_{L^\infty}\right) + \|\sigma\|_{L^\infty} + a\left\|\omega_i(0)\right\|_{L^a}\right).$$

Therefore, for every b larger than or equal to a, we have

$$I'_\epsilon(t) \le \alpha(t)bI_\epsilon(t)^{1-\frac{1}{6}} + R_\epsilon(t). \tag{5.4}$$

Now, assume that $\|v_1(0)-v_2(0)\|_{L^2}^2 < 1$. Let η be a real number such that $0<\eta<1-I(0)$. Define

$$J_{\epsilon,\eta}(t) = \eta + I_\epsilon(t)$$

All the inequalities which follow are true only under the assumption $\eta + I_\varepsilon(t) \le 1$ We infer from inequality (5.4) that

$$J'_{\epsilon,\eta}(t) \le \alpha(t)bJ_{\epsilon,\eta}(t)^{1-\frac{1}{t}} + R_\epsilon(t).$$

Choosing $b = a - \log J_{\varepsilon,\eta}(t)$, we obtain

$$\begin{aligned}J'_{\epsilon,\eta}(t) &\le \alpha(t)\left(a-\log J_{\epsilon,\eta}(t)\right)J_{\epsilon,\eta}(t)\exp\left(\frac{-\log J_{\epsilon,\eta}(t)}{a-\log J_{\epsilon,\eta}(t)}\right) + R_\epsilon(t)\\ &\le e\alpha(t)\left(a-\log J_{\epsilon,\eta}(t)\right)J_{\epsilon,\eta}(t) + R_\epsilon(t).\end{aligned}$$

For every differentiable function f from $\mathbb{R}$ to $(0,1)$, we have, for every $\lambda \in (0,1)$

$$-\frac{1}{\lambda}\frac{d}{dt}\log(1-\lambda\log f(t)) = \frac{f'(t)}{f(t)(1-\lambda\log f(t))} \tag{5.5}$$

Let $\lambda = \frac{1}{a}$ and, using the inequalities

$$J_{\epsilon,\eta}(t) > \eta, \text{ and } a - \log J_{\epsilon,\eta}(t) > a,$$

we have

$$-\frac{d}{dt}\log\left(a - \log J_{\epsilon,\eta}(t)\right) \le e\alpha(t) + \frac{R_\epsilon(t)}{a\eta}$$

Set

$$\bar{\beta}_{\varepsilon,\eta}(t) = \beta(t) + \int_0^t \frac{R_\varepsilon(\tau)}{a\eta} d\tau$$

From the definition of β, we have, after integration,

$$-\log\left(1 - \frac{1}{a}\log J_{\epsilon,\eta}(t)\right) + \log\left(1 - \frac{1}{a}\log J_{\epsilon,\eta}(0)\right) \le \bar{\beta}_{\epsilon,\eta}(t)$$

Taking the exponential of the previous expression, we obtain

$$\begin{aligned}
&\frac{1}{1 - \frac{1}{a}\log J_{\epsilon,\eta}(t)} \le \frac{e^{\bar{\beta}_{\epsilon,\eta}(t)}}{1 - \frac{1}{a}\log J_{\epsilon,\eta}(0)} \\
&\Longrightarrow 1 - \frac{1}{a}\log J_{\epsilon,\eta}(t) \ge \left(1 - \frac{1}{a}\log J_{\epsilon,\eta}(0)\right) e^{-\bar{\beta}_{\epsilon,\eta}(t)} \\
&\Longrightarrow \frac{1}{a}\log J_{\epsilon,\eta}(t) \le 1 - \left(1 - \frac{1}{a}\log J_{\epsilon,\eta}(0)\right) e^{-\bar{\beta}_{\epsilon,\eta}(t)} \\
&\Longrightarrow \log J_{\epsilon,\eta}(t) \le a - \left(a - \log J_{\epsilon,\eta}(0)\right) e^{-\bar{\beta}_{\epsilon,\eta}(t)} \\
&\qquad\qquad = a\left(1 - e^{-\bar{\beta}_{\epsilon,\eta}(t)}\right) + e^{-\bar{\beta}_{\epsilon,\eta}(t)} \log J_{\epsilon,\eta}(0).
\end{aligned}$$

Therefore, again taking the exponential,

$$J_{\epsilon,\eta}(t) \le e^{a\left(1 - \exp\left(-\bar{\beta}_{\epsilon,\eta}(t)\right)\right)} J_\epsilon(0)^{\exp\left(-\bar{\beta}_{\epsilon,\eta}(t)\right)}$$

From the definition of R_ϵ we have

$$R_\varepsilon(t) \le C(t)\epsilon \quad \text{with} \quad C \in L^\infty_{loc}.$$

Passing to the limit when ϵ goes to 0 and, afterwards, when η goes to 0, we complete the proof of the lemma. ∎

Lemma 5.2 *Let σ be a stationary vector field (in the sense of Definition 1.3.2). Consider a function ρ belonging to $\mathcal{S}\left(\mathbb{R}^2\right)$, of integral 1. Next, we define the sequence $(\sigma_n)_{n\in\mathbb{N}}$ by $\sigma_n = \rho_n \star \sigma$ where $\rho_n(x) = (1+n)^2\rho((1+n)x)$ Then, we have*

$$\lim_{n\infty} \|\sigma - \sigma_n\|_{L^2} = 0$$

Proof: I cant follow in detail the thread of Chemins argument, I give a new proof.

By the mean value theorem, given $\xi \in \mathbb{R}^2$, there exists an η on the line segment between the origin and $\xi/(n+1)$ such that

$$\frac{\widehat{\rho}(\xi/(n+1)) - \widehat{\rho}(0)}{|\xi/(n+1)|} = \widehat{\xi} \cdot \nabla\widehat{\rho}(\eta)$$

where $\widehat{\xi}$ is a unit vector in the direction of ξ. From this it follows that

$$\left|1 - \widehat{\rho}\left(\frac{\xi}{n+1}\right)\right| \le \frac{|\xi|}{1+n}\|D\widehat{\rho}\|_{L^\infty}$$

If σ were in L^2, then we could write

$$\begin{aligned}
\|\widehat{\sigma} - \widehat{\sigma_n}\|_{L^2} &= \|\widehat{\sigma} - \widehat{\rho_n * \sigma}\|_{L^2} = \|\widehat{\sigma} - \widehat{\rho_n\widehat{\sigma}}\|_{L^2} = \|\widehat{\sigma}(1 - \widehat{\rho_n})\|_{L^2} \\
&= \left\|\widehat{\sigma}(\xi)\left(1 - \widehat{\rho}\left(\frac{\xi}{1+n}\right)\right)\right\|_{L^2} \le \left\|\widehat{\sigma}(\xi)\frac{|\xi|}{1+n}\right\| D\widehat{\rho}\|_{L^\infty}\|_{L^2} \\
&= \frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\xi\widehat{\sigma}(\xi)\|_{L^2} = \frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\widehat{\nabla\sigma}\|_{L^2} \\
&= \frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\nabla\sigma\|_{L^2} \le C\frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\omega(\sigma)\|_{L^2}
\end{aligned}$$

the last inequality being by Theorem 3.1. This would work fine, since $1/(1+n)$ approaches zero as n approaches infinity. But σ is only in weak- L^2, so the first inequality above is not valid. This is a point that is, perhaps, quite subtle (it is to me, anyway), but the inequality in question is of the form

$$|B| \leq |C| \Longrightarrow \|AB\|_{L^2} \leq \|AC\|_{L^2} \tag{5.6}$$

where

$$A = \widehat{\sigma}, B = 1 - \widehat{\rho}\left(\frac{\xi}{1+n}\right), \text{ and } C = \frac{|\xi|}{1+n}\|D\widehat{\rho}\|_{L^\infty}.$$

The problem is that $\|A\|_{L^2}$ is not finite, while $\|AC\|_{L^2}$ is finite. We need to have $\|A\|_{L^2}$ be finite for Equation (5.6) to be valid; that is, we need to be able to treat $\widehat{\sigma}$ as a regular function in L^2.

To get around this problem, we use a frequency decomposition, and write

$$\|\sigma_n - \sigma\|_{L^2} = \|\chi(D)(\sigma_n - \sigma)\|_{L^2} + \|(1-\chi(D))(\sigma_n - \sigma)\|_{L^2} \tag{5.7}$$

Then,

$$\begin{aligned}
&\|\chi(D)(\sigma_n - \sigma)\|_{L^2} \\
&\quad = \|\mathcal{F}(\chi(D)(\sigma_n - \sigma))\|_{L^2} = \|\chi(\widehat{\sigma_n} - \widehat{\sigma})\|_{L^2} = \|\chi(\widehat{\sigma} - \widehat{\rho_n * \sigma})\|_{L^2} \\
&\quad = \|\chi(\widehat{\sigma} - \widehat{\rho_n}\widehat{\sigma})\|_{L^2} = \|\chi\widehat{\sigma}(1 - \widehat{\rho_n})\|_{L^2} \\
&\quad = \left\|\chi\widehat{\sigma}\left(1 - \widehat{\rho}\left(\frac{\xi}{1+n}\right)\right)\right\|_{L^2} \leq \left\|\chi\widehat{\sigma}\frac{|\xi|}{1+n}\right\| D\widehat{\rho}\|_{L^\infty}\|_{L^2} \\
&\quad = \frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\chi(\xi\widehat{\sigma})\|_{L^2} = \frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\chi\widehat{\nabla\sigma}\|_{L^2} \\
&\quad \leq \frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\widehat{\nabla\sigma}\|_{L^2} = \frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\nabla\sigma\|_{L^2} \leq C\frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\omega(\sigma)\|_{L^2}.
\end{aligned}$$

The use of the equivalent of Equation (5.6) in the critical inequality above is valid, because we are using $A = \chi\widehat{\sigma}$, which is in L^2.

The second term of Equation (5.7) is bounded by

$$\begin{aligned}
\|(1-\chi(D))(\sigma_n - \sigma)\Big\|_{L^2} &\leq \sum_{q=0}^{\infty}\Big\|\Delta_q(\sigma_n - \sigma)\|_{L^2} \\
&= \sum_{q=0}^{\infty}\|\Delta_q(\rho_n * \sigma - \sigma)\|_{L^2} \leq \sum_{q=0}^{\infty} C2^{-q}\|\nabla\Delta_q(\rho_n * \sigma - \sigma)\|_{L^2} \\
&\leq \sum_{q=0}^{\infty} C2^{-q}\|\omega(\Delta_q(\rho_n * \sigma - \sigma))\|_{L^2} \\
&= \sum_{q=0}^{\infty} C2^{-q}\|\Delta_q(\rho_n * \omega(\sigma) - \omega(\sigma))\|_{L^2} \\
&= \sum_{q=0}^{\infty} C2^{-q}\left\|\mathcal{F}^{-1}(\chi\mathcal{F}((\rho_n * \omega(\sigma) - \omega(\sigma))))\right\|_{L^2} \\
&\leq \sum_{q=0}^{\infty} C2^{-q}\|\rho_n * \omega(\sigma) - \omega(\sigma)\|_{L^2} \leq C\|\rho_n * \omega(\sigma) - \omega(\sigma)\|_{L^2}
\end{aligned}$$

In the second inequality we used the first inequality in the second part of Lemma 2.1, Bernstein's inequality, which required Fourier support on an annulus; this is another reason for decomposing v into low and high frequencies before bounding its norm. In the third inequality, we used Theorem 3.1. We also used the fact that derivatives commute with the Littlewood-Paley operators (up to a possible factor of i), including the derivatives involved in the definition of the vorticity. Finally, we used $\partial_k(f*g) = f*\partial_k g$ to move the vorticity operator past the convolution.

It follows from Equation (5.7) that

$$\|\sigma_n - \sigma\|_{L^2} \leq C\frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\omega(\sigma)\|_{L^2} + C\|\rho_n * \omega(\sigma) - \omega(\sigma)\|_{L^2}$$

The first term clearly approaches zero as n grows large. The second term approaches zero because ω is in L^2 and ρ_n is an approximation of the identity, though written with an integer-valued parameter approaching infinity rather than a positive real-valued parameter approaching zero, as is more customary. This completes the proof. ∎

Now, let us finish the proof of the Eulerian part of Theorem 5.1. In order to prove the existence, we will regularize the initial data and we will prove that the sequence of solutions so obtained is a Cauchy sequence in $L^\infty_{loc}(\mathbb{R}; E_m)$.

Proof of Theorem 5.1: Part A.

Proof:

Let T be an arbitrary strictly positive real number. The regularization of the initial data is very classical, because

$$\begin{aligned}\partial_k v_{0,n}(x) &= \partial_k \int_{\mathbb{R}^2} \chi_n(x-y) v_0(y)dy = \int_{\mathbb{R}^2} \left(\partial_k \chi_n(x-y)\right) v_0(y)dy \\ &= (\partial_k \chi_n) * v_0.\end{aligned}$$

which exists since χ_n is smooth and compactly supported. In fact, we can take as many derivatives as we wish, and they are each bounded, so v_0 is in all C^r spaces. We define

$$v_{0,n} = \chi_n * v_0$$

where $\chi_n(x) = (1+n)^2 \chi((1+n)x)$ and integral 1. Thus, if we write,

$$\begin{aligned}\|v_0 - v_{0,n}\|_{L^2} &= \|\sigma + (v_0 - \sigma) - \chi_n * (\sigma + (v_0 - \sigma))\|_{L^2} \\ &= \|\sigma - \chi_n * \sigma + (v_0 - \sigma) - \chi_n * (v_0 - \sigma)\|_{L^2} \\ &\le \|(v_0 - \sigma) - \chi_n * (v_0 - \sigma)\|_{L^2} + \|\sigma - \chi_n * \sigma\|_{L^2}.\end{aligned} \tag{5.8}$$

the second term approaches zero. And the first term approaches zero because χ_n is an approximation of the identity and $(v_0 - \sigma)$ is in L^2.

According to the definition of the space E_m (see Definition 1.3) and to Lemma 2.4, we have the fact that the sequence $(v_{0,n})_{n\in\mathbb{N}}$ converges to v_0 in the space E_m.

According to Theorem 4.4, we have at our disposal a sequence $(v_n)_{n\in\mathbb{N}}$ of solutions of the incompressible Euler system (E). According to Biot-Savart's law, the sequence $(v_n)_{n\in\mathbb{N}}$ satisfies the following estimates:

$$\forall b \ge a, \|\omega_n(t)\|_{L^b} = \|\omega_0\|_{L^b},$$
$$\|v_n(t)\|_{L^\infty} \le C \|v_0 - \sigma\|_{L^2} e^{t\|\nabla\sigma\|_{L^\infty}} + \|\omega_0\|_{L^\infty}.$$

Taking n and m large enough, we can assume that

$$\|v_n(0) - v_m(0)\|_{L^2} \le e^{-a(\exp \beta(T) - 1)}.$$

Hence, Lemma 5.1 ensures that, if n and m are large enough, then we have, for every t less than or equal to T

$$\|v_n(t) - v_m(t)\|_{L^2} \le \|v_n(0) - v_m(0)\|_{L^2}^{\exp(-\beta(t))}.$$

The completeness of $C(\mathbb{R}; E_m)$ follows from the completeness of L^2 and the fact that for any divergence-free sequence (u_n) converging to u in L^2 $\operatorname{div} u_n \to \operatorname{div} u$ as a distribution, so $\operatorname{div}\, u = 0$ as a distribution.

Hence, $v_n \to v$ in $C(\mathbb{R}; E_m)$, and v satisfies the initial condition, $v(0) = v_0$. Therefore, it, along with its corresponding pressure, satisfy the Euler equations. Uniqueness of the velocity follows immediately from Lemma 5.1, so the proof of the first part of Theorem 5.1 is now complete. ∎

5.2 On ordinary differential equations

The aim of this section is to prove the Lagrangian part of Yudovich's theorem 5.1. First, we define the space of logarithmic Lipschitzian (in short, log-Lipschitzian) vector fields.

Definition 5.1 *The set of log-Lipschitzian vector fields on $\mathbb{R}^d$, denoted by LL is the set of bounded vector fields v such that*

$$\|v\|_{LL} \stackrel{def}{=} \sup_{0<|x-x'|\leq 1} \frac{|v(t,x)-v(t,x')|}{|x-x'|(1-\log|x-x'|)} < \infty.$$

Theorem 5.2 *Let v be a vector field belonging to the space $L^1_{loc}(\mathbb{R};LL)$ and $L^1_{loc}(\mathbb{R};L^\infty)$ then there exists a unique mapping ψ, continuous in $\mathbb{R}\times\mathbb{R}^d$, with values in $\mathbb{R}^d$, such that*

$$\psi(t,x) = x + \int_0^t v(s,\psi(s,x))ds.$$

Moreover, the flow ψ is such that, for all t

$$\psi(t) - \mathrm{Id} \in C^{\exp\left(-\int_0^t \|v(s)\|_{LL} ds\right)}.$$

More precisely, we have

$$|x-y| \leq e^{1-\exp\left(\int_0^t \|v(s)\|_{LL} ds\right)} \Rightarrow |\psi(t,x)-\psi(t,y)| \leq |x-y|^{\exp\left(-\int_0^t \|v(s)\|_{LL} ds\right)} e^{\left(1-\exp - \int_0^t \|v(s)\|_{LL} ds\right)}.$$

Let us digress a little on the ordinary differential equations associated with non-Lipschitzian vector fields. This will lead us very simply to the main theorem. Throughout this section, μ will denote a continuous, increasing function from $\mathbb{R}^+$ to itself, vanishing in 0, and strictly positive elsewhere.

Definition 5.2 *Consider two metric spaces (X,d) and (Y,δ). We will denote by $\mathcal{C}_\mu(X,Y)$ the set of bounded functions u from X to Y such that there exists a constant C such that, for every $x \in X$ and $y \in X$*

$$\delta(u(x),u(y)) \leq C\mu(d(x,y)).$$

Remark 5.4 *Remark If (Y,δ) is a Banach space (denoted by $(E,\|\cdot\|)$), the space $\mathcal{C}_\mu(X,E)$ is a Banach space under the norm*

$$\|u\|_\mu = \|u\|_{L^\infty} + \sup_{(x,y)\in X\times X, x\neq y} \frac{\|u(x)-u(y)\|}{\mu(d(x,y))}.$$

The following theorem describes the simple hypotheses which imply existence and uniqueness for the integral curves of an ordinary differential equation.

Theorem 5.3 *Let E be a Banach space, Ω be an open set of E, I an open interval of $\mathbb{R}$ and (t_0,x_0) an element of $I\times\Omega$. Consider a function F belonging to $L^1_{loc}(I;\mathcal{C}_\mu(\Omega;E))$. We assume, in addition, that*

$$\int_0^1 \frac{dr}{\mu(r)} = +\infty \tag{5.9}$$

Then, there exists an interval J such that $t_0 \in J \subset I$ and such that the equation

$$(ODE) \quad x(t) = x_0 + \int_{t_0}^t F(s,x(s))ds$$

has a unique continuous solution defined on the interval J.

Remark 5.5 *For the condition that $F \in L^1_{loc}(I;C_\mu(\Omega;E))$, we note that*

$$\begin{aligned} F \in L^1_{loc}(I;C_\mu(\Omega;E)) &\Longrightarrow F(s,\cdot) \in C_\mu(\Omega;E) \\ &\Longrightarrow F(s,\cdot) : \Omega \to E \\ &\Longrightarrow F(s,x(s)) \in E \text{ and } x(s) \in \Omega \\ &\Longrightarrow \|F(s,x(s))\| = \|F(s,x(s))\|_E \end{aligned}$$

Also, I think that $F \in L^1_{loc}(I;C_\mu(\Omega;E))$ means that

$$\int_{I'} \|F(s,\cdot)\|_\mu ds < \infty$$

where I' is any closed subinterval of I. Note that this implies that there exists a locally integrable function $\gamma(t)$, which serves as the constant C in Definition 5.2. (The function γ depends on the function $F(t,\cdot) \in C_\mu(X,Y)$, and so on the parameter t.)

Remark 5.6 *As before, we defined* μ *so that* $\mu \in C(\mathbb{R}^+, \mathbb{R}^+)$, μ *increasing,* $\mu(0) = 0$, $\mu(x) > 0$ *for all* $x > 0$ *Thus, it is also true that*

$$\int_x^1 \frac{dr}{\mu(r)} < \infty, \text{ for all } x > 0$$

That is to say, the integrand has a singularity at zero only.

In order to prove this theorem, we start by establishing the uniqueness of trajectories. Let $x_1(t)$ and $x_2(t)$ be two solutions of (ODE) defined on a neighbourhood $\tilde{J}$ of t_0 with the same initial data x_0. Define

$$\rho(t) = \|x_1(t) - x_2(t)\|$$

We calculate,

$$\begin{aligned}
\rho(t) = \|x_1(t) - x_2(t)\| &= \left\| \int_{t_0}^t F(s, x_1(s)) - F(s, x_2(s))\, ds \right\|_E \\
&\leq \int_{t_0}^t \|F(s, x_1(s)) - F(s, x_2(s))\|_E\, ds \\
&= \int_{t_0}^t d(F(s, x_1(s)), F(s, x_2(s)))\, ds \leq \int_{t_0}^t \gamma(s)\mu(d(x_1(s), x_2(s)))\, ds \\
&= \int_{t_0}^t \gamma(s)\mu(\|x_1(s) - x_2(s)\|_E)\, ds = \int_{t_0}^t \gamma(s)\mu(\rho(s))ds.
\end{aligned}$$

Since F is in $L^1_{loc}(I; \mathcal{C}_\mu(\Omega, E))$, we immediately deduce that

$$0 \leq \rho(t) \leq \int_{t_0}^t \gamma(s)\mu(\rho(s))ds \quad \text{with} \quad \gamma \in L^1_{loc}(I) \quad \text{and} \quad \gamma \geq 0 \tag{5.10}$$

The function γ comes from Remark 5.5.

Lemma 5.3 *Let* ρ *be a measurable, positive function,* γ *a positive, locally integrable function and* μ *a continuous, increasing function. Assume that, for a positive real number* a, *the function* ρ *satisfies*

$$\rho(t) \leq a + \int_{t_0}^t \gamma(s)\mu(\rho(s))ds \tag{5.11}$$

If a *is different from zero, then we have*

$$-\mathcal{M}(\rho(t)) + \mathcal{M}(a) \leq \int_{t_0}^t \gamma(s)ds \quad \textit{where} \quad \mathcal{M}(x) = \int_x^1 \frac{dr}{\mu(r)} \tag{5.12}$$

If a *is zero and if* μ *satisfies* (5.9), *then the function* ρ *is identically zero.*

Proof: We first define

$$R_a(t) = a + \int_{t_0}^t \gamma(s)\mu(\rho(s))ds.$$

The function R_a is continuous and increasing. Hence, we have in the sense of distributions,

$$\dot{R}_a(t) = \gamma(t)\mu(\rho(t)).$$

Then

$$\begin{aligned}
\dot{R}_a(t) &= \gamma(t)\mu(\rho(t)) \leq \gamma(t)\mu(R_a(t)) \\
&\iff \mu(\rho(t)) \leq \mu(R_a(t)) \\
&\iff \rho(t) \leq R_a(t).
\end{aligned} \tag{5.13}$$

and the last inequality holds by assumption (5.11). We used the positivity of γ as well as the increasing nature of μ.

Assume that a is strictly positive. Then the function R_a is strictly positive. As the mapping $\mathcal{M}$ is continuously differentiable on the set of strictly positive real numbers s, it follows from (5.13) that

$$-\frac{d}{dt}\mathcal{M}\left(R_a(t)\right) = \frac{\dot{R}_a(t)}{\mu\left(R_a(t)\right)} \leq \gamma(t)$$

and integrating gives

$$\mathcal{M}\left(R_a\left(t_0\right)\right) - \mathcal{M}\left(R_a(t)\right) \leq \int_{t_0}^{t} \gamma(s)ds.$$

But $R_a\left(t_0\right) = a$, and $-\mathcal{M}$ is increasing, being the integral of a positive function in the forward direction, so

$$\rho(t) \leq R_a(t) \Longrightarrow -\mathcal{M}(\rho(t)) \leq -\mathcal{M}\left(R_a(t)\right),$$

hence,

$$\mathcal{M}(a) - \mathcal{M}(\rho(t)) \leq \mathcal{M}\left(R_a\left(t_0\right)\right) - \mathcal{M}\left(R_a(t)\right) \leq \int_{t_0}^{t} \gamma(s)ds$$

which is Equation (5.12). Now let $a = 0$. Then, in fact, (5.11) holds for any $a' \geq 0$

$$\rho(t) \leq a' + \int_{t_0}^{t} \gamma(s)\mu(\rho(s))ds.$$

Hence, the first part of the lemma applies, giving

$$\mathcal{M}\left(a'\right) \leq \int_{t_0}^{t} \gamma(s)ds + \mathcal{M}(\rho(t)).$$

Assume, by way of contradiction, that ρ is not identically zero. Then there exists some t_1 such that $\rho\left(t_1\right) > 0$, hence

$$\mathcal{M}\left(a'\right) = \int_{a'}^{1} \frac{dr}{\mu(r)} \leq \int_{t_0}^{t_1} \gamma(s)ds + \mathcal{M}\left(\rho\left(t_1\right)\right) = C,$$

where C is a constant. The critical point is that $C < \infty$, which follows, by Remark 5.6, from $\rho\left(t_1\right) > 0$.

Letting $a' \to 0^{+}$, we conclude that

$$\int_{0}^{1} \frac{dr}{\mu(r)} \leq C$$

contradicting the assumption in Equation (5.9). This completes the proof of the lemma. ∎

Proof of Theorem 5.3

Proof:

Thanks to inequality (5.11), the uniqueness of the integral curves passing through a given point is an immediate consequence of Lemma 5.3. Let us prove the existence. Consider the classical Picard scheme

$$x_{k+1}(t) = x_0 + \int_{t_0}^{t} F\left(\tau, x_k(\tau)\right) d\tau.$$

Next, we verity the fact that, for J sufficiently small, we remain in the domain of definition of the function F and that the sequence $\left(x_k\right)_{k\in\mathbb{N}}$ is bounded in $L^{\infty}(J)$. We verify these two things. Now,

$$\left\|x_{k+1}(t) - x_0\right\|_E \leq \int_{t_0}^{t} \left\|F\left(\tau, x_k(\tau)\right)\right\| d\tau \leq \int_{t_0}^{t} \gamma(\tau)\mu\left(\left\|x_k(\tau)\right\|_E\right) d\tau,$$

as long as $x_k(\tau)$ remains in the domain of $F(\tau, \cdot)$ (i.e., in Ω).

We then have,

$$\begin{aligned} f_{k+1}(t) := \left\|x_{k+1}(t) - x_0\right\|_E &\leq \left(\int_{t_0}^{t} \gamma(s)ds\right) \mu\left(\sup_{\tau\in[t_0,t]} \left\|x_k(\tau)\right\|_E\right) \\ &= C(t)\mu\left(\sup_{\tau\in[t_0,t]} \left\|x_k(\tau)\right\|_E\right), \end{aligned}$$

where

$$C(t)=\int_{t_0}^{t}\gamma(s)ds.$$

Where we used the fact that μ is increasing to bring the supremum inside μ. $C(t)$ is continuous by the continuity of the integral, and μ is continuous by definition. Also, $C(t)$ is nondecreasing. Hence, $f_{k+1}(t)$ is continuous. Since Ω is open, there exists an $r>0$ such that an r -neighborhood of x_0 remains in the domain of definition of F. It will be our goal to show that for a suitable interval J, $\|x_k(t)-x_0\|_E$ remains less than r, for then $x_k(t)$ will remain within the domain of definition of F, and $\|x_k\|_E$ will be bounded by $\|x_0\|_E+r$.

Let

$$R=\mu\left(\|x_0\|_E+r\right).$$

Choose $t_1>t_0$ so that $C\left(t_1\right)R<r$, which is possible by the continuity of $C(t)$ and the fact that $C\left(t_0\right)=0$. Let $J=[t_0,t_1]$.

Let the induction hypothesis be that $f_k(t)=\|x_k(t)-x_0\|_E<r$ for all $t\in J$. The function x_1, which is constant, certainly obeys the induction hypothesis, so assume that the hypothesis is satisfied for k. Then for any $t\in J$

$$f_{k+1}(t)<C(t)\mu\left(\sup_{\tau\in[t_0,t]}\|x_k(\tau)\|_E\right)<C(t)R<C\left(t_1\right)R<r,$$

where we used the fact that $C(t)$ is increasing, and where we used the induction hypothesis to know that $\sup_{\tau\in[t_0,t]}\|x_k(\tau)\|<\|x_0\|_E+r$ and so conclude that

$$\mu\left(\sup_{\tau\in[t_0,t]}\|x_k(\tau)\|_E\right)<R.$$

This completes the proof by induction and, as observed above, this shows that on the interval J, x_k remains within the domain of definition of F and $\|x_k\|_E$ is bounded by $\|x_0\|_E+r$.

Now, we are going to prove that the sequence so defined is a Cauchy sequence in the space of continuous functions from the interval J into E. To do so, we define

$$\rho_{k+1,n}(t)=||x_{k+1+n}(t)-x_{k+1}(t)\|.$$

It follows that

$$0\leq\rho_{k+1,n}(t)\leq\int_{t_0}^{t}\gamma(\tau)\mu\left(\rho_{k,n}(\tau)\right)d\tau.$$

Defining $\rho_k(t)=\sup_n\|x_{k+1+n}(t)-x_{k+1}(t)\|$, as μ is increasing, we deduce that

$$0\leq\rho_{k+1}(t)\leq\int_{t_0}^{t}\gamma(\tau)\mu\left(\rho_k(\tau)\right)d\tau.$$

Since μ is increasing and by Fatou's lemma, we obtain from the inequality above that

$$\tilde{\rho}(t)\overset{\text{def}}{=}\limsup_{k\to+\infty}\rho_k(t)\leq\int_{t_0}^{t}\gamma(\tau)\mu(\tilde{\rho}(\tau))d\tau.$$

Applying Lemma 5.3 again, we find that $\tilde{\rho}(t)$ vanishes in a neighbourhood of t_0 so the proof of Theorem 5.3 is complete. ∎

Proof of Theorem 5.2

Proof:

We define the flow, ψ, from the integral curves. An integral curve, $\alpha(t)$, is a curve, $\alpha:I\to\mathbb{R}^d$ for some interval I that satisfies

$$\frac{d\alpha}{dt}=v(t,\alpha(t))$$

for all $t \in I$. Given an initial value, this is written equivalently as

$$\alpha(t) = \alpha(0) + \int_0^t v(s, \alpha(s)) ds.$$

We can then define $\psi(t, x) = \alpha(t)$ under the assumption that $\alpha(0) = x$.

From Theorem 5.3, there exists a unique flow ψ. The continuity of ψ follows from the continuity of α and the bound on $\|x_1(t) - x_2(t)\| = \|\psi(t, x_1) - \psi(t, x_2)\|$ as follow. An important point is that the integral curves α, and hence the flow ψ have all of $\mathbb{R}$ as their domain. This would follow from the extra condition we imposed in the statement of Theorem 5.2. Thus the integral, above, defining $\alpha(t)$ is valid for all $t \in \mathbb{R}$, since

$$\left\| \int_0^t v(s, \alpha(s)) ds \right\| \leq \int_0^t \|v(s, \cdot)\|_{L^\infty} ds < \infty.$$

It is now sufficient to study the regularity in the variable x of the flow ψ. To do this, consider two integral curves of v, denoted $x_1(t)$ and $x_2(t)$, starting respectively from two distinct points x_1 and x_2 such that

$$\|x_1 - x_2\| < 1.$$

The following inequalities are true only if

$$\|x_1(t) - x_2(t)\| < 1.$$

It follows that

$$\begin{aligned} \|x_1(t) - x_2(t)\| &\leq \|x_1 - x_2\| + \int_0^t \|v(\tau, x_1(\tau)) - v(\tau, x_2(\tau))\| d\tau \\ &\leq \|x_1 - x_2\| + \int_0^t \|v(\tau)\|_{LL} \times \mu(\|x_1(\tau) - x_2(\tau)\|) d\tau. \end{aligned}$$

For the first part of the inequality comes from subtracting the expression for two integral curves coming from the expression for the flow in Theorem 5.2. That is, $x_i(t)$ is the value of the curve generated by starting at the point $x_i(0) = x_i$ and flowing for time t. As for the second part of the inequality, we observe that

$$\begin{aligned} &\|v(\tau, x_1(\tau)) - v(\tau, x_2(\tau))\| \\ &= \frac{\|v(\tau, x_1(\tau)) - v(\tau, x_2(\tau))\|}{\|x_1(\tau) - x_2(\tau)\| (1 - \log \|x_1(\tau) - x_2(\tau)\|)} \\ &\times \|x_1(\tau) - x_2(\tau)\| (1 - \log \|x_1(\tau) - x_2(\tau)\|) \\ &\leq \left(\sup_{0<|x-x'|\leq 1} \frac{\|v(\tau, x)) - v(\tau, x')\|}{\|x - x'\| (1 - \log \|x - x'\|)} \right) \mu(\|x_1(\tau) - x_2(\tau)\|) \\ &= \|v(\tau)\|_{LL} \times \mu(\|x_1(\tau) - x_2(\tau)\|) \end{aligned}$$

where

$$\mu(r) = r(1 - \log r).$$

Observe that $\lim_{r \to 0+} \mu(r) = 0, \mu$ is increasing, and

$$\int_0^1 \frac{dr}{\mu(r)} = \lim_{a \to 0+} [-\log(1 - \log(x))]_a^1 = \infty$$

so the required conditions on μ, are satisfied.

Let us apply Lemma 5.3 with $\rho(t) = \|x_1(t) - x_2(t)\|, a = \|x_1 - x_2\|$ and $\gamma(t) = \|v(t)\|_{LL}$. We infer that

$$-\log(1 - \log \|x_1(t) - x_2(t)\|) + \log(1 - \log \|x_1 - x_2\|) \leq \int_0^t \|v(\tau)\|_{LL} d\tau$$

Taking a double exponential, as in the previous section, it follows that

$$\|x_1(t) - x_2(t)\| \leq \|x_1 - x_2\|^{\exp\left(-\int_0^t \|v(s)\|_{LL} ds\right)} e^{1 - \exp\left(-\int_0^t \|v(s)\|_{LL} ds\right)},$$

as long as $\|x_1(t) - x_2(t)\| < 1$. Up to now, we have completed the "more precisely" part of the Theorem. For the rest part, let $r = \exp\left(-\int_0^t \|v(s)\|_{LL} ds\right)$. Then $0 < r < 1$, so

$$\|(\psi(t) - \mathrm{Id})(x)\|_{C^r} = \|\psi(t) - \mathrm{Id}\|_{L^\infty} + \sup_{x \neq y} \left\{ \frac{\|\psi(t,x) - \psi(t,y)\|}{\|x-y\|^r} \right\}.$$

But

$$\begin{aligned} \|\psi(t) - \mathrm{Id}\|_{L^\infty} &= \sup_{x \in \mathbb{R}^d} \|\psi(x,t) - \psi(x,0)\| \\ &= \sup_{x \in \mathbb{R}^d} \left\| \int_0^t v(s, \psi(s,x)) ds \right\| \\ &\leq \int_0^t \|v(s,\cdot)\|_{L^\infty} ds \end{aligned}$$

We need this last expression to be finite, which follow from the extra assumption that we introduced in the statement of Theorem 5.2.

Also,

$$\sup_{x \neq y} \left\{ \frac{\|\psi(t,x) - \psi(t,y)\|}{\|x-y\|^r} \right\} \leq \exp\left(1 - \exp\left(-\int_0^t \|v(s)\|_{LL} ds\right)\right) \leq e$$

Thus, $\|(\psi(t) -$ Id $)(x)\|_{C^r}$ is finite, so $\psi(t)-$ Id $\in C^r$,, we complete the proof of Theorem 5.2. ∎

Proof of the Theorem 5.1: Part B

Proof: In Part A we have obtained the existence and uniqueness solution. From Theorem 5.2, we get the flow

$$\psi(t,x) = x + \int_0^t v(s, \psi(s,x)) ds$$

and some Hölder estimate. What remains to show is that the vector fields are in $L^1_{loc}(R; LL)$ and that, specifically,

$$\|v(s)\|_{LL} \leq C \|\omega_0\|_{L^\infty \cap L^a}$$

so that we have membership in the Hölder space claimed in the statement of Theorem 5.1 (Yudovich's theorem).

From the Biot-Savart lawwhich requires that a be strictly less than 2we have

$$\begin{aligned} I := &|v(t,x) - v(t,x')| \\ &= \frac{1}{2\pi} \int \omega(t,y) \left[\frac{(x-y)^\perp}{|x-y|^2} - \frac{(x'-y)^\perp}{|x'-y|^2} \right] dy \\ &\leq \frac{1}{2\pi} \sqrt{(I_1)^2 + (I_2)^2}, \end{aligned}$$

where

$$I_2 := \int |\omega(t,y)| \left| \frac{x^1 - y^1}{|x-y|^2} - \frac{(x')^1 - y^1}{|x'-y|^2} \right| dy,$$

and where I_1 is defined similarly.

The technique we use to bound I_2 will clearly apply equally well to I_1, so we will deal only with I_2.

Let $a = |x - x'|/2$, and let R be some fixed real number much greater than 1. Then we can split I_2 into three integrals:

$$I_2 = J + K + L,$$

where

$$J := \int_{B_{10a}} |\omega(t,y)| f(y) dy, \quad K := \int_{B_R \setminus B_{10a}} |\omega(t,y)| f(y) dy,$$

and

$$L := \int_{\mathbb{R}^2 \setminus B_R} |\omega(t,y)| f(y) dy$$

with

$$f(y) := \left| \frac{x^1 - y^1}{|x-y|^2} - \frac{(x')^1 - y^1}{|x'-y|^2} \right|.$$

We bound J, K, and L differently. First, because $1/|y|$ is locally integrable and increases linearly with the radius of the ball we integrate over, we have

$$\begin{aligned} J &\le \|\omega(t)\|_{L^\infty} \int_{B_{10a}} \frac{|x^1 - y^1|}{|x-y|^2} + \frac{\left|(x')^1 - y^1\right|}{|x'-y|^2} dy \\ &\le 2 \left\|\omega^0\right\|_{L^\infty} \int_{B_{20a}} \frac{|y^1|}{|y|^2} dy \le 2 \left\|\omega^0\right\|_{L^\infty} \int_{B_{20a}} \frac{1}{|y|} dy \\ &\le 80\pi \left\|\omega^0\right\|_{L^\infty} a, \end{aligned}$$

where we use the equality $\|\omega(t)\|_{L^\infty} = \left\|\omega^0\right\|_{L^\infty}$, which as in [3] equation (3) p. 1775, by the maximum principle. (If we knew we had a flow, it would follow by that; but we are trying to establish the existence of such a flow.)

To bound K, place the origin halfway between x and x', with x' placed at $(a, 0)$ and x at $(-a, 0)$. Then

$$\begin{aligned} (x')^1 - y^1 = a - r\cos\theta, &\quad |x' - y|^2 = a^2 + r^2 - 2ar\cos\theta \\ x^1 - y^1 = -a - r\cos\theta, &\quad |x-y|^2 = a^2 + r^2 + 2ar\cos\theta \end{aligned}$$

since $10a >> a$, we can approximate the integrand, f, in K by observing that both denominators are dominated by the r^2 term, so

$$\begin{aligned} f(y) = f(r,\theta) &= \left| \frac{-a - r\cos\theta}{a^2 + r^2 + 2ar\cos\theta} - \frac{a - r\cos\theta}{a^2 + r^2 - 2ar\cos\theta} \right| \\ &\cong \left| \frac{-a - r\cos\theta}{r^2} - \frac{a - r\cos\theta}{r^2} \right| = \frac{2a}{r^2}. \end{aligned}$$

Then, to within some order of accuracy (this should be refined, obviously),

$$\begin{aligned} K &\le 2\pi \left\|\omega^0\right\|_{L^\infty} \int_{10a}^{R} \frac{2a}{r^2} r\,dr = 4\pi \left\|\omega^0\right\|_{L^\infty} a[\log R - \log(10a)] \\ &= 4\pi \left\|\omega^0\right\|_{L^\infty} a(C - \log a). \end{aligned}$$

As for L, we have, again to within some order of accuracy,

$$L \le \|\omega(t)\|_{L^p(\mathbb{R}^2 \setminus B_R)} \|f\|_{L^q(\mathbb{R}^2 \setminus B_R)} \le 2a \|\omega(t)\|_{L^p(\mathbb{R}^2)} \left\| \frac{1}{r^2} \right\|_{L^q(\mathbb{R}^2 \setminus B_R)},$$

where $1/p + 1/q = 1$. The only restriction on q is that it be strictly greater than 1, meaning that we can choose any $1 \le p < \infty$, giving

$$L \le C' a \|\omega(t)\|_{L^p} = C' a \left\|\omega^0\right\|_{L^p},$$

where the constant C' depends only upon our choice of p. We used the observation as before, which it is important to note does not depend upon the existence of a flow. From our bounds on J, K, and L, we have

$$\begin{aligned} I_2 &\le 80\pi \left\|\omega^0\right\|_{L^\infty} a + 4\pi \left\|\omega^0\right\|_{L^\infty} a(C - \log a) + C' \left\|\omega^0\right\|_{L^p} a \\ &= C \left\|\omega^0\right\|_{L^\infty \cap L^p} a \left(C' - \log a\right). \end{aligned}$$

With the identical bound on I_1, we can write

$$I \le \frac{1}{2\pi} \sqrt{(I_1)^2 + (I_2)^2} \le C \left\|\omega^0\right\|_{L^\infty \cap L^p} a \left(C' - \log a\right).$$

Putting this all together, we have

$$
\begin{aligned}
&\frac{|v(t,x)-v\left(t,x'\right)|}{|x-x'|\,|1-\log\left(x-x'\right)|}\\
&=\frac{I}{2a|1-\log(2a)|}\leq\frac{C\left\|\omega^{0}\right\|_{L^{\infty}\cap L^{p}}a\left(C'-\log a\right)}{2a|1-\log(2a)|}\\
&=C\left\|\omega^{0}\right\|_{L^{\infty}\cap L^{p}}\frac{C'-\log a}{|1-\log(2a)|}\\
&=C\left\|\omega^{0}\right\|_{L^{\infty}\cap L^{p}}\frac{C'-\log\left(|x-x'|/2\right)}{|1-\log\left(|x-x'|\right)|}
\end{aligned}
$$

The supremum over all $0<|x-x'|\leq 1$ is

$$C\left\|\omega^{0}\right\|_{L^{\infty}\cap L^{p}},$$

(with a new choice of C), which gives the norm

$$\|v(t)\|_{LL}=C\left\|\omega^{0}\right\|_{L^{\infty}\cap L^{p}},$$

with p in place of a. ∎

5.3 An example

The aim of this section is to present a solution of the incompressible Euler system, which shows that the Yudovich theorem is optimal.

We are going to construct a solution satisfying the following properties:

- the vorticity ω of the vector field v solution is, at all time t, bounded, and equal to zero outside a compact set;
- at all time t, the flow $\psi(t)$ of v does not belong to the Hölder class $C^{\exp(-t)}$.

We will start by constructing the initial data. Let ω_0 be the function defined on the plane $\mathbb{R}^2$, equal to zero outside $[-1,1]\times[-1,1]$, odd with respect to both variables x_1 and x_2, and whose value is 2π on $[0,1]\times[0,1]$. Let us consider the vector field v_0 defined by

$$v_0\left(x_1,x_2\right)=\left\{\begin{array}{l}-\frac{1}{2\pi}\int\frac{x_2-y_2}{|x-y|^2}\omega_0(y)dy\\ \frac{1}{2\pi}\int\frac{x_1-y_1}{|x-y|^2}\omega_0(y)dy.\end{array}\right.$$

We are going to prove the following theorem.

Theorem 5.4 *Let v be the solution of the Euler equation associated with the initial data v_0 defined above. At the time t, the flow $\psi(t)$ of the vector field v belongs to no C^{α}, for any $\alpha>\exp -t$.*

We are now going to spend some time studying the vector field v_0. This vector field is of course not Lipschitzian. The example constructed in section 3.2, the properties of which are described in Proposition 3.2 shows that the size of some of the partial derivatives of v is equivalent to the logarithm of the distance to the corner of the square.

Here, the vector field v_0 has certain symmetries. This is going to enable us to describe it more explicitly. Indeed, the vector field v_0 is symmetric with respect to the two coordinate axes. As a result, that vector field is tangential to those two axes and therefore vanishes at the origin. We are going to prove the following proposition.

Proposition 5.1 *There exists a constant C such that for all x_1 in $[0,C]$, we have*

$$v_0^1\left(x_1,0\right)\geq -2x_1\log x_1.$$

Proof: Let $\tilde{\omega}_0(x_1) = 2H(x_1) - 1$($H$ is the Heaviside function), we get

$$\begin{aligned} v_0^1(x_1, 0) &= \frac{1}{2\pi} \int \frac{y_2}{|x-y|^2} \omega_0(y) dy \\ &= \int_{-1}^{1} dy_1 \widetilde{\omega}_0(y_1) \int_0^1 \frac{2y_2}{(x_1 - y_1)^2 + y_2^2} dy_2 \\ &= \int_{-1}^{1} dy_1 \widetilde{\omega}_0 \left[\log((x_1 - y_1)^2 + 1) - \log((x_1 - y_1)^2) \right]. \end{aligned}$$

An immediate computation leads to

$$\begin{aligned} v_0^1(x_1, 0) &= \widetilde{v}_0^1(x_1, 0) + \bar{v}_0^1(x_1, 0) \quad \text{with} \\ \widetilde{v}_0^1(x_1, 0) &= -\int_0^1 \log(x_1 - y_1)^2 dy_1 + \int_0^1 \log(x_1 + y_1)^2 dy_1 \quad \text{and} \\ \bar{v}_0^1(x_1, 0) &= \int_0^1 \log \frac{1 + (x_1 - y_1)^2}{1 + (x_1 + y_1)^2} dy_1. \end{aligned}$$

It is obvious that the function $x_1 \mapsto \bar{v}_0^1(x_1, 0)$ is an odd, infinitely differentiable function. Furthermore, an elementary integral evaluation guarantees that we have, for $0 \le x_1 < 1$

$$\tilde{v}_0^1(x_1, 0) = -4x_1 \log x_1 + 2(1 + x_1) \log(1 + x_1) - 2(1 - x_1) \log(1 - x_1).$$

Therefore, when $0 \le x_1 < 1$, we have

$$v_0^1(x_1, 0) = -4x_1 \log x_1 + f(x_1),$$

where f is

$$\begin{aligned} f(x_1) := 2(1 + x_1) \log(1 + x_1) - 2(1 - x_1) \log(1 - x_1) \\ + \int_0^1 \log \frac{1 + (x_1 - y_1)^2}{1 + (x_1 + y_1)^2} dy_1 \end{aligned}$$

which we see by inspection is, in fact, odd. since f is smooth (infinitely differentiable) in a neighborhood of 0, it's derivative is bounded there, so $|f(x_1)| \le C|x_1| + f(0) = C|x_1|$ in that neighborhood ($f(0) = 0$ because f is odd). But the function $-4x_1 \log x_1$ has an infinite derivative at $x_1 = 0$ so it increases faster than any linear function. Therefore, we can absorb the $C|x|$ into $-4x_1 \log x_1$, reducing the constant 4 by any amount we wish. This ensures the conclusion of the proposition. ∎

Proof of Theorem 5.4

Proof: Let us now return to the Euler equation, and its solution v, corresponding to the initial data v_0. According to Yudovich's theorem, the flow of the vector field v is a continuous function of the variable (t, x). Moreover, we know that at all time, the vector field v is symmetric with respect to the two coordinate axes. Therefore both these axes are globally invariant under the flow. The origin, which is their intersection point, is therefore stable under the flow ψ of the vector field v. Hence we have, for all t

$$\psi(t, 0) = 0, \quad \psi^1(t, 0, x_2) = 0 \quad \text{and} \quad \psi^2(t, x_1, 0) = 0 \tag{5.14}$$

Let T be an arbitrary, strictly positive real number. The vorticity is preserved along the flow lines. The identity (5.14) above implies therefore the existence of a neighbourhood W of the origin such that we have, for all $t \in [0, T]$

$$\omega(t)_{|W} = \omega_{0|W}.$$

First, equation (5.14) ensures that the flow lines don't cross quadrants, so that the vorticity maintains it values of $\pm 2\pi$ or zero in each quadrant. Second, since $\psi(t)$ is a diffeomorphism that fixes the origin, it maps any open neighborhood A of the origin to another open neighborhood $A(t)$ of the origin. These two facts together would seem to suggest that as long as we choose A so that it is contained in the support of ω_0- the square $[-1, 1] \times [-1, 1]-$ that

$$W = \bigcap_{t \in (0, T)} A(t)$$

$t \in [0, T]$ would suffice for the neighborhood.

The problem is, it is not obvious that W remains open. Fortunately, it turns out that all we need it the existence of a $T > 0$ such that such a W exists, and to obtain this we can use the boundedness of the velocity over any finite time to insure that for small enough T the flow lines cannot carry any of the zero vorticity outside the square $[-1, 1] \times [-1, 1]$ to within a fixed finite distance of the origin, which will ensure that W is open.

Another possible approach is to let $r : [0, T] \to \mathbb{R}^+$ be the distance from the origin to the image of the boundary of the square $[-1, 1] \times [-1, 1]$ under $\psi(t)$. Because ψ is continuous in time and space, r is continuous and so achieves its infimum on $[0, T]$. This infimum then cannot be zero, since $\psi(t)$ is a diffeomorphism fixing the origin. But the origin is an internal point of the square and so its image (again the origin) must be an internal point of the image of the square; hence, $r(t) > 0$ for all time t. This allows us to choose W to be the ball centered at the origin with a radius equal to $\inf_{t\in[0,T]} r(t)$.

The divergence-free vector field $\tilde{v}(t) = v(t) - v_0$ is symmetric with respect to the two coordinate axes. Its vorticity is identically zero on W. Therefore there exists a constant A such that for all $t \in [0, T]$, we have

$$|v(t, x) - v_0(x)| \leq A|x|.$$

In fact, since $v(t) - v_0$ is divergence-free we can apply the Biot-Savart law in the form $v(t) - v_0 = K * \omega$ where K is the Biot-Savart kernel and $\omega = \omega\left(v(t) - v_0\right)$. Since ω is zero in the neighborhood W of the origin, it follows that the partial derivatives of $v(t) - v_0$ exist and are continuous (in fact, $v(t) - v_0$ is smooth) in W, since

$$\partial_k \left(v(t) - v_0\right)(x) = \left(\partial_k K\right) * \omega(x)$$

and the singularity in the convolution is avoided as long as x is in W, because ω is zero near where the singularity of $\partial_k K$ occurs. Thus $D\left(v(t) - v_0\right)$ is bounded on W (or rather, on a smaller neighborhood than W which we relabel as W). Using the fact that $v(t, 0) = v_0(0) = 0$, and arguing as we did earlier, it follows that $|v(t, x) - v_0(x)| \leq A|x|-$ though only on W which is all we actually need.

From Proposition 5.1 we get the existence of a constant C' such that, for any couple of real numbers $(t, x_1) \in [0, T] \times [0, C']$, we have

$$v^1\left(t, x_1, 0\right) \geq -x_1 \log x_1.$$

Since $\psi^1(t, 0) = 0$ and ψ continuous, let $x_1 \in [0, 1)$ such that, for all $t \in [0, T]$, we have

$$\psi^1\left(t, x_1, 0\right) \in [0, C'].$$

It follows from the estimate above that

$$\psi^1\left(t, x_1(0), 0\right) \geq x_1(t) \quad \text{with} \quad \dot{x}_1(t) = -x_1(t) \log x_1(t)$$

That is, choose a value of x_1 and consider $(x_1, 0)$ to be the initial position $(x_1(0), 0)$ of a point at time zero, and let $(x_1(t), 0)$ be the position of the point at time t moving under the flow. Then $v^1\left(t, x_1, 0\right) \geq -x_1 \log x_1$ means that the point moves at least as far as what would result if it moved horizontally to the right (or the left if $x_1 > 1$) at a speed given by $-x_1 \log x_1$ for all t in $[0, T]$. That is, the position at time t, $\left(\psi^1\left(t, x_1, 0\right), 0\right)$ is at least as far to the right as the solution to

$$\dot{x}_1(t) = -x_1(t) \log x_1(t).$$

Integrating $\dot{x}_1(t) = -x_1(t) \log x_1(t)$ gives

$$\begin{aligned} \log\log x_1(t) - \log\log x_1(0) = -t \Longrightarrow \log x_1(t) &= \ln x_1(0) e^{-t} \\ \Longrightarrow x_1(t) &= \exp\left(\ln x_1(0) e^{-t}\right) = x_1(0)^{\exp(-t)}. \end{aligned}$$

So it follows that

$$\psi^1\left(t, x_1, 0\right) \geq x_1^{\exp(-t)}.$$

Since $\psi(t,0)=0$, let $f:\mathbb{R}^2\to\mathbb{R}^2$ be given by $f=\psi(t)-$ Id. To complete the proof of Theorem 5.4, we must show that f is not in C^α for any $\alpha>e^{-t}$. To see this, observe that for $x_1>0$

$$\begin{aligned}\frac{|f(t,x_1,0)-f(t,0,0)|}{|(x_1,0)-(0,0)|^\alpha}&=\frac{|(\psi(t,x_1,0)-(x_1,0))-(\psi(t,0,0)-(0,0))|}{x_1^\alpha}\\&=\frac{|\psi(t,x_1,0)-(x_1,0)|}{x_1^\alpha}\geq\frac{x_1^{\exp(-t)}-x_1}{x_1^\alpha}\\&=x_1^{\exp(-t)-\alpha}-x_1^{1-\alpha}\end{aligned}$$

which is unbounded when $\alpha>\exp(-t)$, since then $x_1^{\exp(-t)-\alpha}$ approaches infinity as x_1 approaches zero. Then Theorem 5.4 is proved. ■

5.4 The vortex patch problem

The vortex patch problem is as follows: let us suppose that the vorticity is, initially, the characteristic function of an open bounded set D_0, with a boundary of Hölder class $C^{k+\epsilon}$, where k is a strictly positive integer and ϵ is a real number in the interval $(0,1)$. According to Theorem 5.1, there exists a unique vector field, a solution of the Euler equations on $\mathbb{R}\times\mathbb{R}^2$, whose vorticity belongs to $L^\infty\left(\mathbb{R}^3\right)$. Such a vector field has a flow ψ with exponentially decreasing regularity in time, that is to say $\psi(t)$ is a homeomorphism of Hölder class $C^{\exp(-\alpha t)}$. According to the identity (1.12) which states the preservation of the vorticity along the flow lines, the vorticity at time t is thus the characteristic function of an open bounded set D_t, whose topology is unchanged. On the other hand, the boundary of that open set is now a priori only of class $C^{\exp(-\alpha t)}$. So two very natural questions arise: does the boundary of that open set remain smooth during a small time interval? If so, what happens for large time intervals?

Remark 5.7 *Any vector function v with $divv=0$ will be called a flow. If $v_n|_\Gamma=0$ (the outward normal component of v on Γ) in addition, it will be called a tangential flow.*

In the case when the vorticity is initially the characteristic function of the interior of a closed curve of the plane, simple and of class $C^{1+\epsilon}$, one can be tempted to use the following approach. Let γ_0 be an embedding of the circle $\mathbf{S}^1$ of class $C^{1+\epsilon}$ whose range is the boundary of the open set D_t. The solution vector field is then completely known when the boundary of the open set is known. Let us then look for a parametrization of that boundary by the function γ defined by by

$$\partial_t\gamma(t,s)=v(t,\gamma(t,s)). \tag{5.15}$$

But according to Biot-Savart's law, the solution vector field is defined by

$$v(t)=\nabla^\perp f(t)\quad\text{ with }\quad f(t,x)=\frac{1}{2\pi}\int_{D_t}\log|x-y|dy$$

If we assume that $\gamma(t,\cdot)$ is an embedding of class $C^{1+\epsilon}$ of the circle, it follows, from Green's formula, that

$$v(t,x)=\frac{1}{2\pi}\int_0^{2\pi}\log|x-\gamma(t,\sigma)|\partial_\sigma\gamma(t,\sigma)d\sigma.$$

In fact, let $F_i:\mathbb{R}^2\to\mathbb{R}^2, i=1,2$, where $F_1(x)=(0,\log|x|)$ and $F_2(x)=(-\log|x|,0)$. From the divergence theorem,

$$\int_{D_t}\operatorname{div}F_i(x-y)dy=\int_{\partial D_t}F_i(x-y)\cdot n d\sigma(y)$$

where n is a unit normal vector to the boundary. But,

$$(\operatorname{div}F_1(x-y),\operatorname{div}F_2(x-y))=\nabla^\perp\log|x-y|,$$

while

$$(F_1(x-y)\cdot n,F_2(x-y)\cdot n)=\log|x-y|\tau,$$

where τ, a unit tangent vector, is n rotated 90 degrees counter-clockwise, so

$$\nabla^{\perp} \int_{D_t} \log|x-y| dy = \int_{\partial D_t} \log|x-y| \tau d\sigma(y).$$

(We also used the fact that $\nabla_x^{\perp} \log|x-y| = \nabla_y^{\perp} \log|x-y|$.) Applying this to to unit-speed parameterization of the boundary gives the above equation.

According to (5.15), we have to solve, in the set of embeddings of class $C^{1+\varepsilon}$ the following equation:

$$\partial_t \gamma(t,s) = \frac{1}{2\pi} \int_0^{2\pi} \log|\gamma(t,s) - \gamma(t,\sigma)| \partial_\sigma \gamma(t,\sigma) d\sigma. \tag{5.16}$$

In section 5.5, we will prove a theorem which will in particular lead to the following theorem.

Theorem 5.5 *Let ϵ be a real number in the interval $(0,1)$ and let γ_0 be a function in the space $C^{1+\epsilon}(\mathbf{S}^1; \mathbb{R}^2)$, one to one, and whose derivative does not vanish. Then there exists a unique solution $\gamma(t,s)$ to equation (5.15) belonging to $L^\infty_{loc}(\mathbb{R}; C^{1+\epsilon}(\mathbf{S}^1; \mathbb{R}^2))$ and which is, for all time, an embedding of the circle.*

In order to understand this problem, we give some explanations. The Euler equations (without forcing) in velocity form can be written,

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases}$$

where u is the velocity field and p is the pressure. The operator $u \cdot \nabla = u^i \partial_i$, where we follow the usual convention that repeated indices are summed over. These equations model the flow of an incompressible inviscid fluid.

By introducing the 2D vorticity,

$$\omega = \partial_1 u^2 - \partial_2 u^1,$$

we obtain the vorticity formulation,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \\ u = K * \omega \end{cases}$$

Here,

$$K(x) = \frac{1}{2\pi} \frac{x^{\perp}}{|x|^2}, \quad x^{\perp} := (-x_2, x_1)$$

is the Biot-Savart kernel, which can also be written

$$K = \nabla^{\perp} \mathcal{F}, \quad \mathcal{F}(x) = \frac{1}{2\pi} \log|x|, \quad \nabla^{\perp} := (-\partial_2, \partial_1)$$

$\mathcal{F}$ being the fundamental solution to the Laplacian.

Let $\eta(t,x)$ be the flow map associated to the velocity field u, so that

$$\partial_t \eta(t,x) = u(t, \eta(t,x)), \quad \eta(0,x) = x.$$

Then vorticity formulation tells us that the vorticity is transported by the flow map, so that

$$\omega(t,x) = \omega_0\left(\eta^{-1}(t,x)\right) \tag{5.17}$$

is the vorticity of the solution to the Euler equations at time t, where ω_0 is the initial vorticity.

All this presupposes that sufficiently regular solutions exist and are unique. In fact, it all can be made sense of for initial vorticity in $L^1 \cap L^\infty$, in which case the vorticity remains in $L^1 \cap L^\infty$, as first shown by Yudovich. One must, however, use a weak formulation continue to hold.

If the vorticity is initially the characteristic function of a bounded domain, it will remain so for all time as the Euler solution evolves, since $\eta(t,\cdot)$ is a diffeomorphism. A (classical) vortex patch is such a bounded domain. So if

$$\omega_0 = \mathbf{1}_\Omega, \tag{5.18}$$

where Ω is a bounded domain, then

$$\omega(t)=\mathbf{1}_{\Omega_t}, \quad \Omega_t:=\eta(t,\Omega)$$

The bounded domain, Ω_t, is the vortex patch at time t. The regularity of the boundary of Ω will be specified using a parameter, ϵ. Throughout this paper we fix $\epsilon\in(0,1)$. We can now state the result of more precisely.

Theorem 5.6 *Let Ω be a bounded domain whose boundary is the image of a simple closed curve $\gamma_0\in C^{1+\epsilon}\left(\mathbf{S}^1\right)$ and let ω_0 be as in (5.20). There exists a unique solution u to the 2D Euler equations, with*

$$\nabla u(t)\in L^{\infty}\left(\mathbb{R}^2\right), \quad \gamma(t,\cdot):=\eta\left(t,\gamma_0(\cdot)\right)\in C^{1+\epsilon}\left(\mathbf{S}^1\right) \text{ for all } t\in\mathbb{R}.$$

5.5 Proof of the persistence

In what follows, we denote by ϵ an arbitrary real number in the interval $(0,1)$ and by Σ an arbitrary closed set of the plane (eventually empty).

Definition 5.3 *Let $X=(X_\lambda)_{\lambda\in\Lambda}$ be a family of vector fields such that both they and their divergences are of class C^ϵ. Such a family is said to be admissible outside Σ if and only if we have*

$$I(\Sigma,X)=\inf_{x\notin\Sigma}\sup_{\lambda\in\Lambda}|X_\lambda(x)|>0$$

We now define the concept of tangential regularity with respect to such a family of vector fields.

Definition 5.4 *Let X be a regular family of vector fields such that both they and their divergences are of class C^ϵ and which is admissible outside Σ. We denote by $C^\varepsilon(\Sigma,X)$ the set of distributions u belonging to L^∞ such that, for all λ, we have*

$$X_\lambda(x,D)u \overset{def}{=} \operatorname{div}(uX_\lambda)-u\operatorname{div}X_\lambda\in C^{\epsilon-1}$$

We are going to state a general theorem of persistence of the geometrical structures for the incompressible Euler system. This theorem will of course contain the global existence result for the traditional vortex patch problem. As suggested by Theorem 3.3.2, the important concept is the tangential regularity with respect to a set X of vector fields of class C^ϵ, admissible outside Σ. Here, unlike in Chapter 9, the set Σ will always be empty (we will then say that the set is admissible). For the rest of this chapter we settle on the following notation:

$$\begin{aligned}
I(X)&=\inf_{x\in\mathbb{R}^2}\sup_{\lambda\in\Lambda}|X_\lambda(x)| \quad (>0),\\
N_\epsilon(X)&=\frac{1}{\epsilon}\sup_{\lambda\in\Lambda}\frac{\|X_\lambda\|_\epsilon+\|\operatorname{div}X_\lambda\|_\epsilon}{I(X)},\\
\|u\|_{\epsilon,X}&=N_\epsilon(X)\|u\|_{L^\infty}+\sup_{\lambda\in\Lambda}\frac{\|X_\lambda(x,D)u\|_{\epsilon-1}}{I(X)}.
\end{aligned}$$

We can now state the main theorem of this section.

Theorem 5.7 *Let ϵ be a real number in the interval $(0,1)$, a be a real number, greater than 1, and $X_0=(X_{0,\lambda})_{\lambda\in\Lambda}$ be an admissible set of class C^ϵ on the plane. Let us consider a vector field v_0 on $\mathbb{R}^2$ belonging to $C^1_\star$, whose gradient is in L^a. If ω_0 belongs to $C^\epsilon(X_0)$, then there exists a unique solution v of (E) such that*

$$v\in L^\infty_{loc}(\mathbb{R};Lip) \quad \text{and} \quad \nabla v\in L^a.$$

Moreover, if ψ is the flow of v, then for all λ

$$X_{0;\lambda}(x,D)\psi\in L^\infty_{loc}\left(\mathbb{R};C^e\right).$$

Finally, if $X_{t,\lambda}=\psi(t)^\star X_{0,\lambda}==\left(X_{0,\lambda}(x,D)\psi(t)\right)\left(\psi^{-1}(t,x)\right)$, then the set $X_t=(X_{t,\lambda})_{\lambda\in\Lambda}$ is admissible and we have

$$N_\epsilon\left(X_t\right)\in L^\infty_{loc}(\mathbb{R}) \quad \text{and} \quad \|\omega(t)\|_{\epsilon,X_t}\in L^\infty_{loc}(\mathbb{R})$$

Proof: The approach we will follow is the same as the one inspiring the proof of Theorem 5.1. Let us regularize the initial data. We set

$$v_{0,n} = S_n v_0 \quad \text{and} \quad \omega_{0,n} = S_n \omega_0$$

Theorem 4.2.3, on the global existence of smooth solutions, states the existence of a global solution v_n of the system (E). The important point of this proof consists in proving an a priori estimate on the Lipschitz norm of a smooth solution of the system (E), and then in taking the limit. ∎

Before proving this theorem, let us make sure that it leads to Theorem 5.4 .1 Let f_0 be a function of class $C^{1+\epsilon}$ such that, in a neighbourhood of the curve γ_0 that curve is the set of zeros of f_0. The gradient of the function f_0 is supposed never to vanish on γ_0. Now let α be a real-valued function, identically equal to 1 near the curve γ_0, and supported in a neighbourhood of γ_0 where the gradient of f_0 does not vanish. We then define the following three vector fields:

$$X_{0,0} = \nabla^{\perp} f_0, \quad X_{0,1} = (1-\alpha)\partial_1 \quad \text{and} \quad X_{0,2} = (1-\alpha)\partial_2$$

It is trivial to check that the set of vector fields defined above is an admissible set of class C^{ϵ} since ω_0 is the characteristic function of the interior domain of the curve γ_0 it is clear that $X_{0,i}(x, D)\omega_0 = 0$. The hypotheses of Theorem 5.5 .1 are therefore satisfied.

Let σ_0 be a point on the circle and x_0 a point on the curve γ_0. Let us consider the following ordinary differential equation:

$$\begin{cases} \partial_\sigma \widetilde{\gamma}_0(\sigma) = X_{0,0}\left(\widetilde{\gamma}_0(\sigma)\right) \\ \widetilde{\gamma}_0\left(\sigma_0\right) = x_0 \end{cases}$$

The function $\tilde{\gamma}_0$ is an embedding of the circle of class $C^{1+\epsilon}$. Let $\tilde{\gamma}(t)$ be the function defined by $\tilde{\gamma}(t,\sigma) = \psi\left(t, \tilde{\gamma}_0(\sigma)\right)$. According to the persistence theorem 5.5 .1 we know that

$$X_{0,0}(x, D)\psi \in L^{\infty}_{loc}\left(\mathbb{R}; C^{\epsilon}\right)$$

Differentiating the identity defining $\tilde{\gamma}$, we find

$$\partial_\sigma \widetilde{\gamma}(t,\sigma) = \left(X_{0,0}(x, D)\psi\right)\left(t, \widetilde{\gamma}_0(\sigma)\right)$$

Therefore $\partial_\sigma \widetilde{\gamma}$ belongs to $L^{\infty}_{loc}\left(\mathbb{R}; C^{\epsilon}\right)$. The fact that it is an embedding of the circle results immediately from the fact that ψ is Lipschitzian. Hence Theorem 5.4 .1 is proved.

Theorem 5.8 *There exists a constant C satisfying the following property. Let $\epsilon \in (0,1)$ be a real number, let $a > 1$ be a real number, and let $X_0 = (X_{0,\lambda})_{\lambda\in\Lambda}$ be an admissible set of class C^{ϵ} on the plane. Let us consider a vector field v solution of the Euler system, and belonging to the space $L^{\infty}_{loc}\left(\mathbb{R}; C^{\infty}_b\right)$. Then, at all time t, we have*

$$\|\nabla v(t)\|_{L^\infty} \leq \tilde{N}\left(X_0, \epsilon, \omega_0\right) \exp\left(\frac{Ct\left\|\omega_0\right\|_{L^\infty}}{\epsilon^2}\right) \quad \textit{with}$$

$$\widetilde{N}\left(X_0, \epsilon, \omega_0\right) = Ca\left\|\omega_0\right\|_{L^a} + \frac{C}{\epsilon}\left\|\omega_0\right\|_{L^\infty} \log\left(\frac{\left\|\omega_0\right\|_{\epsilon, X_0}}{\left\|\omega_0\right\|_{L^\infty}}\right)$$

Proof: Let us assume this lemma to be true for the time being. From the definition of $\|\cdot\|_{\epsilon,X_t}$ it follows that

$$\frac{\|\omega(t)\|_{\epsilon,X_t}}{\|\omega(t)\|_{L^\infty}} \leq C\frac{\left\|\omega_0\right\|\, \epsilon, X_\circ}{\left\|\omega_0\right\|_{L^\infty}} \exp\left(\frac{C}{\epsilon}\int_0^t \|\nabla v(\tau)\|_{L^\infty d\tau}\right)$$

Let us now apply 'Theorem 3.3.2. since the vector field v is divergence-free, we can state that

$$\|\nabla v(t)\|_{L^\infty} \leq Ca\left\|\omega_0\right\|_{L^a} + \frac{C}{\epsilon}\left\|\omega_0\right\|_{L^\infty} \log(e + \frac{\left\|\omega_0\right\|_{\epsilon,X_0}}{\left\|\omega_0\right\|_{L^\infty}} \times \exp\left(\frac{C}{\epsilon}\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\right)\bigg)$$

since the quotient

$$\frac{\|\omega_0\|_{\varepsilon,X_0}}{\|\omega_0\|_{L^\infty}}$$

is greater than 1, we can write $\|\nabla v(t)\|_{L^\infty} \leq Ca\,\|\omega_0\|_{L^\circ} + \frac{C}{\epsilon}\,\|\omega_0\|_{L^\infty} \log\left(\frac{\|\omega_0\|_{\epsilon,X_0}}{\|\omega_0\|_{L^\infty}}\right)$

$$+\frac{C}{\epsilon^2}\,\|\omega_0\|_{L^\infty}\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau$$

Then Gronwall's lemma leads to Theorem 5.5.2. ∎

Lemma 5.4 *There exists a constant C such that*

$$I\left(X_t\right) \geq I\left(X_0\right)\exp\left(-\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\right) \tag{5.19}$$

$$\|X_{t,\lambda}(x,D)\omega(t)\|_{\epsilon-1} \leq C\,\|X_{0,\lambda}(x,D)\omega_0\|_{\epsilon-1}\exp\left(\frac{C}{\epsilon}\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\right) \tag{5.20}$$

$$\|\operatorname{div} X_{t,\lambda}\|_{\epsilon} \leq \|\operatorname{div} X_{0,\lambda}\|_{\epsilon}\exp\left(C\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\right) \tag{5.21}$$

$$\|X_{t,\lambda}\|_{\epsilon} \leq C\left(\|X_{0,\lambda}\|_{\epsilon} + \|\operatorname{div} X_{0,\lambda}\|_{\epsilon} + \frac{\epsilon\,\|X_{0,\lambda}(x,D)\omega_0\|_{\epsilon-1}}{\|\omega_0\|_{L^\infty}}\right)\times\exp\left(\frac{C}{\epsilon}\int_0^t \|\nabla v(\tau)\|_{L\infty} d\tau\right). \tag{5.22}$$

Proof: If we differentiate the equation defining the flow along the vector field $X_{0,\lambda}$, it follows that

$$\begin{cases} \partial_t X_{0,\lambda}(x,D)\psi(t,x) = \nabla v(t,\psi(t,x))X_{0,\lambda}(x,D)\psi(t,x) \\ X_{0,\lambda}(x,D)\psi(0,x) = X_{0,\lambda}(x) \end{cases}$$

Integrating the equation above between t and 0 yields

$$|X_{0,\lambda}(x)| \leq |X_{0,\lambda}(x,D)\psi(t,x)|\exp\left(\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\right).$$

As a result, for all x in the plane,

$$\sup_{\lambda\in\Lambda}|X_{0,\lambda}(x)| \leq \sup_{\lambda\in\Lambda}|X_{0,\lambda}(x,D)\psi(t,x)|\exp\left(\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\right).$$

Since $X_{t,\lambda}(x) = \left(X_{0,\lambda}(x,D)\psi(t)\right)\left(\psi^{-1}(t,x)\right)$, Definition 3.3 .1 for $I(X)$ ensures the inequality (5.18) The relation (5.22) can be written as

$$\partial_t X_{t,\lambda} + v\cdot\nabla X_{t,\lambda} = X_{t,\lambda}(x,D)v$$

This relation means that the two vector fields $\partial_t + v\cdot\nabla$ and $X_{t,\lambda}$ commute. Owing to the preservation of the vorticity along the flow lines of v, we find that

$$\partial_t X_{t,\lambda}(x,D)\omega(t) + v\cdot\nabla X_{t,\lambda}(x,D)\omega(t) = 0$$

The estimate of the propagation of the H?lderian norm proved in Lemma 4.1.1 yields

$$\|X_{t,\lambda}(x,D)\omega(t)\|_{\epsilon-1} \leq C\,\|X_{0,\lambda}(x,D)\omega_0\|_{\epsilon-1}\exp\left(\frac{C}{\epsilon}\int_0^t \|\nabla v(\tau)\|_{L\infty} d\tau\right)$$

which is nothing more than inequality (5.19). Applying the divergence operator to equation (5.23) yields

$$\partial_t \operatorname{div} X_{t,\lambda} + v\cdot\nabla\operatorname{div} X_{t,\lambda} = X_{t,\lambda}(x,D)\operatorname{div} v$$

since the vector field v is a solution of the Euler equation, its divergence is zero. Therefore the divergence of the vector fields $X_{t,\lambda}$ is preserved along the flow lines, that is

$$\partial_t \operatorname{div} X_{t,\lambda} + v \cdot \nabla \operatorname{div} X_{t,\lambda} = 0$$

Inequality (5.20) follows immediately from Lemma 4.1.1. To prove estimate (5.21), slightly more delicate, we will use the transport equation (5.23) of X_t and Lemma 3.3.2. By means of this lemma, we can state that we have, with the same notation,

$$\partial_t X_{t,\lambda} + v \cdot \nabla X_{t,\lambda} = W_1 \left(X_{t,\lambda}, v(t)\right) + W_2 \left(X_{t,\lambda}, v(t)\right) + A(t) X_{t,\lambda}$$

where $A(t)$ is a continuous operator mapping C^ϵ to itself such that

$$\|A(t)\|_{\mathcal{L}(\mathcal{C};\mathcal{C})} \leq \frac{C}{\epsilon} \|\nabla v(t)\|_{L^\infty}$$

From the propagation estimate of Lemma 4.1.1, as well as the upper bounds of W_1 and W_2 stated in Lemma 3.3 .2 and the inequalities (5.19) and (5.20), we infer the existence of a constant C such that

$$\begin{aligned} \left\|X_{t,\lambda}\right\|_\epsilon \leq \left\|X_{0,\lambda}\right\|_\epsilon e^{\frac{\mathcal{C}}{\epsilon} \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau} \\ + \left(C \left\|X_0(x,D)\omega_0\right\|_{\epsilon-1} + \tfrac{C}{\epsilon} \left\|\operatorname{div} X_{0,\lambda}\right\|_{\epsilon-1}\right) \int_0^t e^{\frac{\mathcal{C}}{\epsilon} \int_0^\tau \left\|\nabla v(\tau')\right\|_{L^\infty} d\tau'} d\tau \end{aligned}$$

It is clear that one can find an upper bound for the last term of the inequality above:

$$Ct \left(\left\|X_0(x,D)\omega_0\right\|_{\epsilon-1} + \frac{\left\|\operatorname{div} X_{0,\lambda}\right\|_{\varepsilon} \left\|\omega_0\right\|_{L^\infty}}{\epsilon} \right) e^{\frac{\mathcal{C}}{\epsilon} \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau}$$

If we observe that

$$\left\|\omega_0\right\|_{L^\infty} = \|\omega(t)\|_{L^\infty} \leq 2 \|\nabla v(t)\|_{L^\infty}$$

we can assert that

$$t \leq \frac{\epsilon}{C \left\|\omega_0\right\|_{L^\infty}} e^{\frac{\mathcal{C}}{\epsilon} \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau}$$

We then infer immediately that

$$\left\|X_{t,\lambda}\right\|_\epsilon \leq \left(\left\|X_{0,\lambda}\right\|_\epsilon + C \left\|\operatorname{div} X_{0,\lambda}\right\|_{\varepsilon} + \frac{C\epsilon \left\|X_{0,\lambda}(x,D)\omega_0\right\|_{\epsilon-1}}{\left\|\omega_0\right\|_{L^\infty}} \right) e^{\frac{c}{\epsilon} \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau}$$

which is exactly the inequality (5.21) we wanted. Lemma 5.5 .1 and hence Theorem 5.5 .2 are proved. ■

We now have to take the limit. This will be very easy, considering the estimates proved above. Let us consider the sequence of regularized initial data, defined by equation (5.16) at the beginning of this section. According to Theorem 5.5.2, which we have just proved, we can write

$$\|\nabla v_n(t)\|_{L^\infty} \leq C \tilde{N} \left(X_0, \epsilon, \omega_{0,n}\right) \exp\left(\frac{Ct \left\|\omega_{0,n}\right\|_{L^\infty}}{\epsilon^2} \right)$$

remembering that

$$\tilde{N} \left(X_0, \epsilon, \omega_{0,n}\right) = C \left(\left\|\omega_{0,n}\right\|_{L^\infty} + a \left\|\omega_{0,n}\right\|_{L^a} \right) + \frac{C}{\epsilon} \left\|\omega_{0,n}\right\|_{L^\infty} \log\left(\frac{\left\|\omega_{0,n}\right\|_{\epsilon, X_0}}{\left\|\omega_{0,n}\right\|_{L^\infty}} \right)$$

Inequality (3.22) states that

$$\left\|S_n \omega\right\|_{\epsilon, X} \leq \frac{C}{1-\epsilon} \left(N_\epsilon(X) \|\omega\|_{L^\infty} + \|\omega\|_{\epsilon, X}\right)$$

Then the fact that $\left\|\omega_{0,n}\right\|_{L^\infty} \leq C \left\|\omega_0\right\|_{L^\infty}$ and $\left\|\omega_{0,n}\right\|_{L^a} \leq C \left\|\omega_0\right\|_{L^\alpha}$ leads to

$$\|\nabla v_n(t)\|_{L^\infty} \leq C \widetilde{N} \left(X_0, \epsilon, \omega_0\right) \exp\left(\frac{Ct \left\|\omega_0\right\|_{L^\infty}}{\epsilon^2} \right)$$

Let us prove that $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $L^\infty_{loc}(\mathbb{R};C^{-\alpha})$, for all α in the interval $(0,1)$. Indeed we have

$$\partial_t (v_n - v_m) + v_n \cdot \nabla (v_n - v_m) = \pi (v_n - v_m, v_n + v_m) + (v_m - v_n) \cdot \nabla v_m$$

If we split the term $(v_m - v_n) \cdot \nabla v_m$ into a paraproduct and a remainder, and then if we use Theorem 2.4 .1 which describes the action of the paraproduct and the remainder in H?lder spaces, we find that

$$\|(v_n - v_m) \cdot \nabla v_m\|_{-\alpha} \leq C_{\alpha,\epsilon} \|\nabla v_m(t)\|_{L^\infty} \|v_n - v_m\|_{-\alpha}$$

Proposition 2.5.1 states that

$$\|\pi (v_n - v_m, v_n + v_m)\|_{-\alpha} \leq C_{\alpha,\epsilon} \left(\|v_n(t)\|_{Lip} + \|v_m(t)\|_{Lip}\right) \|v_n - v_m\|_{-\alpha}$$

Let us define

$$V(t) = C\widetilde{N}(X_0, \epsilon, \omega_0) \exp\left(\frac{Ct\|\omega_0\|_{L^\infty}}{\epsilon^2}\right)$$

If we apply the propagation estimate of Lemma 4.1.1; it follows that

$$\|(v_n - v_m)(t)\|_{-\alpha} \leq \|v_{0,n} - v_{0,m}\|_{-\alpha} \exp\left(C_{\alpha,\epsilon} \int_0^t V(\tau) d\tau\right)$$

By interpolation we infer that the sequence $(v_n)_{n\in\mathbb{N}}$ is, for any r strictly smaller than 1, a Cauchy sequence of $L^\infty_{loc}(\mathbb{R};C^r)$

The important point consists now in proving that the solution v of the Euler system constructed in this way satisfies the tangential regularity properties with respect to an admissible set of vector fields to be defined.

The first step consists in the proof of an easy property of stability of the flow, in the setting of the vector fields introduced in section 5.2.

Lemma 5.5 *Let $(F_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^1([0,T];C_\mu)$, where μ satisfies the assumptions of Theorem 5.3. Furthermore, suppose that*

$$\lim_{n\to\infty} F_n = F \quad \text{in} \quad L^1([0,T];L^\infty).$$

Let $(\psi_n)_{n\in\mathbb{N}}$ be the sequence of solutions of

$$\psi_n(t,x) = x + \int_0^t F_n(s, \psi_n(s,x))\, ds$$

Then $(\psi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{Id} + L^\infty\left([0,T]\times\mathbb{R}^d;\mathbb{R}^d\right)$ and its limit ψ is a solution of

$$\psi(t,x) = x + \int_0^t F(s,\psi(s,x)) ds$$

Moreover, if the sequence $(F_n)_{n\in N}$ is bounded in the space $L^1([0,T];Lip)$ (resp. $L^1([0,T];LL)$), then

$$\forall \epsilon > 0, \lim_{n\to\infty} \psi_n = \psi \quad \text{in} \quad L^\infty\left([0,T]; \mathrm{Id} + C^{1-\epsilon}\right)$$

$$\left(\text{resp.} \quad \text{in} \quad L^\infty\left([0,T]; \mathrm{Id} + C^{\exp\left(-\epsilon + \int_0^T \|v(\tau)\|_{LL} d\tau\right)}\right)\right)$$

and the result remains true for $\left(\psi_n^{-1}\right)_{n\in\mathbb{N}}$ and ψ^{-1}.

The proof of this lemma uses the same ingredients as that of Theorem 5.2.2. We have

$$|\psi_{n+k}(t,x) - \psi_n(t,x)| \leq \int_{t_0}^t \|F_{n+k}(s) - F_n(s)\|_{L^\infty ds}$$
$$+ \int_{t_0}^t |F_n(s, \psi_{n+k}(s,x)) - F_n(s, \psi_n(s,x))|\, ds$$
$$\leq \int_{t_0}^t \|F_{n+k}(s) - F_n(s)\|_{L^\infty} ds$$
$$+ \int_{t_0}^t \mu\left(\|\psi_{n+k}(s) - \psi_n(s)\|_{L^\infty}\right) \|F_n(s)\|_{C_\mu} ds$$

Then let us define

$$\rho_n(t) = \sup_{\substack{0 \le \tau \le t \\ k \ge 0}} \|\psi_{n+k}(\tau) - \psi_n(\tau)\|_{L^\infty}$$

In a rigorously similar way as in the proof of Theorem 5.2.2, we find that

$$\rho_n(t) \le a_n + \int_0^t \mu\left(\rho_n(s)\right) \|F_n(s)\|_{C_\mu} ds$$

with

$$a_n = \sup_{k \ge 0} \int_0^t \|F_{n+k}(s) - F_n(s)\|_{L^\infty} ds$$

From inequality (5.9) we get

$$-\mathcal{M}\left(\rho_n(t)\right) + \mathcal{M}\left(a_n\right) \le \int_0^t \|F_n(s)\|_{C_\mu} ds$$

By assumption, the right-hand side of the inequality above is bounded by a constant C, independent of n. Thus

$$\mathcal{M}\left(a_n\right) \le C + \mathcal{M}\left(\rho_n(t)\right)$$

But the sequence $\left(a_n\right)_{n\in\mathbb{N}}$ goes to 0, and thus the sequence $\left(\mathcal{M}\left(a_n\right)\right)_{n\in\mathbb{N}}$ goes to infinity, as does the sequence $\left(\mathcal{M}\left(\rho_n(t)\right)\right)_{n\in\mathbb{N}}$. From the definition of $\mathcal{M}$, this leads to the fact that the sequence $\left(\rho_n(t)\right)_{n\in\mathbb{N}}$ goes to 0. Therefore we have

$$\lim_{n\to\infty} \psi_n = \psi \quad \text{ in } \quad \mathrm{Id} + L^\infty\left([0,T] \times \mathbb{R}^d; \mathbb{R}^d\right)$$

The proof of the lemma ends with an obvious interpolation. Let us now prove the first part of Theorem 5.5 .1 , namely that we have for all λ

$$X_{0,\lambda}(x,D)\psi \in L^\infty_{loc}\left(\mathbb{R}; C^\epsilon\right)$$

We know that

$$\begin{aligned} X_{0,\lambda}(x,D)\psi_n &= X_{0,\lambda}(x,D)\left(\psi_n - \mathrm{Id}\right) \\ &= \sum_j \partial_j\left(X^j_{0,\lambda}(x)\left(\psi_n - \mathrm{Id}\right)\right) - \left(\psi_n - \mathrm{Id}\right)\operatorname{div} X_{0,\lambda} + X_{0,\lambda} \end{aligned}$$

According to the stability lemma 5.5 .2 above, and since C^ϵ is a normed algebra, the sequence $\left(X_{0,\lambda}(x,D)\psi_n\right)_{n\in\mathbb{N}}$ converges to $X_{0,\lambda}(x,D)\psi$ in the space $L^\infty_{loc}(\mathbb{R};\ C^{\epsilon-1})$ But the inequality (5.21) of Lemma 5.5.1, together with the fact that the sequence $\left(\psi_n\right)_{n\in\mathbb{N}}$ is bounded in $L^\infty_{loc}(\mathbb{R}; Lip)$, ensures that for all r strictly smaller than ϵ

$$\lim_{n\to\infty} X_{0,\lambda}(x,D)\psi_n = X_{0,\lambda}(x,D)\psi \quad \text{ in } \quad L^\infty_{loc}\left(\mathbb{R}; C^r\right)$$

It follows that

$$X_{0,\lambda}(x,D)\psi \in L^\infty_{loc}\left(\mathbb{R}; C^e\right)$$

which is the first part of Theorem 5.5 .1 To prove the whole theorem, it is necessary now to prove that the initial smoothness is propagated. The first thing to be done is to define, for each time t of the evolution, an admissible set of vector fields. Let us define, for all λ

$$X_{t,\lambda}(x) = \left(X_{0,\lambda}(x,D)\psi\right)\left(t, \psi^{-1}(t,x)\right)$$

It follows from the definition of the vector fields $X_{n,t,\lambda}$ and $X_{t,\lambda}$ that

$$\begin{aligned} X_{t,n,\lambda}(x) - X_{t,\lambda}(x) = \left(X_{0,\lambda}(x,D)\psi_n\right)\left(t, \psi_n^{-1}(t,x)\right) - \left(X_{0,\lambda}(x,D)\psi_n\right)\left(t, \psi^{-1}(t,x)\right) \\ + \left(X_{0,\lambda}(x,D)\psi_n - X_{0,\lambda}(x,D)\psi\right)\left(t, \psi^{-1}(t,x)\right) \end{aligned}$$

Therefore

$$\|X_{t,n,\lambda} - X_{t,\lambda}\|_{L^\infty} \le \|X_{0,\lambda}(x,D)\psi_n\|_\epsilon \left(\left\|\psi_n^{-1}(t) - \psi^{-1}(t)\right\|_{L^\infty}\right)^\varepsilon$$

$$+\left\|X_{0,\lambda}(x,D)\psi_n - X_{0,\lambda}(x,D)\psi\right\|_{L^\infty}$$

Furthermore, the inequality (5.21) of Lemma 5.5 .1 states that the sequence $\left(X_{t,n,\lambda}\right)_{n\in\mathbb{N}}$ is bounded in the space $L^\infty_{loc}\left(\mathbb{R};C^c\right)$. Therefore it is true that for all

λ and all r strictly smaller than ϵ, we have

$$\lim_{n\to\infty} X_{t,n,\lambda} = X_{t,\lambda} \quad \text{in} \quad L^\infty_{loc}\left(\mathbb{R};C^r\right)$$

since the sequence $\left(X_{t,n,\lambda}\right)_{n\in\mathbb{N}}$ is bounded in $L^\infty_{loc}\left(\mathbb{R};C^e\right)$, the vector fields $X_{t,\lambda}$ are locally bounded in time, with values in C^ϵ. Similarly, from inequality (5.20) the function div $X_{t,\lambda}$ belongs to the space $L^\infty_{loc}\left(\mathbb{R};C^\epsilon\right)$

Finally, since the sequence $\left(X_{t,n,\lambda}\right)_{n\in\mathbb{N}}$ converges to $X_{t,\lambda}$ in $L^\infty_{loc}\left(\mathbb{R};L^\infty\right)$, it follows from inequality (5.18) that

$$I\left(X_t\right) \geq I\left(X_0\right)\exp\left(-\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\right)$$

The set of vector fields X_t is therefore an admissible set of vector fields of class C^ϵ since the sequence $\left(v_n\right)_{n\in\mathbb{N}}$ converges to v in $L^\infty_{loc}\left(\mathbb{R};C^r\right)$ for all r strictly smaller than 1, it follows, for all strictly negative r, that

$$\lim_{n\to\infty} \omega_n = \omega \quad \text{in} \quad L^\infty_{loc}\left(\mathbb{R};C^r\right)$$

From Theorem 2.4.1, stating the way the paraproduct and the remainder operate, we infer that for all λ, the sequences $\left(\omega_n X_{t,n,\lambda}\right)_{n\in\mathbb{N}}$ and $\left(\omega_n \operatorname{div} X_{t,n,\lambda}\right)_{n\in\mathbb{N}}$ converge respectively to $\omega X_{t,\lambda}$ and $\omega \operatorname{div} X_{t,\lambda}$ in the space $L^\infty_{loc}\left(\mathbb{R};C^r\right)$, and this is true for all strictly negative r. We then deduce that for all λ

$$\lim_{n\to\infty} X_{t,n,\lambda}(x,D)\omega_n = X_{t,\lambda}(x,D)\omega \quad \text{in} \quad L^\infty_{loc}\left(\mathbb{R};C^r\right)$$

From the estimate (5.19), the sequence $\left(\left\|X_{t,n,\lambda}(x,D)\omega(t)\right\|_{\epsilon-1}\right)_{n\in\mathbb{N}}$ is, for all λ a bounded sequence of locally bounded functions. The identity above enables us to state that

$$X_{t,\lambda}(x,D)\omega \in L^\infty_{loc}\left(\mathbb{R};C^{\epsilon-1}\right)$$

This completes the proof of Theorem 5.5 .1

6 Vortex Sheets

References

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