Seminar on Nilpotent Lie groups

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1 Introduction to Lie groups

This section is a summary of the content in [Kir08, p.1-44].

1.1 Basics of Lie groups

Definition 1.1 (Lie group). A **real Lie group** is a set G equipped with a smooth manifold structure and a group structure, such that multiplication and the inversion are smooth maps, with a **morphism of Lie groups**: $f\colon G \to G'$ is a smooth map of manifolds and morpisms of groups $G \to G'$. A **complex Lie group** is a complex analytic manifold equipped with an analytic group structure. A **Lie subgroup** $H \leq G$ is a Lie group $H \subset G$ that is an immersed submanifold and subgroup. In particular, H is a Lie group, while a **closed Lie subgroup** is a subgroup and submanifold.

Example 1.1. $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$ and many subgroups, eg. SL_n, O_n, SO_n, U_n, SU_n, $\ldots$

Corollary 1.2. If G is a connected Lie group, then any neighbourhood $U \ni 1$ generates G.

Theorem 1.3. *If $H \trianglelefteq G$ normal closed Lie subgroup, then G/H is a Lie group.*

Theorem 1.4. *Let $f\colon G_1 \to G_2$ be a morphism of Lie groups. Then $H = \ker f$ is a normal closed subgroup. f induces an injective morphism $G_1/H \to G_2$ and $\operatorname{Im} f \leq G_2$ is a Lie subgroup. If $\operatorname{Im} f$ is a submanifold, then it is a closed Lie subgroup and $f\colon G_1/H \xrightarrow{\cong} \operatorname{Im} f$.*

Example 1.2. The homomorphism $(\mathbb{R},+) \to (\mathbb{R}^2,+)$, $t \mapsto \begin{pmatrix} t \\ t\theta \end{pmatrix}$ induces an immersion $(\mathbb{R},+) \to (\mathbb{R}^2/\mathbb{Z}^2)$ if θ is irrational. This is a Lie subgroup that is not closed.

Theorem 1.5. *The universal covering of a Lie group is a Lie group.*

Definition 1.6 (Left, Right, Adjoint Action). A Lie group G acts on itself by the following actions: Left action $L_g(h) = gh$, Right action $R_g(h) = hg^{-1}$, Adjoint action $\operatorname{Ad}_g(h) = ghg^{-1}$.

Corollary 1.7. L_g and R_g are transitive, they commute and $\operatorname{Ad}_g = L_g R_g$.

Definition 1.8. A vector field $v \in \operatorname{Vect}(G)$ is left-invariant, if $dL_g(X(h)) = X(L_g(h)) = X(gh)$ for all $g \in G$. Similarly for right- and bi-invariant vector fields and differential forms.

Theorem 1.9. *The map $v \mapsto v(1)$ defines an isomorphism between the vector space of left-invariant vector fields and T_1G.*

Proposition 1.10. *Let $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$, G be a $\mathbb{K}$-Lie group, $\mathfrak{g} = T_eG$ and $x \in \mathfrak{g}$. Then there exists a unique morphism of Lie groups $\gamma_x\colon \mathbb{K} \to G$ such that $\dot{\gamma}_x(0) = x$.*

Definition 1.11 (Exponential Map). $\exp\colon \mathfrak{g} \to G$, $\exp(X) = \gamma_X(1)$.

Example 1.3. For $G = \mathbb{R}$ we have $\mathfrak{g} = \mathbb{R}$, $\gamma_a(t) = ta$ and hence $\exp(a) = a$.

Example 1.4. For $G = S^1$ we have $\mathfrak{g} = \mathbb{R}$ and $\exp(a) = e^{2\pi i a}$.

Example 1.5. For Lie groups $G \subset GL_n(\mathbb{K})$ this agrees with the exponential map on matrices.

Theorem 1.12. *The exponential map satisfies: (1) it is a diffeomorphism between a neighbourhood of $0 \in \mathfrak{g}$ and $1 \in G$. The local inverse map will be denoted $\log$. (2) $\exp((t+s)x) = \exp(tx)\exp(sx)$ for any $s,t \in \mathbb{K}$, (3) for any morphism of Lie groups $\varphi\colon G_1 \to G_2$ and any $x \in \mathfrak{g}_1$ we have $\exp(\varphi_*(x)) = \varphi(\exp(x))$.*

Proposition 1.13. *For sufficiently small $x, y \in G$ we have $\exp(x)\exp(y) = \exp(\mu(x,y))$ for some smooth map $\mu\colon G \times G \to G$ defined on a neighbourhood of $(0,0)$.*

Lemma 1.14. The Taylor series for μ is given by $\mu(x,y) = x + y + \lambda(x,y) + \text{order}{\geq}3$, where $\lambda\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bilinear and skew-symmetric.

Definition 1.15 (Commutator). We define $[x,y] = 2\lambda(x,y)$.

Remark 1.6. $\exp(x)\exp(y) = \exp(x + y + \frac{1}{2}[x,y] + \ldots)$

Example 1.7. Let $G \subset GL_n(\mathbb{K})$, so that $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{K})$, then $[x, y] = xy - yx$.

Theorem 1.16 (Jacobi-Identity). $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$.

Definition 1.17 (Lie Algebra). A Lie Algebra over $\mathbb{K}$ is a vector space $\mathfrak{g}$ over $\mathbb{K}$ with a $\mathbb{K}$-bilinear skew-symmetric map $[,]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity.

Definition 1.18 (Morphism of Lie Algebras). $\mathbb{K}$-linear maps preserving the commutator.

Example 1.8. Any vector space $\mathfrak{g}$ with commutator $[x, y] = 0$.

Example 1.9. Any associative algebra over $\mathbb{K}$ with $[x, y] = xy - yx$.

Theorem 1.19. *Let G be a Lie group. Then $\mathfrak{g} = T_1G = \mathrm{Lie}(G)$ has a canonical structure of a Lie algebra. Every morphisms of Lie groups $\varphi: G_1 \to G_2$ defines a morphism of Lie algebras $\varphi_*: \mathfrak{g}_1 \to \mathfrak{g}_2$. In particular there is a map $\mathrm{Hom}(G_1, G_2) \to \mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$. If G_1 is connected, this map is injective.*

Example 1.10. Let M be a manifold, then $\mathfrak{X}(M)$, the vector fields on M, with the commutator bracket of vector fields is a Lie algebra.

Definition 1.20 (Subalgebra, Ideal). Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, a subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a Lie subalgebra if it is closed under commutator, $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. $\mathfrak{h}$ is called an ideal if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

Theorem 1.21. *Let H be a Lie subgroup of G. Then $\mathfrak{h} = T_1H$ is a Lie subalgebra of $\mathfrak{g} = T_1G$.*

Theorem 1.22. *Let H be a normal closed Lie subgroup of G. Then $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ and $\mathrm{Lie}(G/H) = \mathfrak{g}/\mathfrak{h}$. Conversely, if H is a closed subgroup, H, G connected and $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, then H is normal.*

Definition 1.23 (Center). The center $\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] = 0\}$, an ideal in $\mathfrak{g}$.

Theorem 1.24. *Connected subgroups $H \subset G \overset{1:1}{\leftrightarrow}$ Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$.*

Theorem 1.25. *If G_1 is connected and simply connected, then $\mathrm{Hom}(G_1, G_2) = \mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$.*

Theorem 1.26. *Any fin-dim Lie algebra is isomorphic to a Lie algebra of a Lie group.*

Corollary 1.27. For any finite dimensional Lie algebra $\mathfrak{g}$ there is a unique up to isomorphism connected simply connected Lie group G with $\mathrm{Lie}(G) = \mathfrak{g}$. Any other connected Lie group G' with Lie algebra $\mathfrak{g}$ must be of the form G/Z for some discrete central subgroup $Z \leq G$.

2 Nilpotent Lie groups and Algebras

For details, compare [CG90, p.1-24] and [FS82, p.1-8].

Definition 2.1 (Descending Central Series). $\mathfrak{g}^{(1)} = \mathfrak{g}, \quad \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}]$.

Lemma 2.2. For all $p, q \in \mathbb{N}$ one has $[\mathfrak{g}^{(p)}, \mathfrak{g}^{(q)}] \subset \mathfrak{g}^{(p+q)}$. In particular $\mathfrak{g}^{(p)}$ is an ideal in $\mathfrak{g}$.

Definition 2.3 (Nilpotent Lie Algebra and group). A Lie algebra $\mathfrak{g}$ is called nilpotent if there is $n \in \mathbb{N}$ such that $\mathfrak{g}^{(n+1)} = 0$. If n is the smallest such integer, then $\mathfrak{g}$ is called n-step nilpotent. A Lie group G is called nilpotent if its Lie algebra is.

Definition 2.4 (Graded Nilpotent Lie Algebra). A nilpotent Lie algebra is called graded, if it splits in $\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i$ with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.

Example 2.1 (Heisenberg group of rank 3). The Heisenberg group and its Lie algebra:

$$\mathbb{H}^3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\} \quad \mathfrak{g} = T_{\mathrm{id}}\mathbb{H}^3(\mathbb{R}) = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} \middle| \, \alpha, \beta, \gamma \in \mathbb{R} \right\}.$$

This Lie algebra has a basis $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ und $Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The only non-trivial commutator relation is $[X, Y] = Z$. In particular, its descending central series is $\mathfrak{g}^{(1)} = \mathfrak{g}$, $\mathfrak{g}^{(2)} = \langle Z \rangle$, $\mathfrak{g}^{(k)} = 0$ otherwise. So $\mathfrak{g}$ is a 3-step nilpotent Lie algebra and $\mathbb{H}^3(\mathbb{R})$ a nilpotent Lie group.

Example 2.2. The same remains true for the $(2n+1)$-dimensional Heisenberg group:

$$\mathbb{H}^n(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x_1 & \dots & \dots & x_n & z \\ & 1 & 0 & \dots & 0 & y_1 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & 0 & \vdots \\ & & & & 1 & y_n \\ & & & & & 1 \end{pmatrix} \middle| \, x_1, \dots, x_n, y_1, \dots y_n, z \in \mathbb{R} \right\} \subset GL_{2n+1}(\mathbb{R}),$$

ie. this is also a nilpotent Lie group with 3-step nilpotent Lie algebra.

Lemma 2.5. If $\mathfrak{g}$ is nilpotent, so are all subalgebras and quotient algebras of $\mathfrak{g}$.

Theorem 2.6 (Engel's Theorem). [CG90, p.4]*Let $\mathfrak{g}$ be a Lie algebra and let $\alpha\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a homomorphism such that $\alpha(X)$ is nilpotent for all $X \in \mathfrak{g}$. Then there exists a flag (Jordan-Hölder series) of subspaces $0 = V_0 \subset V_1 \subset \dots V_n = V$ with $\dim V_j = j$ such that $\alpha(X)V_j \subset V_{j-1}$ for all $j \geq 1$ and all $X \in \mathfrak{g}$. In particular $\alpha(\mathfrak{g})$ is a nilpotent Lie algebra.*

Corollary 2.7. If $\mathfrak{g}$ is a Lie algebra such that $\operatorname{ad} X\colon \mathfrak{g} \to \mathfrak{g}$, $Y \mapsto [X, Y]$ is nilpotent for every $X \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

Theorem 2.8 (Birkhoff Embedding Theorem). [CG90, p.7] *Let $\mathfrak{g}$ be a nilpotent Lie algebra over $\mathbb{K}$. Then there is a fin-dim vector space V together with an injection $\iota\colon \mathfrak{g} \to \mathfrak{gl}(V)$ such that $\iota(X)$ is nilpotent for all $X \in \mathfrak{g}$.*

2.1 Nilpotent Lie groups

We only consider *connected simply-connected* Lie groups for now.

Remark 2.3. A Lie group G is called nilpotent if its Lie algebra $\mathfrak{g}$ is nilpotent.

Lemma 2.9. A Lie group is nilpotent if and only if for its decending central series given by $G^{(1)} = G$ and $G^{(j+1)} = [G, G^{(j)}]$ one has $G^{(j)} = 0$ for some j.

Theorem 2.10. *Let $X, Y \in \mathfrak{g}$ such that $[X, Y] = 0$. Then* $\exp(X)\exp(Y) = \exp(X + Y) = \exp(Y)\exp(X)$.

Theorem 2.11. *For small $X, Y \in \mathfrak{g}$ one has exp$(X)\exp(Y) = \exp(\mu(X, Y))$ for some $\mathfrak{g}$ valued function μ which is given by the series convergent in some neighbourhood of $(0, 0)$:*
$\mu(X, Y) = X + Y + \sum_{n \geq 2} \mu_n(X, Y)$ where $\mu_n(X, Y)$ is a Lie polynomial in X, Y of degree n.

Corollary 2.12. The group operation in a connected Lie group G can be recovered from $\mathfrak{g}$.

Proof. If G is connected, then it is generated by any nbhd of 1. In small nbhds one has:

$$\begin{aligned} xy &= \exp(X)\exp(Y) = \exp(\mu(X,Y)) \\ &= \exp\left(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y]] + [Y,[Y,X]]\right) + \dots\right), \end{aligned}$$

where $X = \log(x)$ and $Y = \log(y)$.

Remark 2.4. For nilpotent Lie groups, the sum is finite.

Theorem 2.13 ([CG90, p.13], [FS82, p.3]). *G a nilpotent Lie group. Then:*

(1) $\exp\colon \mathfrak{g} \to G$ *is a diffeomorphism;*

(2) The Campbell-Baker-Hausdorff formula holds for all $X, Y \in \mathfrak{g}$*;*

(3) Identifying $\mathfrak{g} \xrightarrow{\exp} G$*, the group law* $(x,y) \to xy$ *is a polynomial map;*

(4) If λ *denotes a Lebesgue measure on* $\mathfrak{g}$*, then* $\lambda \circ \log$ *is a bi-invariant Haar measure on* G*.*

3 Homogeneous Lie Group

In this section we discuss nilpotent Lie algebras and groups in the spirit of Folland and Stein's book [FS82] as well as introduce homogeneous (Lie) groups. For more analysis and details in this direction we refer to the recent open access books [FS82, FR17, RS19].

Proposition 3.1 (Exponential mapping and Haar measure). [RS19] *Let* $\mathbb{G}$ *be a connected and simply-connected nilpotent Lie group with Lie algebra* $\mathfrak{g}$*. Then:*

1. *The exponential map* $\exp$ *is a diffeomorphism from* $\mathfrak{g}$ *to* $\mathbb{G}$*. Moreover, if* $\mathbb{G}$ *is identified with* $\mathfrak{g}$ *via* $\exp$*, then the group law* $(x,y) \mapsto xy$ *is a polynomial map.*

2. *If* λ *denotes a Lebesgue measure on* $\mathfrak{g}$*, then* $\lambda \circ \exp^{-1}$ *is a bi-invariant Haar measure on* $\mathbb{G}$*.*

Definition 3.2 (Dilations on a Lie algebra). A family of a Lie algebra $\mathfrak{g}$ is a family of linear mappings $\{\delta_r : \mathfrak{g} \to \mathfrak{g} \mid r > 0\}$ which satisfies:

1. the mappings are of the form $\delta_r = \exp(A \log r)$ for some fixed A being a diagonalisable linear operator on $\mathfrak{g}$ with positive eigenvalues.

2. each δ_r is a morphism of the Lie algebra $\mathfrak{g}$, that is, a linear mapping from $\mathfrak{g}$ to itself which respects the Lie bracket:

$$\forall X, Y \in \mathfrak{g}, r > 0 \quad [D_r X, D_r Y] = D_r[X,Y]$$

3. In particular, $\delta_{rs} = \delta_r \delta_s$ for all $r, s > 0$. If $\alpha > 0$ and $\{\delta_r\}$ is a family of dilations on $\mathfrak{g}$, then so is $\{\widetilde{\delta_r}\}$, where

$$\widetilde{\delta_r} := \delta_{r^\alpha} = \exp(\alpha A \log r).$$

By adjusting α we can always assume that the minimum eigenvalue of A is equal to 1.

Remark 3.1. We call the eigenvalues of A the dilations' weights. The set of dilations' weights, or in other worlds, the set of eigenvalues of A is denoted by $\mathcal{W}_A$.

Proposition 3.3 (Lie algebras with dilations are nilpotent). *If a Lie algebra* $\mathfrak{g}$ *admits a family of dilations then it is nilpotent.*

Remark 3.2. Not all nilpotent Lie algebras admit a dilation structure: an example of a (nine-dimensional) nilpotent Lie algebra that does not allow any compatible family of dilations was constructed by Dyer [Dye70].

Definition 3.4 (Graded Lie algebras and groups). A Lie algebra $\mathfrak{g}$ is called graded if it is endowed with a vector space decomposition (where all but finitely many of the V_k's are 0)

$$\mathfrak{g} = \oplus_{j=1}^{\infty} V_j \quad \text{such that} \quad [V_i, V_j] \subset V_{i+j}$$

Consequently, a Lie group is called graded if it is a connected simply-connected Lie group whose Lie algebra is graded.

Definition 3.5 (Stratified Lie algebras and groups). A graded Lie algebra $\mathfrak{g}$ is called stratified if V_1 generates $\mathfrak{g}$ an algebra. In this case, if $\mathfrak{g}$ is nilpotent of step m we have

$$\mathfrak{g} = \oplus_{j=1}^{m} V_j, \quad [V_j, V_1] = V_{j+1}$$

and the natural dilations of $\mathfrak{g}$ are given by

$$\delta_r \left(\sum_{k=1}^{m} X_k \right) = \sum_{k=1}^{m} r^k X_k, \quad (X_k \in V_k)$$

Consequently, a Lie group is called stratified if it is a connected simply-connected Lie group whose Lie algebra is stratified.

Definition 3.6 (Homogeneous Lie groups). Let δ_r be dilations on $\mathbb{G}$. We say that a Lie group $\mathbb{G}$ is a homogeneous Lie group if:

(a) It is a connected and simply-connected nilpotent Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ is endowed with a family of dilations $\{\delta_r\}$.

(b) The maps $\exp \circ \delta_r \circ \exp^{-1}$ are group automorphism of $\mathbb{G}$

Remark 3.3. By Proposition Proposition 3.1, the exponential mapping exp is a global diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$, it induces the corresponding family on $\mathbb{G}$ which we may still call the dilations on $\mathbb{G}$ and denote by δ_r. Hence δ can be as an representation of semi-group $(\mathbb{R}, \times)$ on $\mathfrak{g}$. Thus, for $x \in \mathbb{G}$ we will write $\delta_r(x)$ or abbreviate it writing simply rx.

Lemma 3.7. Graded Lie algebras are naturally equipped with dilations. If a Lie algebra $\mathfrak{g}$ has a family of dilations such that the weights are all rational, then $\mathfrak{g}$ has a natural gradation.

Proof. Let $\mathfrak{g} = \bigoplus_{j=1}^{r} \mathfrak{g}_j$ and let $\alpha = \min\{1 \leq j \leq r \mid \mathfrak{g}_j \neq 0\}$. Then $\delta_r = \exp(A \log r)$, where A is the operator defined on $\mathfrak{g}_j$ by $AX = \frac{j}{\alpha} X$ for $X \in \mathfrak{g}_j$ is a family of dilations. $\square$

Proposition 3.8. *The following hold:*

(i) A Lie algebra equipped with a family of dilations is nilpotent.

(ii) A homogeneous Lie group is a nilpotent Lie group.

Proof. Let $\mathcal{A}$ denote the set of eigenvalues of A. For $a \in \mathcal{A}$ let $W_a \subset \mathfrak{g}$ denote the corresponding eigenspace. Then for $X \in W_a$ one has $\delta_r X = r^a X$. Hence, for $X \in W_a$ and $Y \in W_b$ one has $\delta_r[X, Y] = [\delta_r X, \delta_r Y] = r^{a+b}[X, Y]$, thus $[W_a, W_b] \subset W_{a+b}$. Since $a \geq 1$ we see that $\mathfrak{g}^{(j)} \subset \bigoplus_{a \geq j} W_a$. Since $\mathcal{A}$ is finite it follows that $\mathfrak{g}^{(j)} = 0$ for j sufficiently large. $\square$

Example 3.4 (Abelian groups). The Euclidean space $\mathbb{R}^n$ is a homogeneous group with dilation given by the scalar multiplication. The abelian group $(\mathbb{R}^n, +)$ is also graded: its Lie algebra $\mathbb{R}^n$ is trivially graded, i.e. $V_1 = \mathbb{R}^n$.

Example 3.5 (Heisenberg groups). If n is a positive integer, the Heisenberg group $\mathbb{H}^n$ is the group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ and whose multiplication is given by

$$(z_1, \ldots, z_n, t)\,(z_1', \ldots, z_n', t') = \left(z_1 + z_1', \ldots, z_n + z_n', t + t' + 2\,\mathrm{Im} \sum_{k=1}^{n} z_k \bar{z}_k' \right).$$

The Heisenberg group $\mathbb{H}^n$ is a homogeneous group with dilations

$$\delta_r(z_1,\ldots,z_n,t) = \left(rz_1,\ldots,rz_n,r^2t\right).$$

It is also graded: its Lie algebra $\mathfrak{g}_n$ can be decomposed as

$$\mathfrak{g}_n = V_1 \oplus V_2 \quad \text{where} \quad V_1 = \oplus_{i=1}^n \mathbb{R}X_i \oplus \mathbb{R}Y_i \quad \text{and} \quad V_2 = \mathbb{R}T$$

Example 3.6 (Upper triangular groups). Let $\mathbb{G}$ be the group of all $n\times n$ real matrices (a_{ij}) such that $a_{ii}=1$ for $1\le i\le n$ and $a_{ij}=0$ when $i>j$. Then $\mathbb{G}$ is a homogeneous group with dilations

$$\delta_r(a_{ij}) = r^{j-i}a_{ij}$$

Remark 3.7. A gradation over a Lie algebra is not unique: the same Lie algebra may admit different gradations. For example, any vector space decomposition of $\mathbb{R}^n$ yields a graded structure on the group $(\mathbb{R}^n,+)$. More convincingly, we can decompose the 3 dimensional Heisenberg Lie algebra $\mathfrak{h}_1$ as

$$\mathfrak{h}_1 = \bigoplus_{j=1}^{3} V_j \quad \text{with} \quad V_1 = \mathbb{R}X_1, V_2 = \mathbb{R}Y_1, V_3 = \mathbb{R}T$$

This last example can be easily generalised to find several gradations on the Heisenberg groups $\mathbb{H}_{n_o}, n_o = 2,3,\ldots,$ which are not the classical ones given in Example Example 3.5. Another example would be

$$\mathfrak{h}_1 = \bigoplus_{j=1}^{8} V_j \quad \text{with} \quad V_3 = \mathbb{R}X_1, V_5 = \mathbb{R}Y_1, V_8 = \mathbb{R}T$$

and all the other $V_j = \{0\}$.

Remark 3.8. A gradation may not even exist. The first obstruction is that the existence of a gradation implies nilpotency; in other words, a graded Lie group or a graded Lie algebra are nilpotent. Even then, a gradation of a nilpotent Lie algebra may not exist. As a curiosity, let us mention that the (dimensionally) lowest nilpotent Lie algebra which is not graded is the seven dimensional Lie algebra given by the following commutator relations:

$$\begin{aligned} &[X_1,X_j] = X_{j+1} \text{ for } j=2,\ldots,6, \quad [X_2,X_3] = X_6 \\ &[X_2,X_4] = [X_5,X_2] = [X_3,X_4] = X_7 \end{aligned}$$

They define a seven dimensional nilpotent Lie algebra of step 6 (with basis $\{X_1,\ldots,X_7\}$). It is the (dimensionally) lowest nilpotent Lie algebra which is not graded.

Remark 3.9. Different gradations may lead to "morally equivalent" decompositions. For instance, if a Lie algebra $\mathfrak{g}$ is graded by $\mathfrak{g} = \oplus_{j=1}^\infty V_j$ then it is also graded by $\mathfrak{g} = \oplus_{j=1}^\infty W_j$ where $W_{2j'+1} = \{0\}$ and $W_{2j'} = V_{j'}$. This last example motivates the presentation of homogeneous Lie groups: indeed graded Lie groups are homogeneous and the natural homogeneous structure for the graded Lie algebra

$$\mathfrak{g} = \oplus_{j=1}^\infty V_j = \oplus_{j=1}^\infty W_j$$

is the same for the two gradations.

Remark 3.10. There are plenty of graded Lie groups which are not stratified, simply because the first vector subspace of the gradation may not generate the whole Lie algebra (it may be $\{0\}$ for example). Moreover, a direct product of two stratified Lie groups is graded but may be not stratified as their stratification structures may not 'match'.

4 Coadjoint action and unitary dual

In the following section G will always be a Lie group (**not necessarily nilpotent**) with Lie algebra $\mathfrak{g}$, and its dual Lie algebra being $\mathfrak{g}^*$. There are several key features of nilpotent Lie groups which we use here:

1. Because the finite filtration structure, we have both Campbell-Baker-Hausdorff formula and adjoint action on $\mathfrak{g}$ both are polynomials.

2. **The Kirillov's Lemma**: For $\mathfrak{g}$ non-commutative nilpotent Lie algebra with centre $\mathfrak{z}(\mathfrak{g})$ 1-dimensional, then $\mathfrak{g}$ admits a splitting:

$$\mathfrak{g} = \mathbb{R}Z \oplus \mathbb{R}Y \oplus \mathfrak{w} = \mathbb{R}X \oplus \mathfrak{g}_0$$

   Moreover, we have Z spans the center of $\mathfrak{g}$ with $[X,Y] = Z$ and $\mathfrak{g}_0$ the centralizer of Y and an ideal.

First note there is a natural **coadjoint action** of G on $\mathfrak{g}^*$, where for $g \in G, X \in \mathfrak{g}$ and $l \in \mathfrak{g}^*$ we have:

$$(\mathrm{Ad}^*(g)l)(X) = l(\mathrm{Ad}(g)^{-1}(X))$$

with its differential at unit element e gives the derived coadjoint action of $\mathfrak{g}^*$ via:

$$((\mathrm{ad}^* X)l)(Y) = l(\mathrm{ad}(-X)Y) = l([Y,X])$$

Definition 4.1. If $\mathfrak{g}$ is a Lie algebra and $l \in \mathfrak{g}$ with its **radical** defined as:

$$\mathfrak{r}_l := \{Y \in \mathfrak{g} \mid B_l(X,Y) = 0 \quad \forall X \in \mathfrak{g}\}$$

with the natural bilinear form B_l defined as $B_l(X,Y) = l([X,Y])$.

Note B_l defines an antisymmetric form, and the radical is even dimensional, with the isotropic subspace (i.e., space on which B_l is 0) have maximal dimension $\dim \mathfrak{g} - \frac{1}{2}\dim \mathfrak{g}/\mathfrak{r}_l := n-k$. We call such spaces **polarizing subalgebra**.

Theorem 4.2. *for each $l \in \mathfrak{g}$ there always exists a polarizing subalgebra. Moreover, if $\mathfrak{g}_0$ is a subalgebra of codimension 1, then there are two mutually exclusive possibilities:*

1. $\mathfrak{r}_l \subseteq \mathfrak{g}_0 \Leftrightarrow \mathfrak{r}_l \subseteq \mathfrak{r}_{l_0} \Leftrightarrow \mathfrak{r}_l$ *is of codimension 1 in* $\mathfrak{r}_{l_0}$*. In this case, any polarizing subalgebra for* l_0 *is also polarizing for* l*.*

2. $\mathfrak{r}_l \nsubseteq \mathfrak{g}_0 \Leftrightarrow \mathfrak{r}_l \supseteq \mathfrak{r}_{l_0} \Leftrightarrow \mathfrak{r}_{l_0}$ *is of codimension 1 in* $\mathfrak{r}_l$*. In this case, If* $\mathfrak{m}$ *is a polarizing subalgebra then* $\mathfrak{m}_0 = \mathfrak{m} \cap \mathfrak{g}_0$ *is a polarizing subalgebra for* l_0 *with* $\mathfrak{m}_0$ *is of codimension 1 in* $\mathfrak{m}$ *and* $\mathfrak{m} = \mathfrak{r}_l + \mathfrak{m}_0$*.*

Remark 4.1. The proof of this theorem uses Kirillov's lemma.

If we choose a strong Malcev basis, then the polarizing subalgebra for l can be reconstructed in a step-by-step way by taking

$$\mathfrak{m}_l = \sum_{j=1}^{n} \mathfrak{r}(l_j) \quad \text{where} \quad l_j = l|_{\mathfrak{g}_j}$$

for $\{\mathfrak{g}_j\}_j$ a chain of ideals such that $\dim \mathfrak{g}_j = j$

Remark 4.2. For various examples of coadjoint actions, see [CG90, Example 1.3.7 to 1.3.11].

Now Kirillov theory, which founded on the representation of Heisenberg group, gives our the complete description of $\widehat{G}$ for G a nilpotent Lie group. For a starter, we begin with a subgroup M on which B_l vanishes, then we induces the representation to that of a representation of G. To make it more explicit:

Definition 4.3. Given a representation of closed subgroup K of G together with a representation (π, H_π), the **induced representation** of G, which we denote as $\mathrm{Ind}_K^G(\pi)$ is a representation of G with:

$$H_\sigma := \left\{ f : G \to H_\pi \text{ Borel measurable} \mid f(kg) = \pi(k)f(g) \text{ and } \int_{K\backslash G} \|f(g)\|^2 d\dot{g} < \infty \right\}$$

where $d\dot{g}$ is the right invariant measure (which always exists for unimodular groups). Then G acts on the right:

$$\sigma(x)f(g) = f(gx) \quad \forall x \in G$$

We choose now K to be $M := \exp \mathfrak{m}$ for $\mathfrak{m}$ the polarizing subalgebra of $\mathfrak{g}$ with respect to $l \in \mathfrak{g}^*$. Then M admits a 1-dimensional representation:

$$\chi_{l,M} : M \to S^1 \qquad \exp Y \mapsto e^{2\pi i l(Y)}$$

and form induced representation $\pi_{l,M} := \mathrm{Ind}_M^G(\chi_{l,M})$. The following theorems shows this actually characterizes all irreducible unitary representations of G (in the nilpotent case!):

Theorem 4.4 ([CG90, Theorem 2.2.1 to 2.2.4]). *Let G be a nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then:*

1. *For each $l \in \mathfrak{g}^*$ there exists a polarizing subalgebra $\mathfrak{m}$ such that $\pi_{l,M}$ is irreducible. Moreover, this is independent of choice of polarizing subalgebra $\mathfrak{m}$, hence we may write $\pi_l = [\pi_{l,M}]$*
2. *Let $\pi \in \widehat{G}$, then there is a $l \in \mathfrak{g}^*$ such that $\pi \cong \pi_l$. Moreover, $\pi_l = \pi_{l'}$ if and only if they are in the same coadjoint orbit.*

Remark 4.3 (Sketch of proof of Theorem 4.4). One key feature of the nilpotent Lie group is the following "onion-peeling" feature which we describe below. Consider $\mathfrak{h}$ is a one-dimensional central subalgebra of $\mathfrak{g}$ with respective Lie group H. Suppose $l \in \mathfrak{g}^*$ vanishes on $\mathfrak{h}$ (hence every polarizing subalgebra $\mathfrak{m}$ for l contains $\mathfrak{h}$), then we quotient out $\mathfrak{h}$ which fit into the following diagram:

$$\begin{tikzcd}
M \arrow[r, hook] \arrow[d, two heads, "p_M = p|_M"'] \arrow[rrr, dashed, bend left, "\log"] & G \arrow[d, two heads, "p"] & \mathfrak{m} \arrow[r, hook] \arrow[d] & \mathfrak{g} \arrow[d] \\
\overline{M} \arrow[r, hook] & \overline{G} & \bar{\mathfrak{m}} = \mathfrak{m}/\mathfrak{h} \arrow[r, hook] & \bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{h}
\end{tikzcd}$$

and it is straightforward to check:

$$\chi_{l,M} = \chi_{\bar{l},\overline{M}} \circ p_M \qquad \pi_{l,M} = \pi_{\bar{l},\overline{M}} \circ p$$

Hence by using induction we suffices to check the case when $\mathfrak{h}$ has only one-dimensional center, on which $l \in \mathfrak{g}^*$ is nonvanishing. But this is very much captured by Stone-von Neumann theorem [CG90, Theorem 2.2.9]:

Theorem 4.5 (Stone-von Neumann). *Let ρ_1, ρ_2 be two unitary representations of $\mathbb{R}$ in the same Hilbert space H_ρ satisfying the covariance relation:*

$$\rho_1(x)\rho_2(y)\rho_1(x)^{-1} = e^{2\pi i\lambda x\cdot y}\rho_2(y) \qquad \forall x, y \in \mathbb{R} \quad \lambda \neq 0$$

then H_ρ admits a splitting into $\bigoplus_{i\in\mathbb{N}} H_i$ of invariant irreducible subspaces under joint action. Moreover each H_k are isometric to $L^2(\mathbb{R})$, under which ρ_i transforms to 'canonical' actions on $L^2(\mathbb{R})$:

$$[\rho_1(x)f](t) = f(t+x) \qquad [\rho_2 f](t) = e^{2\pi i\lambda yt}f(t)$$

for each $\lambda \neq 0$, these two ρ_i act irreducibly on $L^2(\mathbb{R})$.

Now the killing blow is by the following byproduct of Kirillov's theorem, which allows us to do induction on $\dim G$.

Theorem 4.6 ([CG90, Proposition 2.3.4])**.** *Let G be nilpotent Lie group with one dimensional center $Z(G) = \exp(\mathbb{R}Z)$. Choose splitting $\mathbb{R}Z \oplus \mathfrak{g}_0$ as in Kirillov's Lemma, with $G_0 = \exp(\mathfrak{g}_0)$. Then for each $\pi \in \widehat{G}$ that does not vanishes on the center, then we can find a $\sigma \in \widehat{G_0}$ such that $\mathrm{ind}_{G_0}^{G}(\sigma) \cong \pi$.*

5 Parametrization of coadjoint orbits

Now we have a complete characterization of $\widehat{G}$ and to apply Plancherel technique we further need the Plancherel measure on $\widehat{G}$. Before the key theorem, we see that the orbits of unipotent action can only pass through each pointwise fixed hyperplane once, as vindicated by the following theorem:

Lemma 5.1. Assume G acts on V unipotently with $G \cdot v_0 = v_0$, then $(G \cdot v) \cap (v + \mathbb{R}v_0) = \{v\}$ or $v + \mathbb{R}v_0$ for any $v \in V$.

The key theorem of the following discussion is as follows:

Theorem 5.2 (Chevalley-Rosenlicht). [CG90, Theorem 3.1.4] *Let G be a m-dimensional connected group acting unipotently on a real vector space V. If $v \in V$, then we can parametrize the G-orbit as:*

$$G \cdot v = \{\exp(t_1 X_1) \cdots \exp(t_k X_k) \cdot v \mid t_i \in \mathbb{R}\}$$

*which is a closed submanifolds of V. Choose bases $\{e_i\}$ with respect to Jordan Hölder series V_i as in **Engel's Theorem** such that $V_j = \mathbb{R} - \mathrm{span}\{e_{j+1}, \cdots e_m\}$, we define a map Q in polynomials Q_j:*

$$Q : \mathbb{R}^k \mapsto G \cdot v \quad (t_1, \cdots t_k) \mapsto \exp(t_1 X_1) \cdots \exp(t_k X_k) \cdot v = \sum_{i=1}^{m} Q_j(t_1, \cdots, t_k) e_j$$

Then we can write partition of $\{1, 2, \cdots, m\} = S \sqcup T$ with $S = \{j_1 < \cdots < j_k\}$ such that Q_j depends only on $\{t_i\}_{j_i \le j}$. Moreover, Q_{j_i} is linear in t_i:

$$Q_{j_i} = t_i + Q'_{j_i}(t_1, \ldots, t_{i-1}) for 1 \le i \le k$$

This can be seen as a refinement of CBH-formula, which gives a complete description of polynomial actions on V. In particular, we see $G \cdot v$ is an affine variety with an 'nice' invariant measure like $\mathbb{R}^n$:

Lemma 5.3 ([CG90, Corollary 3.1.5])**.** The inverse map are polynomials by $P_1, \ldots, P_m : \mathbb{R}^k \to \mathbb{R}$ such that $P = \sum_{j=1}^{m} P_j e_j$ where P_j depends only on those $\{u_i \mid j_i \le j\}$. Moreover, P_{j_i}s are orthogonal projections to u_i for all $1 \le i \le k$.

Sketch of Proof. Denote $u_i = Q_{j_i}(t_1, \ldots, t_k)$ and we see $u_1 = t_1$ and recursively the following can be defined. □

Remark 5.1. Note the partition S only depends on the orbit (but not the particular choice of representatives) and these are precisely those dimensions on which the orbits increase dimension by passing from V/V_{j-1} to V/V_j. Moreover, given splitting $V = V_S \oplus V_T$ base on S, T, and the following map:

$$\begin{array}{ccccc} \mathbb{R}^k & \xrightarrow{\quad P \quad} & G \cdot v & \xrightarrow{\text{permutation of basis}} & V_S \oplus V_T \\ u = (u_1, \ldots, u_k) & \longmapsto & (P_1(u), \ldots, P_m(u)) & \longrightarrow & (u_{j_1}, \ldots, u_{j_n}) \oplus (P_{t_1}(u), \ldots, P_{t_{m-k}}(u)) \end{array}$$

and we see from the map that $G \cdot v$ is just the graph of $\{P_j \mid j \in T\}$.

Now recall each set $U \subseteq \mathbb{R}^n$ is said to be **Zariski-open** if it is a union of sets $\{x \in \mathbb{R}^m : P(x) \neq 0\}$ for some polynomial P. We define the set of **generic orbits** in V to be

$$\{v \in V \mid \dim G \cdot v \text{ is maximal in } V/V_j \text{ for all } 1 \leq j \leq m\}$$

which is a Zariski-open G-invariant set by **Chevalley-Rosenlicht**. All orbits in it have maximum dimension, but it need not coincide with the full set of orbits in V of maximal dimension. Upon choosing V_S as "representatives" we see V_T are parameters of orbits. So we want to show U admits a locally trivial structure in V under Zariski topology:

Theorem 5.4 ([CG90, Theorem 3.1.6]). *Use the setting of* ***Chevalley-Rosenlicht****, then there exists a Zariski-open set $U \subseteq V$ such that for rational functions $\{R_i(x,t)\}_{i=1}^m$ of $(x,t) \in \mathbb{R}^m \times \mathbb{R}^k$, if $v = \sum_{i=1}^m x_i e_i$, then:*

1. *The functions $R_i(x,t)$ are rational nonsingular on $U \times \mathbb{R}^k$ and for $v \in U$, $R(x,t) = \sum_{i=1}^m R_i(x,t)e_i$ maps $\mathbb{R}^k$ diffeomorphic onto the orbit $G \cdot v$;*
2. *For fixed x, $R_j(x,t)$ has all behaviours of $Q_j(t)$ in* ***Chevalley-Rosenlicht****, whereas in general:*

$$R_j(x,t) = \begin{cases} x_j + R_j'(x_1, \dots, x_{j-1}, t_1, \dots, t_i) & \text{if } j \notin S \\ t_i + x_{j_i} + R_{j_i}'(x_1, \dots x_{j_i-1}, t_1, \dots, t_{i-1}) & \text{if } j = j_i \in S \end{cases}$$

   with R' rational and i the largest index in S such that $j_i < j$. Moreover, $R_1(x,t) = x_1$.

Remark 5.2. Note The sets S and U are more specific as the complement of indices of generic dimensions. i.e., by choosing the d_j the generic dimension of G-orbits in V/V_j and $k = d_m$ the generic dimension of orbits in V. Now:

$$U := \{v \in V \mid \dim G \cdot \bar{v} = d \text{ where } v \in V/V_j \text{ for } 1 \leq j \leq m\}$$
$$S := \{j_1 < \cdots j_k \mid \text{for each } j_i,\, d_{j_1} \neq d_{j_i} - 1\}$$

Dually we have a completely analogous version of P map in Lemma 5.3 that extends them to $(x,u) \in \mathbb{R}^m \times \mathbb{R}^k$. For details of statements consult [CG90, Corollary 3.1.8]. Having these all settled, we are ready to describe the local structure already:

Theorem 5.5 ([CG90, Theorem 3.1.9]). *Inherited the setting as in this section, we have:*

1. *Every G-orbit in U meets V_T in a unique point. In particular, $U \cap V_T$ is nonempty and Zariski-open in V_T;*
2. *The local trivialization on U, denoted by $\phi : (U \cap V_T) \times V_S \to U$ are birational nonsingular bijections such that*
   (a) *For each $v \in V_T \cap U$, the map*

$$\begin{array}{ccccccc} P_v : V_S & \xhookrightarrow{\iota} & (U \cap V_T) \times V_S & \longrightarrow & U & \xrightarrow{p_T|_U} & U \cap V_T \\ w & \longrightarrow & (v,w) & \longrightarrow & \phi(v,w) & \longrightarrow & p_T(\phi(v,w)) \end{array}$$

   with p_T the orthogonal projection onto V_T-component;
   (b) *The Jacobian determinant of ϕ is 1.*

Summing them up, we see a nice parametrization of orbit $G \cdot v$ by $u = (u_1, \dots, u_l)$ with $du_1 \cdots du_k$ an invariant measure. (See [CG90, 3.1.10])

Example 5.3. Consider simple case $S := \{2,4\} \subseteq \{1,2,3,4\}$, then according to the lemma above, we have:

$$P(u_1,u_2) = (c, P_2(u_1), P_3(u_1), P_4(u_1,u_2)) = (c, u_2, P_3(u_1), u_2)$$
$$Q(t_1,t_2) = (b, Q_2(t_1), Q_3(t_1), Q_4(t_1,t_2)) = (b, t_1, Q_3(t_1), t_1 + Q_4'(t_2))$$

6 Plancherel Formula of nilpotent Lie groups

The highlight of this discussion on representation theory ends in the concrete Plancherel formula on nilpotent group case. We state the main theorem first.

Theorem 6.1 (Plancherel inversion theorem). [CG90, Theorem 4.3.9] *Let $\{X_1, \dots, X_n\}$ be a strong Malcev basis for a nilpotent Lie algebra $\mathfrak{g}$ with dual basis $\{l_1, \dots l_n\}$. Define U to be the set of generic coadjoint orbits with index set $S = \{i_1 < \cdots < i_{2k}\}$ for $\mathfrak{r}_l \backslash \mathfrak{g}$ and T the complement of S. Then we define the **Pfaffian** associated to l by:*

$$|\mathrm{Pf}(l)|^2 = \det B, \quad \textit{where} \quad B_{jk} = B_l(X_{i_j}, X_{i_k})$$

as above. Then for $\phi \in \mathcal{S}(G)$, the function evaluated at e is given by an absolutely convergent integral:

$$\phi(e) = \int_{U \cap V_T} |\mathrm{Pf}(l)| \operatorname{tr} \pi_l(\phi)\, dl$$

with dl the Lebesgue measure on $V_T = \mathbb{R} - \operatorname{span}\{l_i : i \in T\}$ such that the cube determined by T has measure 1.

To validify this theorem we need to first explain $\operatorname{tr}\pi$. It is indeed true that every $\pi \in \widehat{G}$ gives a trace-class operator on $L^2(\mathbb{R}^k)$ for each Schwartz 'test' function $\phi \in \mathcal{S}(G)$, where $k := \frac{\dim R_l \backslash G}{2}$ the induced dimension. To give a more detailed descripion, one however needs to choose parametrization of the orbit space. We first pin down the parametrization:

Let $\pi = \pi_l \in \widehat{G}$ with $\mathfrak{m}$ the polarizing subalgebra for L. Fix a basis $\{X_1, \dots, X_n\}$ a weak Malcev basis through $\mathfrak{m}$. Then we define the parametrization as follows by implicitly using Theorem A.1:

$$\begin{aligned}
\gamma &: \mathbb{R}^n \to G & (s,t) &\mapsto \exp s_1 X_1 \cdot \cdots \cdot \exp(s_p X_p) \cdot \exp(t_1 X_{p+1}) \cdot \cdots \cdot \exp(t_k X_n) \\
\alpha &: \mathbb{R}^p \to M & s &\mapsto \gamma(s, 0) \\
\beta &: \mathbb{R}^k \to M \backslash G & t &\mapsto \gamma(0, t)
\end{aligned}$$

with the invariant measures $dm, dg, d\dot{g}$ corresponding with the Lebesgue measures in such that in light of [CG90, Lemma 1.2.12].

Theorem 6.2 ([CG90, Theorem 4.2.1, Proposition 4.2.2. ff]). *Fix the setting as above. Take the standard basis realization of $\pi = \pi_l$ in $L^2(\mathbb{R}^k)$ relative to the given Malcev basis. Then for each $\phi \in \mathcal{S}(G)$, $\pi(\phi)$ isof trace class with Schwartz kernel $K_\phi \in \mathcal{S}(\mathbb{R}^k \times \mathbb{R}^k)$, i.e.:*

$$\textit{for all } f \in L^2(\mathbb{R}^k) \qquad \pi(\phi) \circ f(s) = \int_{\mathbb{R}^k} K_\phi(s,t) f(t) dt$$

with $\theta_\pi(\phi) := \operatorname{tr}\pi(\phi)$ a tempered distribution on $\mathcal{S}(G)$, and the following integral absolutely convergent:

$$\theta_\pi(\phi) = \int_{\mathbb{R}^k} K_\phi(s,s) ds$$

Moreover, using the parametrization α, β, γ above, the kernel K_ϕ admits the form:

$$K_\phi(t', t) = \int_M \chi_l(m) \phi(\beta(t')^{-1} m \beta(t)) dm$$

The integral being absolutely convergent, and $\chi(\exp Y) = e^{2\pi i l(Y)}$ for $Y \in \mathfrak{m}$.

with the trace of representation accordingly expression as:

$$\operatorname{tr}\pi(\phi) = \int_{\mathbb{R}^k} K_\phi(u,u) du = \int_{\mathbb{R}} \left[\int_{\mathfrak{m}} e^{2\pi i l(H)} \phi(\beta(u)^{-1} \exp H \beta(u)) dH \right] du \qquad \text{(Kernel of Trace)}$$

Now the choice of basis and hence α, β, γ allows us to fix the invariant measure on G and $\mathfrak{g}$ respectively, hence we can define **Euclidean Fourier transform** on G (resp. on $\mathfrak{g}$):

$$\text{for all } l \in \mathfrak{g}^* \begin{cases} \widehat{f}(l) = \int_{\mathfrak{g}} e^{2\pi i l(X)} f(\exp X) dX & \text{for } f \in \mathcal{S}(G) \\ \mathcal{F}f(l) = \int_{\mathfrak{g}} e^{2\pi i l(X)} f(X) dX & \text{for } f \in \mathcal{S}(\mathfrak{g}) \end{cases}$$

As we are using invariant Haar measure now, this brings life to a more intrinsic version of trace formula:

Theorem 6.3 ([CG90, Theorem 4.2.4]). *Given $\pi \in \widehat{G}$ for G simply connected nilpotent Lie group, which corresponds to the coadjoint orbit $\mathcal{O}_l \subseteq \mathfrak{g}^*$ of l, then there is a unique choice of invariant measure μ on $\mathcal{O}_l$ such that:*

$$\operatorname{tr} \pi(\phi) = \int_{\mathfrak{g}^*} \widehat{\phi}(l) \mu(dl)$$

for all $\phi \in \mathcal{S}(G)$. More precisely, it can described by choosing $l \in \mathfrak{g}^*$ such that $\pi = \pi_l$. Then let $\{U_1, \ldots, U_r, X_1, \ldots, X_{2k}\}$ be any weak Malcev basis through $\mathfrak{r}_l$. Choose $B \in M_{2k}(\mathbb{R})$ to be a matrix corresponding to B_l restricted to $\{X_i\}$, then the Lebesgue measure defines an invariant measure $d\dot{g}$ on $R_l \backslash G$ such that for any Schwartz function ϕ:

$$\begin{aligned} \operatorname{tr} \pi(\phi) &= |\det B|^{1/2} \int_{\mathbb{R}^{2k}} \widehat{\phi}(l \cdot \gamma(x)) dx \\ &= |\det B|^{1/2} \int_{R_l \backslash G} \widehat{\phi}(l \cdot g) d\dot{g} \end{aligned}$$

Remark 6.1. In the case G is not simply connected, the case resembles that of aforementioned case, albeit somewhat more complicated. To begin our discussion, we need to fix the Haar measure beforehand. Choose dx_0 on $\widetilde{G}$ with corresponding Lebesgue measure dX on $\mathfrak{g}$.Then for the lattice Λ on $\mathfrak{g}$ we choose $d\dot{X}$ on $\mathfrak{g}/\Lambda$ to be Λ-equivariant, i.e.:

$$\int_{\mathfrak{g}} \phi(X) dX = \int_{\mathfrak{g}/\Lambda} \left(\sum_{Z \in \Lambda} \phi(X + Z) \right) d\dot{X} \qquad \forall \phi \in \mathcal{S}(\mathfrak{g})$$

Using this set of measures to define $L^1(\mathfrak{g}/\Lambda)$ and for $\phi \in L^1(\mathfrak{g}/\Lambda)$ we define the Fourier transform on its dual lattices $\mathfrak{g}_{\mathbb{Z}}^* := \{l \in \mathfrak{g}^* \mid l(\Gamma) \subseteq \mathbb{Z}\}$ to be:

$$\widehat{\phi}(l) = \int_{\mathfrak{g}/\Lambda} \phi(X) e^{2\pi i l(X)} d\dot{X}$$

note $e^{2\pi i l(X)}$ takes constant value on Λ-cosets on dual lattices. Also we make the Haar measure dx on G to be compatible with the Galois covering, i.e.:

$$\int_{\widetilde{G}} \phi(x_0) dx_0 = \int_G \left(\sum_{\gamma \in \Gamma} \phi(x\gamma) \right) dx \qquad \forall \phi \in C_c(\widetilde{G})$$

Theorem 6.4 ([CG90, Theorem 4.4.4]). *Let G be a connected nilpotent Lie group with $\widetilde{G}$ its covering group, and $\Gamma \subseteq \widetilde{G}$ a discrete central subgroup such that $G \cong \widetilde{G}/\Gamma$. If $\mathfrak{g}$ is the corresponding Lie algebra, we define $\Lambda = \log \Gamma$. Fix $dX, d\dot{X}, dx, dx_0$ as above.*

If $\pi \in \widehat{G}$ then π corresponds to some coadjoint orbit $\mathcal{O}_l \subseteq \mathfrak{g}_{\mathbb{Z}}^*$ and:

$$\operatorname{tr} \pi(f) = \int_{\mathcal{O}_1} \widehat{f \circ \exp}(l) d\theta_l(l') \qquad \forall f \in \mathcal{S}(G)$$

with $d\theta_l$ the invariant measurable on $\mathcal{O}_l$ in the trace formula for the representation $\sigma_l \in \widehat{\widetilde{G}}$ associated with $\mathcal{O}_l$.

To make the exposition complete, there are three key items to be ascertained:

1. The measure present in The measure dl in Theorem 6.1. In nilpotent cases this is really canonical, as there is a natural symplectic structure on the orbit, for which case we take the symplectic volume form to be its measure. To be more explicit, fix $l \in \mathfrak{g}^*$, then the map $\varphi = \varphi_l := l\cdot$ the right action gives a map $G \to \mathcal{O}$ which quotients to be a diffeomorphism $R_l\backslash G \to \mathcal{O}$. Differentiate at e gives $d\varphi$ and we define the **canonical symplectic form** *omega* to be the canonical symplectic form on dual tangent space, which is here $\mathfrak{g}^*$ by:

$$\omega(d\varphi(X), d\varepsilon(Y)) = l([X, Y]) \quad \text{for all } X, Y \in \mathfrak{g}$$

It is easy to check ω is $\mathrm{Ad}^* G$-invariant and we take the **canonical measure** to be the measure associated to the volume form $\mu = \omega^{\wedge n}$ on $\mathfrak{g}^*$.

2. Recall a few notions:

   (a) the **self-dual measure** associated with any nondegenerate bilinear form B, i.e., the form on which we define Fourier transform as:

$$\mathcal{F}\phi(v) = \int_V f(v')e^{2\pi i B(v,v')}dm(v') \quad \phi \in \mathcal{S}(G)$$

   and then we say m is **self-dual** if $\|\phi\|_2^2 = \|\mathcal{F}\phi\|_2^2$.

   (b) The **tempered distribution** corresponding to $\pi = \pi_l$, i.e., the distribution θ_π on $\mathcal{S}(G)$ such that $\langle \theta_\pi, \phi\rangle = \operatorname{tr} \pi_\phi$ for all $\phi \in \mathcal{S}(G)$.

   Then the measure on each orbit can be further identified with the self-dual Euclidean measure dm on $\mathbb{R}^{2k}$ w.r.t. B_l, the canonical measure μ on orbit $\mathcal{O}_l$ and the tempered distribution θ_π in an canonical way. i.e., by $\{U_1, \dots, U_r, X_1, \dots, X_{2k}\}$ the weak Malcev basis through the radical $\mathfrak{r}_l$ and identify $x \in \mathbb{R}^{2k}$ with $\overline{X} = \sum x_j \overline{X_j} \in \mathfrak{r}\backslash\mathfrak{g}$. Then:

   (a) $dm = |\det B|^{1/2} dx_1 \cdots dx_{2k}$ where $B_{ij} = B_l(X_i, X_j)$;

   (b) The map $f := x \mapsto \mathrm{Ad}^*(\exp xX)^{-1}l$, which is a map between $\mathbb{R}^{2k}$ and $\mathcal{O}_l$, transforms dm to θ_l on $\mathcal{O}_l$ and to $(2^k k!)^{-1}\mu_l$ the canonical invariant measure on $\mathcal{O}_l$.

3. Parametrization of the tempered distribution θ_l based on actual choice from Theorem 5.5. Other than choosing a weak Malcev basis, we can identify the dual orthogonal basis $\mathfrak{k}_S := V_T^\perp = \mathbb{R} - \operatorname{span}\{X_i : i \in S\}$ and $\mathfrak{k}_T := V_S^\perp$ respectively. THne the basis X_i gives a map:

$$\gamma : \mathfrak{k}_S \to G \qquad \sum_{j=1}^{2k} x_j X_{i_j} \mapsto \exp xX$$
$$f_l := l \cdot \gamma : \mathfrak{k}_S \to \mathcal{O}_l \qquad x \mapsto \mathrm{Ad}^*(\gamma(x))^{-1}l$$

   Then $\gamma(\mathfrak{k}_S)$ defines a cross-section for $R_l\backslash G$ which transforms the Lebesgue measure m_1 on $\mathfrak{k}_S$ (such that the unit cube has mass 1) to a right-invariant measure on $R_l\backslash G$, with the constant given by Pfaffian, i.e., $\theta_l = (f_l)_*(|\mathrm{Pf(l)}|m_1)$.

7 Infinitesimal representations and characters of nilpotent Lie groups

To wrap up the discussion, we will discuss briefly about the infinitesimal character of $\pi \in \widehat{G}$.

Definition 7.1. Let G be a Lie group and let π be a representation of G on a Hilbert space $\mathcal{H}_\pi$. A vector $v \in \mathcal{H}_\pi$ is said to be smooth or of type C^∞ if the function

$$G \ni x \mapsto \pi(x)v \in \mathcal{H}_\pi$$

is of class C^∞. We denote by $\mathcal{H}_\pi^\infty$ the space of all smooth vectors of π.

The following is a necessary preparation to introduce the notion of the infinitesimal representation and of the operator $\pi(X)$. This will be of fundamental importance in the sequel.

Proposition 7.2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let π be a strongly continuous representation of G on a Hilbert space $\mathcal{H}_\pi$. Then for any $X \in \mathfrak{g}$ and $v \in \mathcal{H}_\pi^\infty$, the limit*

$$\lim_{t\to 0} \frac{1}{t} \left(\pi\left(\exp_G(tX)\right) v - v\right)$$

exists in the norm topology of $\mathcal{H}_\pi$ and is denoted by $\pi(X)v$. Each $\pi(X)$ leaves $\mathcal{H}_\pi^\infty$ invariant, and π is a representation of $\mathfrak{g}$ on $\mathcal{H}_\pi^\infty$ satisfying

$$\forall X, Y \in \mathfrak{g} \quad \pi(X)\pi(Y) - \pi(Y)\pi(X) - \pi([X, Y]) = 0$$

Consequently, π extends to a representation of the Lie algebra $U(\mathfrak{g})$ on $\mathcal{H}_\pi^\infty$ with $\pi(0) = 0$ and $\pi(1) = 0$

Definition 7.3. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let π be a strongly continuous representation of G on a Hilbert space $\mathcal{H}_\pi$. The representation π defined above is called the infinitesimal representation associated to π. We will often denote it also by π. Consequently, for $T \in U(\mathfrak{g})$ or for its corresponding left-invariant differential operator.

First we find a suitable parametrization of universal enveloping algebra $U(\mathfrak{g})$, then we see that using this characterization together with a generalized version of Schur's Lemma to yield the fact the the center of $U(\mathfrak{g})$, which we denote as $Z(\mathfrak{g})$, acts on smooth vectors by scalars.

Remark 7.1. The reader should note this pattern of infinitesimal character is generally the same for semisimple Lie algebras, though the details are largely different. In semisimple cases, we construct a **Harish-Chandra homomorphism**. Something to be added later on...

Definition 7.4. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$ and let $T(\mathfrak{g}) := \sum_{r=0}^{\infty} \bigotimes^r \mathfrak{g}$ the **tensor algebra** of $\mathfrak{g}$, and the **universal enveloping algebra** $U(\mathfrak{g})$ is the quotient of $T(\mathfrak{g})$ by two sided ideal generated by:

$$\{(X \otimes Y - Y \otimes X - [X, Y]) \mid X, Y \in \mathfrak{g}\}$$

By **Poincaré-Birkhoff-Witt Theorem** we can find a canonical basis fo $U(\mathfrak{g})$ by monomials:

$$X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n} \qquad \text{where } \alpha = (\alpha_1, \dots, \alpha_n) \text{ is a multi-index}$$

where X_i are a basis of $\mathfrak{g}$. Let G be an analytic group with Lie algebra $\mathfrak{g}$. Now we can identify $U(\mathfrak{g})$ with the space of left-invariant differential operator $D(G)$ via the following algebra isomorphism: Given $X \in \mathfrak{g}$, then assign to it a left-invariant vector field $\widetilde{X}$ via:

$$\widetilde{X} f(x) = \frac{d}{dt} f(x \cdot (\exp tX))\big|_{t=0}$$

One can also take it as an first-order differential operator. Now extend the map to $U(\mathfrak{g})$ via universal property. Conversely, given a differential operator $D \in D(G_\mathbb{R})$, we realize it as an element in $U(\mathfrak{g})$ use the equation above. Denote $Z(\mathfrak{g})$ the centre of $U(\mathfrak{g})$. By [Kna86, Proposition 3.8] we can identify the centre with all G-invariant differential operator, i.e., if one (hence all) of the following holds true:

$$D \in Z(\mathfrak{g}) \Leftrightarrow DX = XD \Leftrightarrow e^{\operatorname{ad} X} D = D \Leftrightarrow \operatorname{Ad}(g)D = D \qquad \text{for all } X \in \mathfrak{g}, g \in G$$

If we take the degree the basis above to be degree 1, then it gives a natural grading structure by index degree. Denote $S(\mathfrak{g})$ to be its associated graded ring, which can be seen to be identified with the symmetric algebra of $\mathfrak{g}$ via extending the map $\mathfrak{g}_{\text{id}} : S_1(\mathfrak{g}) = U_1(\mathfrak{g})/U_0(\mathfrak{g}) \to \mathfrak{g}$. This can be further identified with the polynomial algebra $\mathbb{C}[\mathfrak{g}^*]$ as follows:

$$\phi : S(\mathfrak{g}) \longrightarrow \mathbb{C}[\mathfrak{g}^*] \qquad X_\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n} \mapsto (X^\alpha : l \mapsto l(X_1)^{\alpha_1} \dots l(X_n)^{\alpha_n})$$

This isomorphism ϕ when restricted to $\operatorname{Ad}^* G$-invariant subspace, gives an characterization of $Z(\mathfrak{g})$.

Theorem 7.5 ([CG90, Corollary 3.3.3 and Theorem 3.3.4]). *Define* $\mathbb{C}[\mathfrak{g}^*]^G$ *and* $\mathfrak{Y}(\mathfrak{g})$ *to be the following subspaces of* $\mathbb{C}[\mathfrak{g}^*]$ *and* $S(\mathfrak{g})$ *respectively:*

$$\mathbb{C}[\mathfrak{g}^*]^G := \{f \in \mathbb{C}[\mathfrak{g}^*] \mid (\operatorname{ad} X)f = 0 \quad \forall X \in \mathfrak{g}\}$$
$$\mathfrak{Y}(\mathfrak{g}) := \{P \in S(\mathfrak{g}) \mid (\operatorname{ad} X)P = 0 \quad \forall X \in \mathfrak{g}\}$$

Then $\mathbb{C}[\mathfrak{g}^*]$ *is the set of* $\operatorname{Ad}^* G$*-invariant polynomials on* $\mathfrak{g}^*$*, while* ϕ *above gives an isomorphism (of algebras) between* $\mathbb{C}[\mathfrak{g}^*]^G$ *and* $\mathfrak{Y}(\mathfrak{g})$*. Moreover, the following symmetrization map* S *gives a linear bijection between* $S(\mathfrak{g})$ *and* $U(\mathfrak{g})$*:*

$$S : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) Y_1 \dots Y_r \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} Y_{\sigma_1} \cdots Y_{\sigma_n}$$

averaging the value of all r*-permutations. If we further assume* G *and* $\mathfrak{g}$ *to be nilpotent, then* S *gives an algebra isomorphism between* $\mathfrak{Y}(\mathfrak{g})$ *and* $Z(\mathfrak{g})$*.*

Remark 7.2. The last statement again uses Kirillov's Lemma as a key input of nilpotency.

To end our discussion we are satisfied with quoting the following theorem:

Theorem 7.6 ([Pou72, Corollary 3.5] and [CG90, Theorem 4.6.2]). *Given a continuous unitary representation* π *of Lie group* G*. If* $A \in Z(\mathfrak{g})$*, then* A *acts on smooth vectors of* H_π *as scalars. If furthermore* G *is assumed to be nilpotent, then by identifying the* A *with its image* $S(A) \in \mathbb{C}[\mathfrak{g}^*]$*, which is a polynomial on* $\mathfrak{g}^*$*, we can explicitly computed the infinitesimal character as:*

$$\pi(A) = S(A)(2\pi i l) \cdot I \quad \textit{where } l \in \mathcal{O}_\pi$$

This chapter will be more related to functional analysis as it is dealing with unbounded operators. However, in the end we will give a theorem found by Nelson in [Nel59], which integrates a Lie algebra representation to a unitary Lie group representation iff the Laplace operator is essentially self-adjoint. All the notions just mentioned will be defined properly in the following. We first start with the definition of unbounded operators and some of their properties, before we introduce the absolute value of an operator, which will be crucial for estimations on the way to Nelson's theorem.

Note that everything, which is done in the following, can also be established for Hilbert modules as it was shown in [Pie06].

8 Unbounded Operators and Analytic Vectors

In the following we will introduce (unbounded) operators, which crucially depend on their domain. Because of this it is not possible to apply an operator on every vector in the Banach or Hilbert space. However, in case of operators that map their domain on subspaces, which are not part of the domain, it is not even possible to square the operator. It is the number of repeated applications of the operator, which gives us a definition for smooth and analytic operators. Moreover, in case of a Hilbert space there are even more detailed possibilities to define unbounded operators similar to bounded ones.

Definition 8.1 (Unbounded Operator). Let $\mathfrak{X}$ be a Banach space. Then an unbounded operator on $\mathfrak{X}$ is given by a linear map

$$A\colon \operatorname{Dom} A \longrightarrow \mathfrak{X}, \tag{1}$$

where $\operatorname{Dom} A \subseteq \mathfrak{X}$ is a linear subspace that is the domain of A. The operator A is called densely defined if $\operatorname{Dom} A$ is a dense subspace of $\mathfrak{X}$. The set of (unbounded) operators on $\mathfrak{X}$ is denoted by $\mathcal{O}(\mathfrak{X})$.

Remark 8.1. Compared with bounded (or continuous) linear operators, unbounded operators do not yield an algebra. The reason is the domain which does not coincide for all the operators in $\mathcal{O}(\mathfrak{X})$. This leads to the situation that addition and composition of operators can not be defined as usual. Let $A, B \in \mathcal{O}(\mathfrak{X})$ be two unbounded operators, then the domain of $A + B$ is given by

$$\operatorname{Dom} A + B = \operatorname{Dom} A \cap \operatorname{Dom} B \tag{2}$$

and

$$\operatorname{Dom} AB = \{x \in \mathfrak{X} \mid Bx \in \operatorname{Dom} A \text{ for } x \in \operatorname{Dom} B\}. \tag{3}$$

As the domain of an unbounded operators is crucial we will assume every operator to be unbounded in the following. This means they all have an associated domain, which is not always mentioned.

Example 8.2 (Differentiation operator). The most famous example and the reason why unbounded operators are considered, is the fact that the usual differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$ is an unbounded one. By taking into account the space of continuous functions $C([0, 1])$ on the unit interval endowed with the supremum norm

$$\|f\| = \sup_{x \in [0,1]} |f(x)| \tag{4}$$

we can consider the function $f(x) = x^n$ for $n \in \mathbb{N}$. Then we obtain

$$\frac{mathrmdf}{\mathrm{d}x}(x) = nx^{n-1}, \tag{5}$$

which is indeed unbounded by using operator norm.

Definition 8.2 (Extension and closure). Let $\mathfrak{X}$ be a Banach space and let $A, B \in \mathcal{O}(\mathfrak{X})$.

(i) The operator A is called extension of B if

$$\operatorname{Dom} B \subseteq \operatorname{Dom} A \qquad \text{and} \qquad A|_{\operatorname{Dom} B} = B. \tag{6}$$

This is denoted by $B \subseteq A$.

(ii) The operators A is called closed if its graph

$$\operatorname{Graph} A = \{(x, Ax) \in \operatorname{Dom} A \times \mathfrak{X}\} \subseteq \mathfrak{X} \times \mathfrak{X} \tag{7}$$

is closed.

(iii) The operator A is called closable if it has a closed extension, which is denoted by $\overline{A}$.

(iv) The domain $\operatorname{Dom} A$ is called invariant if $\operatorname{Im} A \subseteq \operatorname{Dom} A$.

Remark 8.3. Let A be an unbounded operator on $\mathfrak{X}$, which is bounded on its domain, i.e.

$$\|A\| = \sup_{x \in \mathfrak{X}} \frac{\|Ax\|}{\|x\|} < \infty. \tag{8}$$

Then there is a unique and bounded extension of A on $\operatorname{Dom} A^{\mathrm{cl}}$. Especially, if A is a densely defined operator then $\operatorname{Dom} A^{\mathrm{cl}} = \mathfrak{X}$ and thus A becomes a bounded operator on the whole Banach space $\mathfrak{X}$. On the other hand if there is a bounded extension on $\operatorname{Dom} A^{\mathrm{cl}} = \mathfrak{X}$, then there A must already be a bounded operator on $\operatorname{Dom} A$.

Definition 8.3 (Smooth and analytic vectors). Let $\mathfrak{X}$ be a Banach space and let $A \in \mathcal{O}(\mathfrak{X})$ be an unbounded operator.

(i) The vector $x \in \operatorname{Dom} A$ is called smooth for A if $x \in \operatorname{Dom} A^k$ for all $k \in \mathbb{N}$ and the set of smooth vectors for A is denoted by $\operatorname{Dom} A^\infty$.

(ii) The vector $x \in \operatorname{Dom} A$ is called analytic for A if there is an $s > 0$ such that

$$\sum_{n=0}^{\infty} \frac{\|A^n x\|}{n!} s^n < \infty. \tag{9}$$

The set of analytic vectors for A is denoted by $\operatorname{Dom} A^\omega$ and the set of analytic vectors for A with respect to one $s > 0$ is denoted by $\operatorname{Dom} A^{\omega,s}$.

Remark 8.4. Let $A \in \mathcal{O}(\mathfrak{X})$ be an operator on a Banach space $\mathfrak{X}$.

(i) Then the smooth vectors for A are equivalently given by the intersection of all $\operatorname{Dom} A^k$ for $k \in \mathbb{N}$, i.e.

$$\operatorname{Dom} A^\infty = \bigcap_{k=0}^{\infty} \operatorname{Dom} A^k. \tag{10}$$

Moreover, note that every analytic vector for A is contained in the set of smooth vectors for A, i.e. $\operatorname{Dom} A^\omega \subseteq \operatorname{Dom} A^\infty$. The reason is that only in this case the series is well-defined.

(ii) Note that the condition given in Equation (9) is the same as requiring $e^{sA}x$ to have a positive radius of absolute value.

Example 8.5 (Analytic functions). Considering the same conditions as in Example 8.2, i.e. the Banach space is given by $C([0,1])$ endowed with the supremum norm. Then the analytic vectors with respect to the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$ are given by the analytic functions on the interval $[0,1]$.

So far we only considered the case of a Banach space $\mathfrak{X}$. In case of a Hilbert space H with its inner-product $\langle \cdot, \cdot \rangle$ there is even more structure for unbounded operators like symmetric and adjoint operators. However, it is not as easy as in the case of bounded operators to define an adjoint of an operator since we need to be very careful with the domains of the operators. For the following definition we also switch to densely defined operators.

Definition 8.4 (Symmetric operator). Let H be a Hilbert space and let A be a densely defined operator on H.

(i) The operator A is called symmetric if

$$\langle x, Ay \rangle = \langle Ax, y \rangle \tag{11}$$

for all $x, y \in \operatorname{Dom} A$.

(ii) The operator A is called skew-symmetric if

$$\langle x, Ay \rangle = -\langle Ax, y \rangle \tag{12}$$

for all $x, y \in \operatorname{Dom} A$.

Remark 8.6. Note that there is still another definition for symmetry for a densely defined operator A on a Hilbert space which only requires the adjoint operator A^* to be an extension of A, i.e. $A \subseteq A^*$. Indeed it turns out that we also arrive at

$$\langle x, Ay \rangle = \langle Ax, y \rangle \tag{13}$$

for all $x, y \in \operatorname{Dom} A$.

Definition 8.5 (Adjoint operator). Let H be a Hilbert space and let A be a densely defined operator on H.

(i) The operator A^* is called the adjoint of A, if it is defined on the domain

$$\operatorname{Dom} A^* = \{x \in H \mid y \mapsto \langle x, Ay\rangle \text{ is continuous}\} \tag{14}$$

such that

$$\langle x, Ay\rangle = \langle A^* x, y\rangle \tag{15}$$

for all $y \in \operatorname{Dom} A$.

(ii) If $A = A^*$, then A is called self-adjoint.

(iii) If $A = -A^*$, then A is called skew-adjoint.

(iv) If $\overline{A} = A^*$, then A is called essentially self-adjoint.

(v) If $\overline{A} = -A^*$, then A is called essentially skew-adjoint.

Remark 8.7. There must be some discussion on the definition of the adjoint of an operator A. First note that due to the continuity of the map $y \mapsto \langle x, Ay\rangle$ is a continuous linear functional on $\operatorname{Dom} A$, but as this is densely contained in H it has a bounded extension on H. By Riesz theorem it follows that there is a $z \in H$ such that $\langle z, y\rangle = \langle x, Ay\rangle$ for all $y \in \operatorname{Dom} A$. Again, the densely contained domain leads to the existence of a unique operator satisfying $z = A^* x$.

9 Calculus of Absolute Value

At this point the actual journey starts aiming at Nelson's theorem. This is neatly presented in Nelson's paper [Nel59]. Nevertheless, we will give a motivation and a step-by-step explanation of (almost) everything but will refer to this mentioned paper for the unimportant details. The reason for talking about the estimations is the following: suppose $A, X \in \mathcal{O}(\mathfrak{X})$ such that $\|Xx\| \leq \|Ax\|$ for all $x \in \operatorname{Dom} A + X$, then there is a theorem, which gives conditions for all analytic vectors of A being analytic vectors of X. However, this involves certain estimations of $\|X^n x\|$ in terms of $\|x\|, \|Ax\|, \ldots, \|A^n x\|$. For two commuting operators it turns out that $\|X^n x\| \leq \|A^n x\|$ for all $n \in \mathbb{N}$, but this does not holds in the general case. Actually, we need to take into account the commutator to be able to obtain such kind of estimation. Note that in the following A will be an elliptic operator and X denotes a first order operator in the enveloping algebra of operators.

In the following we will establish some rules to rewrite the inequality

$$\|Cx\| \leq \|Ax\| + \|Bx\| \tag{16}$$

for all $x \in \operatorname{Dom} A \cap \operatorname{Dom} B \cap \operatorname{Dom} C$ by

$$|C| \leq |A| + |B| \tag{17}$$

for some operators $A, B, C \in \mathcal{O}(\mathfrak{X})$.

Definition 9.1 (Absolute value of an operator). Let $\mathfrak{X}$ be a Banach space and let $A \in \mathcal{O}(\mathfrak{X})$. The symbol $|A|$ is called the absolute value of A. The set of absolute values $|\mathcal{O}|(\mathfrak{X})$ is linearly generated by the formal finite sum of absolute values of all the operators contained in $\mathcal{O}(\mathfrak{X})$.

Proposition 9.2. *Let $\mathfrak{X}$ be a Banach space.*

(i) The space of absolute values $|\mathcal{O}|(\mathfrak{X})$ yields a abelian semigroup.

(ii) For absolute values $\alpha = |A_1| + \cdots + |A_l|$ and $\beta = |B_1| + \cdots + |B_m|$ the multiplication defined by

$$\alpha\beta = \sum_{i=1}^{l} \sum_{j=1}^{m} |A_i B_j| \tag{18}$$

makes $|\mathcal{O}|(\mathfrak{X})$ a semialgebra by the identification of positive numbers a with $|a \operatorname{id}|$ for id being the identity.

Definition 9.3 (Norm of absolute value). Let $\mathfrak{X}$ be a Banach space and let $\alpha, \beta \in |\mathcal{O}|(\mathfrak{X})$. The norm of an absolute value $\alpha = |A_1| + \cdots + |A_l|$ is defined by

$$\|\alpha x\| = \|A_1 x\| + \cdots + \|A_l x\| \tag{19}$$

for all $x \in \mathfrak{X}$ by using the convention $\|Ax\| = \infty$ for $x \notin \operatorname{Dom} A$. For another absolute value $\beta = |B_1| + \cdots + |B_m|$ the ordering $\alpha \leq \beta$ is defined by

$$\|\alpha x\| \leq \|\beta x\| \tag{20}$$

for all $x \in \mathfrak{X}$.

Proposition 9.4. *Let $\mathfrak{X}$ be a Banach space and $A, B, C \in \mathcal{O}(\mathfrak{X})$.*

(i) The inequality $|A + B| \leq |A| + |B|$ holds true.

(ii) If $|A| \leq |B|$ holds then $|AC| \leq |BC|$ holds.

Proof. By setting $\alpha = |A|$ and $\beta = |B|$ the claims hold true. ☐

Definition 9.5 (Power series of absolute values). Let $\mathfrak{X}$ be a Banach space and let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}} \subset |\mathcal{O}|(\mathfrak{X})$.

(i) A power series of absolute values φ is given by

$$\varphi = \sum_{n=0}^{\infty} \alpha_n s^n. \tag{21}$$

For another power series of absolute value ψ with coefficients $(\beta_n)_{n\in\mathbb{N}}$ the ordering $\varphi \leq \psi$ is defined by $\alpha_n \leq \beta_n$ for all $n \in \mathbb{N}$.

(ii) The norm of a power series of absolute value φ coming from $(\alpha_n)_{n\in\mathbb{N}}$ is defined for all $x \in \mathfrak{X}$ by

$$\|\varphi x\| = \sum_{n=0}^{\infty} \|\alpha_n x\| s^n. \tag{22}$$

Definition 9.6 (Analytic vector for absolute value). Let $\mathfrak{X}$ be a Banach space. A vector $x \in \mathfrak{X}$ is called an analytic vector for $\alpha \in |\mathcal{O}|(\mathfrak{X})$ if there is some $s > 0$ such that

$$\|e^{s\alpha} x\| < \infty. \tag{23}$$

The set of analytic vector for α in denoted by $\operatorname{Dom} \alpha^{\omega}$ and the set of analytic vectors for α for a particular $s > 0$ is denoted by $\operatorname{Dom} \alpha^{\omega,s}$.

Remark 9.1. Note that in case of $\alpha = |A|$ we arrive at

$$\|e^{s|A|} x\| = \sum_{n=0}^{\infty} \frac{\|A^n x\|}{n!} s^n, \tag{24}$$

which means that in this case $x \in \operatorname{Dom} |A|^{\omega}$ is also an analytic vector for A.

The commutator of operators $A, X \in \mathcal{O}(\mathfrak{X})$ is given by

$$(\operatorname{ad} X)A = XA - AX. \tag{25}$$

Hence we define the commutator of absolute value in the following way.

Definition 9.7 (Commutator of absolute value). Let $\mathfrak{X}$ be a Banach space and $\alpha, \xi \in |\mathcal{O}|(\mathfrak{X})$ with $\alpha = |A_1| + \cdots + |A_l|$ and $\xi = |X_1| + \cdots + |X_d|$. The commutator is defined by

$$(\operatorname{ad} \xi)\alpha = \sum_{i=1}^{d} \sum_{j=1}^{l} |X_i A_j - A_j X_i|. \tag{26}$$

Theorem 9.8. *Let $\mathfrak{X}$ be a Banach space and let $\alpha, \xi \in |\mathcal{O}|(\mathfrak{X})$ like before. Let $\xi \leq c\alpha$, $(\operatorname{ad}\xi)^n\alpha \leq c_n\alpha$ and*

$$\nu(s) = \sum_{n=1}^{\infty} \frac{c_n}{n!} s^n \qquad \textit{and} \qquad \kappa(s) = \int_0^s \frac{Dt}{1-\nu(t)}. \tag{27}$$

Then $e^{s\xi} \leq e^{c\kappa(s)\alpha}$.

Remark 9.2. In order to give the estimation

$$(\operatorname{ad}\xi)^n\alpha \leq c_n\alpha \tag{28}$$

in the previous theorem, it is nessessary to require A to be elliptic.

Definition 9.9 (Analytic dominance). Let $\mathfrak{X}$ be a Banach space and let $\alpha, \xi \in |\mathcal{O}|(\mathfrak{X})$ be related as in Theorem 9.8 such that $c, c_n < \infty$ and $\nu(s)$ has positive convergence radius. Then it is said that α analytically dominates ξ, which is denoted by $\xi \overset{\omega}{\leq} \alpha$.

Corollary 9.10. Let $\mathfrak{X}$ be a Banach space and let $\alpha, \xi \in |\mathcal{O}|(\mathfrak{X})$. If $\xi \overset{\omega}{\leq} \alpha$ then $\operatorname{Dom}\alpha^\omega \subseteq \operatorname{Dom}\xi^\omega$.

Remark 9.3. Note that so far we did not use the completeness of the Banach space $\mathfrak{X}$ we always have available. Therefore it would be sufficient to consider only a normed vector space. Furthermore, we did not even use the linearity of the operators $\mathcal{O}(\mathfrak{X})$.

10 Nelson's Theorem

From now on let H be a Hilbert space. In then next two lemmas we are aiming to find statements about domains of extended corresponding operators. The problem is that for two operators A and X having closures with $|X| \overset{\omega}{\leq} |A|$, it is not clear that in case $x \in \operatorname{Dom}\overline{A}^\omega$ we can conclude $x \in \operatorname{Dom}\overline{X}^{2^\omega}$.

Proposition 10.1. *Let $X \in H$ be a closed and symmetric operator on a Hilbert space H. Then X is self-adjoint iff $\operatorname{Dom}X^\omega \subseteq H$ is dense.*

This equivalence says that we only need enough analytic vectors for the operator X, which are contained in the Hilbert space H to make it a self-adjoint operator.

Remark 10.1. (i) The previous statement still holds true if "symmetric" is replaced by "skew-symmetric" and "self-adjoint" by "skew-adjoint" by considering ιX, which has the same set of analytic vectors as X.

(ii) The statement having the condition of $\operatorname{Dom}X^\omega$ as a dense subspace in Proposition 10.1 is also said to be Nelson's theorem.

Lemma 10.2. Let $X_1, \dots, X_d, A \in H$ be symmetric operators on a Hilbert space H with common invariant domain Dom and suppose that A is essentially self-adjoint. Let $\xi = |X_1| + \dots + |X_d|$, $\alpha = |A| + |\operatorname{id}|$, $\xi \leq c\alpha$ and $(\operatorname{ad}\xi)^n\alpha \leq c_n\alpha$ with $c < \infty$ and $c_n < \infty$ for all $n \geq 1$. For all finite sequences $i_1, \dots, i_n$ one has

$$\operatorname{Dom}\overline{A}^n \subset \operatorname{Dom}\overline{X}_{i_1} \dots \overline{X}_{i_n}. \tag{29}$$

Let $\widetilde{\operatorname{Dom}} = \bigcap_{n=1}^\infty \operatorname{Dom}\overline{A}^n$ and let $\widetilde{X}_1, \dots, \widetilde{X}_d, \widetilde{A}$ be the restrictions of $\overline{X}_1, \dots, \overline{X}_d, \overline{A}$ to $\widetilde{\operatorname{Dom}}$. Let $\widetilde{\xi} = \widetilde{|X|}_1 + \dots + \widetilde{|X|}_d$, $\alpha = \widetilde{|A|} + |\operatorname{id}|$. Then $\widetilde{\xi} \leq c\widetilde{\alpha}$, $(\operatorname{ad}\widetilde{\xi})^n\widetilde{\alpha} \leq c_n\widetilde{\alpha}$ for all $n \geq 1$.

Moreover, if $\xi \overset{\omega}{\leq} \alpha$, then there is an $s > 0$ such that the set $\operatorname{Dom}\widetilde{\xi}^{\omega,s} \subseteq \operatorname{Dom}$ is dense in H and each X_i is essentially self-adjoint.

Remark 10.2. The essence of this lemma is that we can find a common subspace $\widetilde{\mathrm{Dom}}$, which the closed operators satisfy the same estimations on as given by the original operators before. Moreover, in case of analytic dominance the self-adjointness of the dominating operator passes to other such that they become essentially self-adjoint.

In the following $\mathrm{Dom} \subseteq H$ be a subspace and $\mathfrak{g}$ be a Lie algebra consisting of the skew-symmetric operators having Dom as a common invariant domain. The Lie bracket of $\mathfrak{g}$ is the usual commutator, which maps into the skew-symmetric operators as well.

Let us denote the universal enveloping algebra of $\mathfrak{g}$ by $U(\mathfrak{g})$. An element of $U(\mathfrak{g})$ is said to be of order $\leq n$ if it consists of a real linear combination of operators of the form $Y_1 \cdots Y_k$ with $k \leq n$ and all $Y_j \in \mathfrak{g}$. The set of elements of order $\leq n$ is denoted by $U(\mathfrak{g})_n$ and make $U(\mathfrak{g})$ a filtered algebra, i.e. $U(\mathfrak{g})_0 \subseteq U(\mathfrak{g})_1 \subseteq \cdots \subseteq U(\mathfrak{g})_n \subseteq \ldots$ and $U(\mathfrak{g})_k \cdot U(\mathfrak{g})_l \subseteq U(\mathfrak{g})_{k+l}$. The element Δ of order 2 is also called *Nelson's Laplacian* and given by

$$\Delta = X_1^2 + \cdots + X_d^2 \tag{30}$$

for $X_1, \ldots, X_d$ forming a basis of the Lie algebra $\mathfrak{g}$. The goal in the following is to establish some estimations in terms of Nelson's Laplacian Δ.

Lemma 10.3. For $B \in U(\mathfrak{g})_2$ there is some $k < \infty$ such that $|B| \leq k|\Delta - \mathrm{id}\,|$.

Lemma 10.4. Let $\xi = |X_1| + \cdots + |X_d|$ and let $\alpha = |\Delta - \mathrm{id}\,|$. Then $\xi \overset{\omega}{\leq} \alpha$, especially, $\xi \leq \sqrt{\frac{d}{2}}\alpha$ and there is a $c < \infty$ such that for all $n \geq 1$, $(\mathrm{ad}\,\xi)^n \alpha \leq c^n \alpha$. Furthermore, $\xi \overset{\omega}{\leq} |\Delta| + |\,\mathrm{id}\,|$.

Lemma 10.5. Let $m \in \mathbb{N}$. If $B \in U(A)_{2m}$, then for some $k < \infty$ one has

$$|B| \leq k\alpha^m, \tag{31}$$

where $\alpha^m = |(\Delta - \not\Bbbk|)^{\gtrdot}$. If $\eta = |Y_1| + \cdots + |Y_l|$ for $Y_j \in U(A)_{2m}$ and $\mathrm{ad}\, Y_j$ maps into $U(A)_{2m}$ into itself, for $j = 1, \ldots, l$, then $\eta \overset{\omega}{\leq} \alpha^m$ for some $c < \infty$, $(\mathrm{ad}\,\eta)^n \alpha^m \leq c^n \alpha^m$ for all $n \geq 1$.

In the following G be a simply-connected Lie group of the Lie algebra $\mathfrak{g}$ leading to an answer for the question: when does a representation of $\mathfrak{g}$ come from a unitary representation of G? The next statement gives a relation of unitary representation of G if there are enough analytic vectors.

Lemma 10.6. Let $\mathfrak{g}$ be a Lie algebra of skew-symmetric operators on a Hilbert space H having a common invariant domain Dom. Let $X_1, \ldots, X_d$ be a basis for $\mathfrak{g}$ and let $\xi = |X_1| + \cdots + |X_d|$. If for some $s > 0$ the set $\mathrm{Dom}\,\xi^{\omega,s} \subseteq \mathrm{Dom}$ is dense in H, then there is on H a unique unitary representation U of the simply-connected Lie group G having $\mathfrak{g}$ as a Lie algebra such that for all $X \in \mathfrak{g}$, $\overline{U(X)} = \overline{X}$.

Theorem 10.7. *Let $\mathfrak{g}$ be a Lie algebra of skew-symmetric operators on a Hilbert space H having a common domain Dom. Let $X_1, \ldots, X_d$ be a basis for $\mathfrak{g}$ and $\Delta = X_1^2 + \cdots + X_d^2$. If Δ is essentially self-adjoint, then there is on H a unique unitary representation U of the simply-connected Lie group G having $\mathfrak{g}$ as its Lie algebra such that for all $X \in \mathfrak{g}$ one has $\overline{U(X)} = \overline{X}$.*

Proof. For $\xi = |X_1| + \cdots + |X_d|$ we know by Lemma 10.4 that $\xi \overset{\omega}{\leq} |\Delta| + |\,\mathrm{id}\,|$ and thus by Lemma 10.2 we have $\mathrm{Dom}\,\widetilde{xi}^{\omega,s} \subseteq H$ is dense. Note that for applying Lemma 10.2 Nelson's Laplacian needs to be essentially self-adjoint. By Lemma 10.6 we obtain the unitary representation we were looking for.

$\square$

11 Sobolev Spaces

In this section we introduce abstract Sobolev spaces occuring in connection with Nelson's Laplacian Δ, where Nelson's Laplacian is supposed to be essentially self-adjoint. Thus it is closed and symmetric.

Definition 11.1 (s-th Sobolev space,[Vas06]). Let H be a Hilbert space and $\mathfrak{g}$ the Lie algebra of skew-symmetric operators. Let $X_1, \ldots, X_d \in \mathfrak{g}$ be a basis and $\Delta = X_1^2 + \cdots + X_d^2$ be essentially self-adjoint. Then for $s > 0$ the pre-Hilbert space

$$H^s = \overline{\operatorname{Dom} \overline{\Delta}^{s/2}} \tag{32}$$

with the scalar product given by

$$\langle x, y \rangle_s = \langle x, y \rangle + \langle \overline{\Delta}^{s/2} x, \overline{\Delta}^{s/2} y \rangle \tag{33}$$

for $x, y \in H^s$ is called the s-th Soboloev space of Δ.

Proposition 11.2. *Let H^s be the s-th Sobolev space for some $s > 0$.*

(i) The s-th Sobolev space H^s is a Hilbert space with respect to the scalar product given in Equation (33).

(ii) An operator $\mathsf{P} \in H$ of order m extends to a linear bounded map

$$\mathsf{P}\colon H^s \longrightarrow H^{s-m}. \tag{34}$$

12 Analysis on homogeneous Lie groups

Given now a dilation structure δ on G using diagonalisable linear operator A, and rearrange its eigenvalues $d_1 \leq \cdots \leq d_n$ in ascending order. The mappings $\{\delta_r = \exp(A \log r)\}$ give the dilation structure to an n -dimensional homogeneous group $\mathbb{G}$. Fix a basis $\{X_k\}_{k=1}^n$ of the Lie algebra $\mathfrak{g}$ of the Lie group $\mathbb{G}$ such that

$$AX_k = d_k X_k$$

for each k. There are two types of choice of norm on nilpotent Lie groups:

1. We can force the invariance of norm under group action. By noticing the exponential map in the case of nilpotent (and hence in the case of homogeneous Lie groups) are diffeomorphisms, The Euclidean norm on Lie algebra exp can be pulled back to give a norm on $\mathbb{G}$ (which is the Haar measure on $\mathbb{G}$ by [CG90, Theorem 1.2.10]) i.e.:

$$\|x\|_G := \left\|\exp^{-1} x\right\|$$

we use $|x| = \|x\|_G$ to denote the Haar measure on G.

2. We coerce the invariance for sake of homogeneity with respect to dilation. This gives rise to the following definition:

Definition 12.1 (Homogeneous Quasi-norms). Let us define a homogeneous quasi-norm on a homogeneous group $\mathbb{G}$ to be a continuous function $x \mapsto |x|$ from $\mathbb{G}$ to $[0, \infty)$ that satisfies for all $x \in \mathbb{G}$ and $r > 0$,

(a) (symmetry:) $\left|x^{-1}\right| = |x|$,

(b) (homogeneity:) $|\delta_r(x)| = r|x|$ for all $r > 0$.

(c) (non-degeneracy): $|x| = 0$ if and only if $x = e$.

Here and elsewhere we denote by $rx = \delta_r x$ the dilation of x induced by the dilations on the Lie algebra through the exponential mapping.

We define moreover with respect to the dilation structure:

$$Q := \sum_{k=1}^{n} d_k = \mathrm{Tr}(A)$$

the **homogeneous dimension** of $\mathbb{G}$. From now on Q will always denote the homogeneous dimension of G.

Remark 12.1. There always exist homogeneous quasi-norms on homogeneous groups. Moreover, there always exist quasi-norms that are C^∞-smooth on $\mathbb{G}\backslash\{0\}$.

Observe that

$$X = \sum_{k=1}^{n} c_k X_k \in \mathfrak{g} \quad \text{implies} \quad \|\delta_r X\| = \left(\sum_{k=1}^{n} c_k^2 r^{2d_k}\right)^{1/2}$$

where $\|\cdot\|$ is the Euclidean norm. We can notice that for $X \neq 0$ the function $\|\delta_r X\|$ is a strictly increasing function of r, and it tends to 0 and ∞ as $r \to 0$ and $r \to \infty$, respectively. Now, for $x = \exp X$, we can define a homogeneous quasi-norm on G by setting

$$\|e\| := 0 \quad \text{and} \quad \|x\| := 1/r \quad \text{for} \quad x \neq 0$$

where $r = r(X) > 0$ is the unique number such that

$$\left\|\delta_{r(X)} X\right\| = 1$$

By the implicit function theorem and the fact that the Euclidean unit sphere is a C^∞ manifold we see that this function is C^∞ on $\mathbb{G}\backslash\{0\}$

This two norms also give rises to different scaling constant when measuring subsets E of G. For the Haar measure, we have

$$|\delta_r(E)| = r^Q|E|, \quad d(rx) = r^Q dx$$

In particular, we have $|B(x,r)| = r^Q$ for all $r > 0$ and $x \in \mathbb{G}$, whereas for homogeneous quasi-norms, $\|B(x,r)\| = r^{\dim G}$.

Example 12.2. In the case of Heisenberg groups $\mathbb{H}_1$, the dilation corresponds to the matrix $A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}$. Hence the homogeneous quasi-norm is (up to norm equivalence) the **Koranyi norm**:

$$\|(x,y,t)\| = ((x^2+y^2)^2 + t^2)^{1/4}$$

where as the norm corresponding to the Haar measure is:

$$|(x,y,t+\frac{x \cdot y}{2})| = (x^2+y^2+t^2)^{1/2}$$

Definition 12.2 (Homogeneous functions and operators). A function f on $\mathbb{G}\backslash\{0\}$ is said to be homogeneous of degree λ if it satisfies

$$f \circ \delta_r = r^\lambda f \text{ for all } r > 0.$$

We note that for f and g, we have the formula

$$\int_{\mathbb{G}} f(x)\left(g \circ \delta_r\right)(x) dx = r^{-Q} \int_G \left(f \circ \delta_{1/r}\right)(x) g(x) dx$$

given that the integrals exist. Hence we can extend the mapping $f \mapsto f \circ \delta_r$ to distributions by defining, for any distribution f and any test function ϕ, the distribution $f \circ \delta_r$ by

$$\left\langle f \circ \delta_r, \phi\right\rangle = r^{-Q}\left\langle f, \phi \circ \delta_{1/r}\right\rangle$$

where $\langle\cdot, \cdot\rangle$ denotes the usual duality between functions and distributions. The distribution f is called homogeneous of degree λ if it satisfies

$$f \circ \delta_r = r^\lambda f \text{ for all } r>0$$

Also, a linear operator D on $\mathbb{G}$ is called homogeneous of degree λ if it satisfies

$$D\left(f \circ \delta_r\right) = r^\lambda (D f) \circ \delta_r \text{ for all } r>0$$

for any f. If D is a linear operator homogeneous of degree λ and f is a homogeneous function of degree μ, then $D f$ is homogeneous of degree $\mu-\lambda$

If $0<p \leq \infty$, then L^p will denote the usual Lebesgue space on $\mathbb{G}$. For $0<p<\infty$ we write

$$\|f\|_p := \left(\int_{\mathrm{G}}|f(x)|^p dx\right)^{1/p}$$

despite the fact that this is not a norm for $p<1$. However, the map $(f, g) \mapsto\|f-g\|_p^p$ is a metric on L^p for $p<1$. We recall that if f is a measurable function on $\mathbb{G}$, its distribution function $\lambda_f:[0, \infty] \rightarrow[0, \infty]$ is defined by

$$\lambda_f(\alpha) := |\{x:|f(x)|>\alpha\}|,$$

and its nonincreasing rearrangement $f^*:[0, \infty) \rightarrow[0, \infty)$ is defined by

$$f^*(t) = \inf \left\{\alpha: \lambda_f(\alpha) \leq t\right\}.$$

Moreover,

$$z \int_{\mathrm{G}}|f(x)|^p dx = -\int_0^{\infty} \alpha^p d \lambda_f(\alpha) = p \int_0^{\infty} \alpha^{p-1} \lambda_f(\alpha) d \alpha = \int_0^{\infty} f^*(t)^p dt$$

For $0<p<\infty$, the weak-L^p is the space of functions f such that

$$[f]_p := \sup_{\alpha>0} \alpha^p \lambda_f(\alpha) = \sup_{t>0} t^{1/p} f^*(t)<\infty$$

Proposition 12.3 (Polar decomposition: a special case). *Let f be a locally integrable function on $\mathbb{G} \backslash\{0\}$ and assume that it is homogeneous of degree $-Q$. Then there is a constant μ_f (the "average value" of f) such that for every $g \in L^1\left((0, \infty), r^{-1} dr\right)$ we have*

$$\int_{\mathrm{G}} f(x) g(|x|) dx = \mu_f \int_0^{\infty} g(r) r^{-1} dr$$

Proposition 12.4 (Polar decomposition). *Let*

$$\wp := \{x \in \mathbb{G}:|x|=1\}$$

be the unit sphere with respect to the homogeneous quasi-norm $|\cdot|$. Then there is a unique Radon measure σ on $\wp$ such that for all $f \in L^1(\mathbb{G})$

$$\int_{\mathrm{G}} f(x) dx = \int_0^{\infty} \int_{\varphi} f(r y) r^{Q-1} d \sigma(y) dr$$

Let f and g be two integrable function on $\mathbb{G}$. Then their convolution $f * g$ is well defined by

$$(f * g)(x) := \int_G f(y) g\left(y^{-1} x\right) dy = \int_G f\left(x y^{-1}\right) g(y) dy$$

Proposition 12.5 (Young's inequality). *Suppose*

$$1 \leq p, q, r \leq \infty \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

If $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$ and

$$\|f * g\|_{L^r(\mathbb{G})} \leq \|f\|_{L^p(\mathbb{G})} \|g\|_{L^q(\mathbb{G})}.$$

Now let us summarize some properties of approximations to the identity in terms of the convolution. The following notation will be used throughout this note: if ϕ is a function on $\mathbb{G}$ and $t > 0$, we define ϕ_t by

$$\phi_t := t^{-Q} \phi \circ \delta_{1/t}, \quad \text{that is,} \quad \phi_t(x) := t^{-Q} \phi(x/t)$$

We notice that if $\phi \in L^1(\mathbb{G})$ then $\int_{\mathbb{G}} \phi_t(x) dx$ is independent of t.

Proposition 12.6 (Approximation of identity). *Let $\phi \in L^1(\mathbb{G})$ and let $a := \int_{\mathbb{G}} \phi(x) dx$. Then we have the following properties:*

(i) If $f \in L^p(\mathbb{G})$ for $1 \leq p < \infty$, then $\|f * \phi_t - af\|_p \to 0$ as $t \to 0$.

(ii) If f is bounded and right uniformly continuous, then $\|f * \phi_t - af\|_\infty \to 0$ as $t \to 0$.

(iii) If f is bounded on $\mathbb{G}$ and continuous on an open set $\Omega \subset \mathbb{G}$, then $f * \phi_t - af \to 0$ uniformly on compact subsets of Ω as $t \to 0$.

13 Rockland operators and Sobolev space

In this part, we study a special type of operators: the (homogeneous) Rockland operators. These operators can be viewed as a generalisation of sub-Laplacians to the non-stratified but still homogeneous (graded) setting.

13.1 Rockland operators

We start with the discussion of general Rockland operators, giving definitions, examples, and then relating them to the hypoellipticity questions.

Definition 13.1. A Rockland operator on G is a left-invariant differential operator $\mathcal{R}$ which is homogeneous of positive degree and satisfies the Rockland condition:

(R) for each unitary irreducible representation π on G, except for the trivial representation, the operator $\pi(\mathcal{R}) := \pi(\mathcal{R})$ (infinitesimal representation) is injective on $\mathcal{H}_\pi^\infty$, that is,

$$\forall v \in \mathcal{H}_\pi^\infty \quad \pi(\mathcal{R}) v = 0 \Longrightarrow v = 0.$$

Where $\mathcal{H}_\pi^\infty$ is space of all smooth vectors of π which means $\pi(x) v \in \mathcal{H}_\pi$ is of class C^∞.

Proposition 13.2. *Let G be a homogeneous Lie group. If there exists a Rockland operator on G then the group G is graded. Furthermore, the dilations' weights $v_1, \ldots, v_n$ satisfy*

$$a_1 v_1 = \ldots = a_n v_n$$

for some integers $a_1, \ldots, a_n$

Definition 13.3. If G is a stratified Lie group with a given basis $Z_1, \ldots, Z_p$ for the first stratum of its Lie algebra, then the left-invariant differential operator on G given by

$$Z_1^2 + \ldots + Z_p^2$$

is called a sub-Laplacian.

Example 13.1.

(1) For $\mathbb{G} = \mathbb{R}^n$, $\mathcal{R}$ may be any positive homogeneous elliptic differential operator with constant coefficients. For example, we can take

$$\mathcal{R} = (-\Delta)^m \text{ or } \mathcal{R} = (-1)^m \sum_{j=1}^{n} a_j \left(\frac{\partial}{\partial x_j}\right)^{2m}, \quad a_j > 0, m \in \mathbb{N};$$

(2) For the Heisenberg group $\mathbb{G} = \mathbb{H}^n$, we can take

$$\mathcal{R} = (-\mathcal{L})^m \text{ or } \mathcal{R} = (-1)^m \sum_{j=1}^{n} \left(a_j X_j^{2m} + b_j Y_j^{2m}\right), \quad a_j, b_j > 0, m \in \mathbb{N}$$

where $\mathcal{L} := \sum_{j=1}^{n} \left(X_j^2 + Y_j^2\right)$, with $X_j := \partial_{x_j} - \frac{y_j}{2}\partial_t$, $Y_j := \partial_{y_j} + \frac{x_j}{2}\partial_t$ are the left-invariant vector fields.

(3) For any stratified Lie group with vectors $X_1, \ldots, X_k$ spanning the first stratum, we can take

$$\mathcal{R} = (-1)^m \sum_{j=1}^{k} a_j X_j^{2m}, \quad a_j > 0$$

so that, in particular, for $m = 1$, $\mathcal{R}$ is a positive sub-Laplacian which is homogeneous degree 2;

(4) For any graded Lie group $\mathbb{G}$ with dilation weights $v_1, \ldots, v_n$ let us fix the basis $X_1, \ldots, X_n$ of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ satisfying

$$D_r X_j = r^{v_j} X_j, \quad j = 1, \ldots, n, r > 0$$

where D_r denote the dilations on the Lie algebra. If v_0 is any common multiple of $v_1, \ldots, v_n$, the operator

$$\mathcal{R} = \sum_{j=1}^{n} (-1)^{\frac{v_0}{v_j}} a_j X_j^{2\frac{v_0}{v_j}}, \quad a_j > 0$$

is a Rockland operator of homogeneous degree $2v_0$.

Proof. We only proof the last one satisfies the Rockland condition. The operator $\mathcal{R}$ given in (4) is clearly a homogeneous left-invariant differential operator of homogeneous degree $2\nu_o$. Let $\pi \in \widehat{G}\backslash\{1\}$ and $v \in \mathcal{H}_\pi^\infty$ be such that $\pi(\mathcal{R})v = 0$. Then

$$0 = (\pi(\mathcal{R})v, v)_{\mathcal{H}_\pi} = \sum_{j=1}^{n} (-1)^{\frac{\nu_o}{v_j}} c_j \left(\pi\left(X_j\right)^{2\frac{\nu_o}{v_j}} v, v\right)_{\mathcal{H}_\pi} \tag{35}$$

$$= \sum_{j=1}^{n} c_j \left\|\pi\left(X_j\right)^{\frac{\nu_o}{v_j}} v\right\|_{\mathcal{H}_\pi} \tag{36}$$

and hence $\pi\left(X_j\right)^{\frac{\nu_o}{v_j}} v = 0$ for $j = 1, \ldots, n$ Let us observe the following simple fact regarding any positive integer p and any $Z \in U(\mathfrak{g})$: the hypothesis $\pi(Z)^p v = 0$ implies that

1. if p is odd then $\pi(Z)^{p+1}v = \pi(Z)\pi(Z)^p v = 0$.
2. whereas if p is even then

$$0 = (\pi(Z)^p v, v)_{\mathcal{H}_\pi} = (-1)^{p/2}\left(\pi(Z)^{\frac{p}{2}}v, \pi(Z)^{\frac{p}{2}}v\right)_{\mathcal{H}_\pi} = (-1)^{p/2}\left\|\pi(Z)^{\frac{p}{2}}v\right\|^2_{\mathcal{H}_\pi}$$

and hence $\pi(Z)^{\frac{p}{2}}v = 0$ whereas if p is even then

$$0 = (\pi(Z)^p v, v)_{\mathcal{H}_\pi} = (-1)^{p/2}\left(\pi(Z)^{\frac{p}{2}}v, \pi(Z)^{\frac{p}{2}}v\right)_{\mathcal{H}_\pi} = (-1)^{p/2}\left\|\pi(Z)^{\frac{p}{2}}v\right\|^2_{\mathcal{H}_\pi}$$

and hence $\pi(Z)^{\frac{p}{2}}v = 0$.

Applying this argument inductively on $Z = X_j$ and $p = \nu_o/v_j, \nu_o/2v_j, \ldots$ we obtain that $\pi(X_j)\, v = 0$ for each j. Hence $v = 0$. □

Remark 13.2. By Proposition 3.1, if a homogeneous Lie group $\mathbb{G}$ admits a Rockland operator, then the group $\mathbb{G}$ is graded. Example 4 gives the converse: on such a group, we can always find a Rockland operator.

From one Rockland operator, we can construct many since powers of a Rockland operator or its complex conjugate operator are Rockland:

Lemma 13.4. Let $\mathcal{R}$ be a Rockland operator on a graded Lie group G endowed with a family of dilations with integer weights. Then the operators $\mathcal{R}^k$ for any $k \in \mathbb{N}$ and $\overline{\mathcal{R}}$ are also Rockland operators.

Definition 13.5. Let Ω be an open subset of $\mathbb{R}^n$ and let L be a differential operator on Ω with smooth coefficients. Then L is said to be hypoelliptic if, for any distribution $u \in \mathcal{D}'(\Omega)$ and any open subset Ω' of Ω, the condition $Lu \in C^\infty(\Omega')$ implies that $u \in C^\infty(\Omega')$

Theorem 13.6. *Let $\mathcal{R}$ be a left-invariant and homogeneous differential operator on a homogeneous Lie group $\mathbb{G}$. The hypoellipticity of $\mathcal{R}$ is equivalent to $\mathcal{R}$ satisfying the Rockland condition. In this case, any operator of the form*

$$\mathcal{R} + \sum_{[\alpha]<\nu} c_\alpha X^\alpha$$

where ν is the degree of homogeneity of $\mathcal{R}$ and c_α any complex number, is also hypoelliptic.

Proposition 13.7. *Let $\mathcal{R}$ be a Rockland operator on a graded Lie group $\mathbb{G}$. We assume that $\mathcal{R}$ is formally self-adjoint. Let π be a strongly continuous unitary representation of G. Then the operators $\mathcal{R}$ and $\pi(\mathcal{R})$ densely defined on $\mathcal{D}(G) \subset L^2(G)$ and $\mathcal{H}_\pi^\infty \subset \mathcal{H}_\pi$, respectively, are essentially self-adjoint. In this case we will denote by $\mathcal{R}_2$ the self-adjoint extension and by E its spectral measure:*

$$\mathcal{R}_2 = \int_{\mathbb{R}} \lambda dE(\lambda).$$

Let us fix a positive Rockland operator $\mathcal{R}$ on G which is homogeneous of degree $\nu \in \mathbb{N}$. By functional calculus, we can define the spectral multipliers

$$e^{-t\mathcal{R}_2} := \int_0^\infty e^{-t\lambda} \mathrm{d}E(\lambda), \quad t > 0$$

which form the heat semigroup of $\mathcal{R}$. The operators $e^{-t\mathcal{R}_2}$ are invariant under left-translations and are bounded on $L^2(G)$. Therefore the Schwartz kernel theorem implies that each operator $e^{-t\mathcal{R}_2}$ admits a unique distribution $h_t \in \mathcal{S}'(G)$ as its convolution kernel:

$$e^{-t\mathcal{R}_2} f = f * h_t, \quad t > 0, f \in \mathcal{S}(G)$$

The distributions $h_t, t>0$, are called the heat kernels of $\mathcal{R}$. We summarise their main properties in the following theorem:

Theorem 13.8. *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group G. Then the heat kernels h_t associated with $\mathcal{R}$ satisfy the following properties. Each function h_t is Schwartz and we have*

(1) $h_t * h_s = h_{t+s} \quad \forall s,t>0$,

(2) $h_{r^\nu t}(rx) = r^{-Q} h_t(x) \quad \forall x \in G, t, r > 0$,

(3) $h_t(x) = \overline{h_t\left(x^{-1}\right)} \quad \forall x \in G$,

(4) $\int_G h_t(x)dx = 1 \quad \forall x \in G$ *for all* $t>0$,

(5) The function $h : G \times \mathbb{R} \to \mathbb{C}$ *defined by*

$$h(x,t) := \begin{cases} h_t(x) & \textit{if } t>0 \textit{ and } x \in G \\ 0 & \textit{if } t \leq 0 \textit{ and } x \in G \end{cases}$$

is smooth on $(G\times\mathbb{R})\backslash\{(0,0)\}$ *and satisfies*

$$(\mathcal{R}+\partial_t)\, h = \delta_{0,0}$$

where $\delta_{0,0}$ *is the delta-distribution at* $(0,0) \in G \times \mathbb{R}$,

(6) *Having fixed a homogeneous norm* $|\cdot|$ *on* G, *we have for any* $N \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n$ *and* $\ell \in \mathbb{N}_0$, *that*

$$\exists C = C_{\alpha,N,\ell} > 0 \quad \forall t \in (0,1] \quad \sup_{|x|=1} \left|\partial_t^\ell X^\alpha h_t(x)\right| \leq C_{\alpha,N} t^N$$

Theorem 8.2 shows that the functions h_t provide a commutative approximation of the identity. We already know that $\left\{e^{-t\mathcal{R}_2}\right\}_{t>0}$ is a strongly continuous contraction semi-group. Moreover, we have the following properties for any p :

Proposition 13.9. *The operators $f \mapsto f * h_t, \quad t>0$ form a strongly continuous semi-group on $L^p(G)$ for any $p \in [1,\infty)$ and on $C_o(G)$ if $p=\infty$. This semi-group is also equibounded:*

$$\forall t>0, \forall f \in L^p(G) \text{ or } C_o(G) \quad \|f*h_t\|_p \leqslant \|h_1\|_1 \, \|f\|_p.$$

Furthermore, for any $f \in \mathcal{D}(G)$ and any $p \in [1,\infty]$ (finite or infinite), we have the convergence

$$\left\| \frac{1}{t}\left(f*h_t - f\right) - \mathcal{R}f \right\|_p \longrightarrow_{t\to 0} 0.$$

13.2 Positive Rockland operators

The extension of a positive Rockland operator $\mathcal{R}$ to $L^p(G)$ will be denoted by $\mathcal{R}_p$, and first we discuss the essential properties of such an extension.

Definition 13.10. Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group G. For $p \in [1,\infty)$, we denote by $\mathcal{R}_p$ the operator such that $-\mathcal{R}_p$ is the infinitesimal generator of the semi-group of operators $f \mapsto f*h_t, t>0$, on $L^p(G)$

We also denote by $\mathcal{R}_{\infty_o}$ the operator such that $-\mathcal{R}_{\infty_o}$ is the infinitesimal generator of the semi-group of operators $f \mapsto f*h_t, t>0$, on $C_o(G)$

Theorem 13.11. *Let $\mathcal{R}$ be a positive Rockland operator on G and $p \in [1,\infty]$*

(1) The operator $\mathcal{R}_p$ is closed. The domain of $\mathcal{R}_p$ contains $\mathcal{D}(G)$, and for $f \in \mathcal{D}(G)$ we have $\mathcal{R}_p f = \mathcal{R}f$,

(2) The positive Rockland operator $\overline{\mathcal{R}}_p$ is the infinitesimal generator of the strongly continuous semi-group $\left\{f \mapsto f * \bar{h}_t\right\}_{t>0}$ on $L^p(G)$ for $p \in [1, \infty)$ and on $C_o(G)$ for $p = \infty$,

(3) If $p \in (1, \infty)$ then the dual of $\mathcal{R}_p$ is $\overline{\mathcal{R}}_{p'}$. The dual of $\mathcal{R}_\infty$ restricted to $L^1(G)$ is $\overline{\mathcal{R}}_1$. The dual of $\mathcal{R}_1$ restricted to $C_o(G) \subset L^\infty(G)$ is $\overline{\mathcal{R}}_\infty$,

(4) If $p \in [1, \infty)$, the operator $\mathcal{R}_p$ is the maximal restriction of $\mathcal{R}$ to $L^p(G)$, that is, the domain of $\mathcal{R}_p$ consists of all the functions $f \in L^p(G)$ such that the distributional derivative $\mathcal{R}f$ is in $L^p(G)$ and $\mathcal{R}_p f = \mathcal{R}f$. In particular, the operator $\mathcal{R}_2$ coincides with the self-adjoint extension of $\mathcal{R}$ on $L^2(G)$. The operator $\mathcal{R}_\infty$ is the maximal restriction of $\mathcal{R}$ to $C_o(G)$, that is, the domain of $\mathcal{R}_\infty$ consists of all the function $f \in C_o(G)$ such that the distributional derivative $\mathcal{R}f$ is in $C_o(G)$ and $\mathcal{R}_p f = \mathcal{R}f$,

(5) If $p \in [1, \infty)$, the operator $\mathcal{R}_p$ is the smallest closed extension of $\mathcal{R}|_{\mathcal{D}(G)}$ on $L^p(G)$. For $p = 2, \mathcal{R}_2$ is the self-adjoint extension of $\mathcal{R}$ on $L^2(G)$.

Theorem above has the following couple of corollaries which will enable us to define the fractional powers of $\mathcal{R}_p$.

Corollary 13.12. We keep the same setting and notation as above:

(1) The operator $\mathcal{R}_p$ is injective on $L^p(G)$ for $p \in [1, \infty)$ and $\mathcal{R}_\infty$ is injective on $C_o(G)$, namely, for $p \in [1, \infty) \cup \{\infty\}: \quad \forall f \in \operatorname{Dom}(\mathcal{R}_p) \quad \mathcal{R}_p f = 0 \Longrightarrow f = 0$.

(2) If $p \in (1, \infty)$ then the operator $\mathcal{R}_p$ has dense range in $L^p(G)$. The operator $\mathcal{R}_\infty$ has dense range in $C_o(G)$. The closure of the range of $\mathcal{R}_1$ is the closed subspace $\left\{\phi \in L^1(G) : \int_G \phi = 0\right\}$ of $L^1(G)$.

(3) For $p \in [1, \infty]$ and any $\mu > 0$, the operator $\mu \mathrm{I} + \mathcal{R}_p$ is invertible on $L^p(G), p \in [1, \infty)$, and on $C_o(G)$ for $p = \infty$, and the operator norm of $(\mu \mathrm{I} + \mathcal{R}_p)^{-1}$ is

$$\left\|(\mu \mathrm{I} + \mathcal{R}_p)^{-1}\right\|_{\mathscr{L}(L^p(G))} \leqslant \|h_1\| \, \mu^{-1}$$

or

$$\left\|(\mu \mathrm{I} + \mathcal{R}_\infty)^{-1}\right\|_{\mathscr{L}(C_o(G))} \leqslant \|h_1\| \, \mu^{-1}.$$

Remark 13.3. From (3) we know that the operator $\mathcal{R}_p$ is Komatsu-non-negative, we refer the interested reader to the monograph of Martinez and Sanz [MS01].

13.3 Fractional powers of Rockland operators

Theorem 13.13. *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group G. Let $p \in [1, \infty]$.*

(1) Let $\mathcal{A}$ denote either $\mathcal{R}$ or $\mathrm{I} + \mathcal{R}$.

- *(a) For every $a \in \mathbb{C}$, the operator $\mathcal{A}_p^a$ is closed and injective with $\left(\mathcal{A}_p^a\right)^{-1} = \mathcal{A}_p^{-a}$. We have $\mathcal{A}_p^0 = \mathrm{I}$, and for any $N \in \mathbb{N}, \mathcal{A}_p^N$ coincides with the usual powers of differential operators on $\mathcal{S}(G)$ and $\operatorname{Dom}\left(\mathcal{A}^N\right) \cap \operatorname{Range}\left(\mathcal{A}^N\right)$ is dense in Range($\mathcal{A}_p$).*
- *(b) For any $a, b \in \mathbb{C}$, in the sense of operator graph, we have $\mathcal{A}_p^a \mathcal{A}_p^b \subset \mathcal{A}_p^{a+b}$. If Range $(\mathcal{A}_p)$ is dense then the closure of $\mathcal{A}_p^a \mathcal{A}_p^b$ is $\mathcal{A}_p^{a+b}$.*
- *(c) For every $a \in \mathbb{C}$, the operator $\mathcal{A}_p^a$ is invariant under left translations.*
- *(d) If $p \in (1, \infty)$ then the dual of $\mathcal{A}_p$ is $\overline{\mathcal{A}}_{p'}$. The dual of $\mathcal{A}_\infty$ restricted to $L^1(G)$ is $\overline{\mathcal{A}}_1$. The dual of $\mathcal{A}_1$ restricted to $C_o(G) \subset L^\infty(G)$ is $\overline{\mathcal{A}}_\infty$.*
- *(e) For any $a \in \mathbb{C}_+$, $\operatorname{Dom}\left(\mathcal{A}_p^a\right)$ contains $\mathcal{S}(G)$.*

(2) For each $a \in \mathbb{C}_+$, the operators $(\mathrm{I}+\mathcal{R}_p)^a$ and $\mathcal{R}_p^a$ are unbounded and their domains satisfy for all $\epsilon > 0$

$$\mathrm{Dom}\left[(\mathbf{I}+\mathcal{R}_p)^a\right] = \mathrm{Dom}\left(\mathcal{R}_p^a\right) = \mathrm{Dom}\left[(\mathcal{R}_p+\epsilon\mathrm{I})^a\right].$$

(3) If $0 < \mathrm{Re}\, a < 1$ and $\phi \in \mathrm{Range}\,(\mathcal{R}_p)$ then

$$\mathcal{R}_p^{-a}\phi = \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}e^{-t\mathcal{R}_p}\phi dt$$

in the sense that $\lim_{N\to\infty}\int_0^N$ converges in the norm of $L^p(G)$ or $C_o(G)$.

(4) If $a \in \mathbb{C}_+$, then the operator $(\mathrm{I}+\mathcal{R}_p)^{-a}$ is bounded and for any $\phi \in \mathcal{X}$ with $\mathcal{X} = L^p(G)$ or $C_o(G)$, we have

$$(\mathrm{I}+\mathcal{R}_p)^{-a}\phi = \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}e^{-t(\mathrm{I}+\mathcal{R}_p)}\phi dt$$

in the sense of absolute convergence:

$$\int_0^\infty t^{a-1}\left\|e^{-t(\mathrm{I}+\mathcal{R}_p)}\phi\right\|_{\mathcal{X}} dt < \infty.$$

(5) For any $a, b \in \mathbb{C}$, the two (possibly unbounded) operators $\mathcal{R}_p^a$ and $(\mathrm{I}+\mathcal{R}_p)^b$ commute.

(6) *For any $a \in \mathbb{C}$, the operator $\mathcal{R}_p^a$ is homogeneous of degree νa.*

13.4 Sobolev spaces on graded Lie groups

Definition 13.14. Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν and let $s \in \mathbb{R}$. For any tempered distribution $f \in \mathcal{S}'(G)$, we denote by $(\mathrm{I}+\mathcal{R})^{s/\nu}f$ the tempered distribution defined to be

$$\left\langle(\mathbf{I}+\mathcal{R})^{s/\nu}f, \phi\right\rangle = \left\langle f, (\mathbf{I}+\overline{\mathcal{R}})^{s/\nu}\phi\right\rangle, \quad \phi \in \mathcal{S}(G)$$

Lemma 13.15. For any $s \in \mathbb{R}$ and $p \in [1,\infty]$, the domain of the operator $(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}}$ contains $\mathcal{S}(G)$, and the map

$$f \longmapsto \left\|(\mathbf{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f\right\|_{L^p(G)}$$

defines a norm on $\mathcal{S}(G)$. We denote it by

$$\|f\|_{W^{s,p}(G)} := \left\|(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f\right\|_{L^p(G)}$$

Moreover, any sequence in $\mathcal{S}(G)$ which is Cauchy for $\|\cdot\|_{W^{s,p}(G)}$ is convergent in $\mathcal{S}'(G)$.

Definition 13.16. Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group G. We consider its L^p analogue $\mathcal{R}_p$ and the powers of $(\mathrm{I}+\mathcal{R}_p)^s$. Let $s \in \mathbb{R}$:

(1) If $p \in [1,\infty)$, the Sobolev space $W^{s,p}(G)$ is the subspace of $\mathcal{S}'(G)$ obtained by completion of $\mathcal{S}(G)$ with respect to the Sobolev norm

$$\|f\|_{W^{s,p}(G)} := \left\|(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f\right\|_{L^p(G)}, \quad f \in \mathcal{S}(G).$$

(2) If $p=\infty$, the Sobolev space $W^{s,\infty}(G)$ is the subspace of $\mathcal{S}'(G)$ obtained by completion of $\mathcal{S}(G)$ with respect to the Sobolev norm

$$\|f\|_{W^{s,\infty}(G)} := \left\|(\mathrm{I}+\mathcal{R}_\infty)^{\frac{s}{\nu}} f\right\|_{L^\infty(G)}, \quad f \in \mathcal{S}(G).$$

We can also define the homogeneous version of our Sobolev spaces. For some technical reasons, the definition of homogeneous Sobolev spaces is restricted to the case $p \in (1,\infty)$.

Definition 13.17. Let $\mathcal{R}$ be a Rockland operator of homogeneous degree ν on a graded Lie group G, and let $p \in (1,\infty)$. We denote by $\dot{W}^{s,p}(G)$ the space of tempered distribution obtained by the completion of $\mathcal{S}(G) \cap \mathrm{Dom}\left(\mathcal{R}_p^{\frac{s}{\nu}}\right)$ for the norm

$$\|f\|_{\dot{W}^{s,p}(G)} := \left\|\mathcal{R}_p^{\frac{s}{2}} f\right\|_p, \quad f \in \mathcal{S}(G) \cap \mathrm{Dom}\left(\mathcal{R}_p^{s/\nu}\right).$$

By construction the Sobolev space $W^{s,p}(G)$ endowed with the Sobolev norm is a Banach space which contains $\mathcal{S}(G)$ as a dense subspace and is included in $\mathcal{S}'(G)$. The Sobolev spaces share many properties with their Euclidean counterparts.

Theorem 13.18. *Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν on a graded Lie group G. We consider the associated Sobolev spaces $W^{s,p}(G)$ for $p \in [1,\infty) \cup \{\infty\}$ and $s \in \mathbb{R}$.*

(1) If $s=0$, then $W^{0,p}(G) = L^p(G)$ for $p \in [1,\infty]$.

(2) If $s>0$, then we have

$$W^{s,p}(G) = \mathrm{Dom}\left[(\mathbf{I}+\mathcal{R}_p)^{\frac{s}{\nu}}\right] = \mathrm{Dom}\left(\mathcal{R}_p^{\frac{s}{\nu}}\right) \subsetneq L^p(G),$$

and the following norms are equivalent to $\|\cdot\|_{W^{s,p}(G)}$:

$$f \longmapsto \|f\|_{L^p(G)} + \left\|(\mathbf{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f\right\|_{L^p(G)}, f \longmapsto \|f\|_{L^p(G)} + \left\|\mathcal{R}_p^{\frac{s}{\nu}} f\right\|_{L^p(G)}.$$

(3) Let $s \in \mathbb{R}$ and $f \in \mathcal{S}'(G)$:

(i) Given $p \in (1,\infty)$, we have $f \in W^{s,p}(G)$ if and only if $(\mathrm{I}+\mathcal{R}_p)^{s/\nu} f \in L^p(G)$, in the sense that the linear mapping

$$\mathcal{S}(G) \ni \phi \mapsto \left\langle (\mathbf{I}+\mathcal{R}_p)^{s/\nu} f, \phi \right\rangle = \left\langle f, \left(\mathbf{I}+\overline{\mathcal{R}}_{p'}\right)^{s/\nu} \phi \right\rangle$$

extends to a bounded functional on $L^{p'}(G)$ where p' is the conjugate exponent of p.

(ii) $f \in W^{s,1}(G)$ if and only if $(\mathbf{I}+\mathcal{R}_1)^{s/\nu} f \in L^1(G)$ in the sense that the linear mapping

$$\mathcal{S}(G) \ni \phi \mapsto \left\langle (\mathbf{I}+\mathcal{R}_1)^{s/\nu} f, \phi \right\rangle = \left\langle f, \left(\mathbf{I}+\overline{\mathcal{R}}_\infty\right)^{s/\nu} \phi \right\rangle$$

extends to a bounded functional on $C_o(G)$ and is realised as a measure given by an integrable function.

(iii) $f \in W^{s,\infty}(G)$ if and only if $(\mathrm{I}+\mathcal{R}_\infty)^{s/\nu} f \in C_o(G)$ in the sense that the linear mapping

$$\mathcal{S}(G) \ni \phi \mapsto \left\langle (\mathbf{I}+\mathcal{R}_\infty)^{s/\nu} f, \phi \right\rangle = \left\langle f, \left(\mathbf{I}+\overline{\mathcal{R}}_1\right)^{s/\nu} \phi \right\rangle$$

extends to a bounded functional on $L^1(G)$ and is realised as integration against functions in $C_o(G)$

(4) If $a, b \in \mathbb{R}$ with $a < b$ and $p \in [1, \infty]$ then the following continuous strict inclusions hold

$$\mathcal{S}(G) \subsetneq W^{b,p}(G) \subsetneq W^{a,p}(G) \subsetneq \mathcal{S}'(G)$$

and an equivalent norm for $W^{b,p}(G)$ is

$$W^{b,p}(G) \ni f \longmapsto \|f\|_{W^{a,p}(G)} + \left\| \mathcal{R}_p^{\frac{b-a}{\nu}} f \right\|_{W^{a,p}(G)}$$

(5) For $p \in [1, \infty]$ and any $a, b, c \in \mathbb{R}$ with $a < c < b$, there exists a positive constant $C = C_{a,b,c}$ such that for any $f \in W^{b,p}(G)$, we have $f \in W^{c,p} \cap W^{a,p}$ and

$$\|f\|_{W^{c,p}(G)} \leq C \|f\|_{W^{a,p}(G)}^{1-\theta} \|f\|_{W^{b,p}(G)}^{\theta}$$

where $\theta := (c - a)/(b - a)$.

In the next statement, we show how to produce functions and converging sequences of Sobolev spaces using the convolution:

Proposition 13.19. *Let $a \in \mathbb{R}$ and $p \in [1, \infty]$*

(i) If $f \in L^p(G)$ and $\phi \in \mathcal{S}(G)$, then $f * \phi \in W^{s,p}(G)$ for any s and p.

(ii) If $f \in L^{s,p}_W(G)$ and $\psi \in \mathcal{S}(G)$, then

$$(\mathrm{I} + \mathcal{R}_p)^{\frac{s}{\nu}} (\psi * f) = \psi * \left((\mathrm{I} + \mathcal{R}_p)^{\frac{s}{\nu}} f \right)$$

and $\psi * f \in W^{s,p}(G)$ with

$$\|\psi * f\|_{W^{s,p}(G)} \leq \|\psi\|_{L^1(G)} \|f\|_{W^{s,p}(G)}$$

Furthermore, if $\int \psi = 1$, writing

$$\psi_\epsilon(x) := \epsilon^{-Q} \psi\left(\epsilon^{-1} x\right)$$

for each $\epsilon > 0$, then $\{\psi_\epsilon * f\}$ converges to f in $W^{s,p}(G)$ as $\epsilon \to 0$.

Proposition 13.20 (Hilbert space $H^s(G) = W^{s,2}(G)$). *Let G be a graded Lie group. For any $s \in \mathbb{R}$, the inhomogeneous Sobolev space $H^s(G)$ is a Hilbert space with the inner product given by*

$$(f, g)_{H^s(G)} := \int_G (\mathbf{I} + \mathcal{R}_2)^{\frac{s}{\nu}} f(x) \overline{(\mathbf{I} + \mathcal{R}_2)^{\frac{s}{\nu}} g(x)} dx,$$

and the homogeneous Sobolev space $\dot{H}^s(G)$ is a Hilbert space with the inner product given by

$$(f, g)_{\dot{H}^s(G)} := \int_G \mathcal{R}_2^{\frac{s}{\nu}} f(x) \overline{\mathcal{R}_2^{\frac{s}{\nu}} g(x)} dx.$$

If $s > 0$, an equivalent inner product on $H^s(G)$ is

$$(f, g)_{H^s(G)} := \int_G f(x) \overline{g(x)} dx + \int_G \mathcal{R}_2^{\frac{s}{\nu}} f(x) \overline{\mathcal{R}_2^{\frac{s}{\nu}} g(x)} dx.$$

If $s = \nu \ell$ with $\ell \in \mathbb{N}_0$, an equivalent inner product on $H^s(G)$ is

$$(f, g) = (f, g)_{L^2(G)} + \sum_{[\alpha] = \nu \ell} (X^\alpha f, X^\alpha g)_{L^2(G)},$$

and an equivalent inner product on $\dot{H}^s(G)$ is

$$(f, g) = \sum_{[\alpha] = \nu \ell} (X^\alpha f, X^\alpha g)_{L^2(G)}.$$

13.5 Operators acting on Sobolev spaces

In this section we show that left-invariant differential operators act continuously on homogeneous and inhomogeneous Sobolev spaces.

Theorem 13.21. *Let G be a graded Lie group.*

(1) Let T be a left-invariant differential operator of homogeneous degree ν_T. Then for every $p \in (1, \infty)$ and $s \in \mathbb{R}, T$ maps continuously $L^p_{s+\nu_T}(G)$ to $L^p_s(G)$ Fixing a positive Rockland operator $\mathcal{R}$ in order to define the Sobolev norms, it means that

$$\exists C = C_{s,p,T} > 0 \quad \forall \phi \in \mathcal{S}(G) \quad \|T\phi\|_{L^p_s(G)} \leq C\|\phi\|_{L^p_{s+\nu_T}(G)}.$$

(2) Let T be a ν_T -homogeneous left-invariant differential operator. Then for every $p \in (1, \infty)$ and $s \in \mathbb{R}, T$ maps continuously $\dot{L}^p_{s+\nu_T}(G)$ to $\dot{L}^p_s(G)$. Fixing a positive Rockland operator $\mathcal{R}$ in order to define the Sobolev norms, it means that

$$\exists C = C_{s,p,T} > 0 \quad \forall \phi \in \dot{L}^p_{s+\nu_T}(G) \quad \|T\phi\|_{\dot{L}^p_s(G)} \leq C\|\phi\|_{\dot{L}^p_{s+\nu_T}(G)}.$$

Lemma 13.22. Let $\mathcal{R}$ be a Rockland operator on G of homogeneous degree ν and let $\ell \in \mathbb{N}_0, p \in (1, \infty)$. Then the space $L^p_{\nu\ell}(G)$ is the collection of functions $f \in L^p(G)$ such that $X^\alpha f \in L^p(G)$ for any $\alpha \in \mathbb{N}_0^n$ with $[\alpha] = \nu\ell$. Moreover, the map

$$\phi \mapsto \sum_{[\alpha]=\nu\ell} \|X^\alpha \phi\|_p$$

is a norm on $\dot{L}^p_{\nu\ell}(G)$ which is equivalent to the homogeneous Sobolev norm and the map

$$\phi \mapsto \|\phi\|_p + \sum_{[\alpha]=\nu\ell} \|X^\alpha \phi\|_p$$

is a norm on $L^p_{\nu\ell}(G)$ which is equivalent to the Sobolev norm.

Theorem 13.23. *Let G be a graded Lie group and $p \in (1, \infty)$. The homogeneous L^p Sobolev spaces on G associated with any positive Rockland operators coincide. The inhomogeneous L^p Sobolev spaces on G associated with any positive Rockland operators coincide. Moreover, in the homogeneous and inhomogeneous cases, the Sobolev norms associated to two positive Rockland operators are equivalent.*

In the first part of this talk, index theory for elliptic differential operators on compact manifolds is recalled. The second part explains why filtered manifold are a good framework to study certain hypoelliptic operators. Instead of ellipticity, the Rockland condition for model operators acting on graded nilpotent Lie groups is used. In the last part, index theory for the special case of contact manifolds is discussed and we have a glimpse on how this could be generalized to arbitrary filtered manifolds.

14 Index theory for differential operators on compact manifolds

For a detailed introduction to index theory see for example [HR04]. In the following, let M denote a compact, smooth manifold. Let P be a differential operator of order d acting on $C^\infty(M)$, understood as an unbounded operator on $L^2(M)$ (with respect to some Riemannian metric on M).

We want to solve differential equations of the form $Pu = f$, which is in general hard. The dimension of the cokernel and kernel of P describe how many constraints f has to satisfy in order to have a solution u and how unique this solution is.

Definition 14.1. A closed unbounded operator $P\colon \mathcal{D}(P) \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is *Fredholm* if it has closed range and $\dim KerP < \infty$ and $\dim \operatorname{coker} P < \infty$. In this case, its *Fredholm index* is defined by

$$\operatorname{ind}(P) = \dim KerP - \dim \operatorname{coker} P.$$

The Fredholm index is a useful invariant because it is robust under small perturbations of the operator. We want to find conditions under which a differential operator is Fredholm, one example is ellipticity.

Let $(x_1, \dots, x_n)$ be local coordinates on $U \subset M$. Then P can be written in these coordinates as

$$P = \sum_{|\alpha| \leq d} c_\alpha(x) \partial^\alpha \quad \text{with } c_\alpha \in C^\infty(U).$$

Here, we use multi-index notation: For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ let

$$\begin{aligned} |\alpha| &= \alpha_1 + \alpha_2 + \ldots + \alpha_n, \\ \partial^\alpha &= \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}. \end{aligned}$$

Definition 14.2. The *principal symbol* of P on U is defined to be

$$\sigma(P)(x, \xi) = \sum_{|\alpha| = d} c_\alpha(x)(i\xi)^\alpha \quad \text{for } x \in U \text{ and } \xi \in T_x^* M.$$

One can show that the value $\sigma(P)(x, \xi)$ does not depend on the chosen coordinates, so $\sigma(P)$ is a well-defined function in $C^\infty(T^*M)$.

Definition 14.3. A differential operator P on M is *elliptic* if $\sigma(P)(x, \xi) \neq 0$ for all $x \in M$ and $\xi \neq 0$.

Proposition 14.4. *Let P be an elliptic differential operator on a compact manifold M. Then it is closable and $\overline{P}$ is a Fredholm operator. Moreover, its Fredholm index only depends on the principal symbol.*

Even more is true, its index just depends on a certain K-theory class $[\sigma(P)] \in K_0(C_0(T^*M))$ one builds from the principal symbol (using that it is invertible).

Theorem 14.5 (Atiyah-Singer [AS63]). *The Fredholm index of an elliptic differential operator P on a compact manifold M can be computed as follows*

$$\operatorname{ind}(\overline{P}) = \int_{T^*M} \operatorname{ch}([\sigma(P)]) \cup \operatorname{Td}(TM \otimes \mathbb{C})).$$

Remark 14.1. The above results hold, more generally, also for differential operators $P\colon \Gamma^\infty(E) \to \Gamma^\infty(F)$ that act on the smooth sections of vector bundles E, F over the manifold M.

15 Filtered manifolds

Apart from the elliptic operators, there are more differential operators that are Fredholm.

Proposition 15.1 ([van05]). *Let P be a symmetric hypoelliptic differential operator on a compact manifold M. Suppose that the range of $\overline{P}$ is closed, then $\overline{P}$ is Fredholm.*

Open Question 1. *Given a hypoelliptic differential operator, can we find a (pseudo)-differential calculus in which the operator has an 'invertible principal symbol'? Here, the principal symbol might be no longer a function on T^*M, but an element of a (possibly noncommutative) algebra.*

The answer to this question is positive for some hypoelliptic differential operators that arise in the framework of filtered manifolds. Filtered manifolds were introduced by Melin and recently considered in [vEY17, CP19, SH16].

Definition 15.2. A *filtered manifold* (M, H) is a smooth manifold with a filtration of its tangent bundle $0 = H^0 \subseteq H^1 \subseteq H^2 \subseteq \ldots \subseteq H^r = TM$ consisting of smooth subbundles such that

$$\left[\Gamma^\infty(H^i), \Gamma^\infty(H^j)\right] \subseteq \Gamma^\infty(H^{i+j}) \quad \text{for all } i, j. \tag{37}$$

Here, we set $H^i = TM$ for all $i \geq r$. A manifold is filtered of *step* r, if $H^i \subsetneq TM$ for all $i < r$.

For each point $m \in M$ one can define a graded nilpotent Lie algebra $\mathfrak{g}_m$ by

$$\mathfrak{g}_m := \bigoplus_{i=1}^{r} H^i_m / H^{i-1}_m.$$

The Lie bracket on $\mathfrak{g}_m$ is defined in the following way: For $\langle X_m \rangle \in H^i_m/H^{i-1}_m$ and $\langle Y_m \rangle \in H^j_m/H^{j-1}_m$ extend them to sections $X \in \Gamma^\infty(H^i)$ and $Y \in \Gamma^\infty(H^j)$ and set

$$[\langle X_m \rangle, \langle Y_m \rangle] := \langle [X, Y]_m \rangle \in H^{i+j}_m / H^{i+j-1}_m.$$

This is well-defined by (37) and can be extended bilinearly. The graded nilpotent Lie algebras $\mathfrak{g}_m$ integrate to graded nilpotent Lie groups G_m, these are called the *osculating groups*.

Remark 15.1. For the vector bundle $\mathfrak{t}_H M := \bigoplus_{i=1}^r H^i/H^{i-1}$ the bracket defines a vector bundle morphism $[\,\cdot\,,\,\cdot\,]\colon \mathfrak{t}_H M \otimes \mathfrak{t}_H M \to \mathfrak{t}_H M$, that restricts to a Lie bracket in each fibre. In this sense, we get a bundle of Lie algebras (not in the sense of fibre bundles as the fibres might be non-isomorphic Lie algebras).

Equipping the fibres of the bundle with the Dynkin product, we get group structures on the fibres which vary smoothly along M. Denote this bundle of Lie groups by $T_H M$.

Example 15.2. A filtered manifold of step $r = 1$ is just a smooth manifold. The osculating group at $m \in M$ is $T_m M \cong \mathbb{R}^n$, where $n = \dim M$.

Example 15.3. A filtered manifold of step $r = 2$ is a smooth manifold with a subbundle $H \subseteq TM$. This is called a *Heisenberg structure* for M.

If H has codimension one, we call (M, H) a *Heisenberg manifold*. In this case, all osculating groups G_m are isomorphic to $\mathbb{R}^{l(m)} \times \mathbb{H}_{k(m)}$ for some $l(m), k(m) \in \mathbb{N}_0$ such that $l(m) + 2k(m) + 1 = \dim M$. Here, $\mathbb{H}_k$ denotes the $(2k+1)$-dimensional Heisenberg group. Note that $l(m)$ and $k(m)$ can vary along M, so it can happen that not all osculating groups are isomorphic.

Definition 15.3. A *contact manifold* is a filtered manifold of step 2 and dimension $2k + 1$ such that all osculating groups are isomorphic to the Heisenberg group $\mathbb{H}_k$.

Remark 15.4. Usually, one defines a contact manifold of dimension $2k + 1$ by requiring that there is a codimension 1 bundle $H \subseteq TM$ such that all non-vanishing local 1-forms θ with $\theta(H) = 0$ satisfy that $\theta \wedge (d\theta)^k$ is a volume form. Then θ is called a (local) *contact form*. This definition is equivalent to definition 15.3 by [van05].

Example 15.5. Each graded nilpotent Lie group G of step r can be viewed as a (noncompact) filtered manifold of step r. Let the Lie algebra of G be graded by $\mathfrak{g} = \bigoplus_{i=1}^r \mathfrak{g}_r$ and fix a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ such that $\{X_1, \ldots, X_{\dim \mathfrak{g}_1}\}$ is a basis of $\mathfrak{g}_1$ and $\{X_{\dim \mathfrak{g}_{i-1}+1}, \ldots, X_{\dim \mathfrak{g}_i}\}$ is a basis for $\mathfrak{g}_i$ for all $i > 1$. Viewing them as left-invariant vector fields on G, let H_i be the vector bundle spanned by $\{X_1, \ldots, X_{\dim \mathfrak{g}_i}\}$. This defines a filtration as $\mathfrak{g}$ is graded nilpotent. Moreover, all osculating groups are isomorphic to G.

The filtration on M allows to define a new notion of order for differential operators on M. The basic idea is that a section in $\Gamma^\infty(H^i) \setminus \Gamma^\infty(H^{i-1})$ has order i when viewed as a differential operator on M.

More concretely, let $X_1, \ldots, X_n \colon U \to TM$ let be an *H-frame* on a coordinate chart $U \subset M$, which means that $\{X_1, \ldots, X_{\dim H^i}\}$ is a frame for H^i for all $1 \leq i \leq r$. We use again multi-index notation for differential operators but define the *graded length* for a multiindex $\alpha \in \mathbb{N}_0^n$:

$$[\alpha] = \alpha_1 \cdot 1 + \ldots + \alpha_{\dim H^1} \cdot 1 + \alpha_{\dim H^1 + 1} \cdot 2 + \cdots + \alpha_{\dim H^2} \cdot 2 + \ldots + \alpha_n \cdot r.$$

We define the *graded order* of $X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$ to be $[\alpha]$. Each differential operator P can be written locally as

$$P = \sum_{[\alpha] \leq d} c_\alpha(x) X^\alpha.$$

Definition 15.4. The model operator of P at $m \in U$ is defined to be

$$P_m := \sum_{[\alpha]=d} c_\alpha(m) \langle X(m) \rangle^\alpha$$

viewed as a left invariant differential operator on the osculating group G_m.

One can show that P_m does not depend on the chosen H-frame. In the following, we will replace ellipticity by H-ellipticity:

Definition 15.5. A differential operator P on a filtered manifold (M, H) is H*-elliptic* if all P_m are hypoelliptic operators on G_m.

By the last talk, a differential operator P is H-elliptic if and only if all model operators P_m satisfy the Rockland condition as operators on the graded nilpotent Lie groups G_m.

Example 15.6. To see how this relates to usual ellipticity, recall that for a usual manifold ($r = 1$) of dimension n, the osculating groups are all isomorphic to $\mathbb{R}^n$. The model operators are the constant coefficient operators

$$P_m = \sum_{|\alpha|=d} c_\alpha(m) \partial^\alpha.$$

Every unitary irreducible representation of $\mathbb{R}^n$ is unitarily equivalent to a one-dimensional representation π_ξ given by $\pi_\xi(x) = e^{i\langle \xi, x\rangle}$ for some $\xi \in \mathbb{R}^n$. One computes that $d\pi_\xi(\frac{\partial}{\partial x_j}) = i\xi_j$. Therefore, $d\pi_\xi(P_m) = \sigma(P)(m, \xi)$ holds. Hence, H-ellipticity is, in this case, the same as ellipticity.

In the following, we consider two differential operators which are not elliptic but H-elliptic with respect to certain step 2 filtrations. In both cases, a term which would be of lower order in the usual differential calculus is crucial to make the operator H-elliptic.

Example 15.7. Consider the heat operator on $\mathbb{R}^n \times \mathbb{R}$

$$P = -\Delta_x + \frac{\partial}{\partial t} = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \frac{\partial}{\partial t}.$$

By considering its principal symbol, we see that the operator is not elliptic in the ordinary calculus. Consider the step 2 filtration $0 \subseteq T\mathbb{R}^n \subseteq T(\mathbb{R}^n \times \mathbb{R})$. The osculating groups are still isomorphic to $\mathbb{R}^n \times \mathbb{R}$, but the order of $\frac{\partial}{\partial t}$ is now also 2. As $d\pi_{(\xi,\eta)}(P_{(x,t)}) = |\xi|^2 + i\eta$ is only zero for $(\xi, \eta) = (0, 0)$ for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, it follows that P is H-elliptic with respect to the filtration above.

Example 15.8 ([EPS04]). Let $\{X_1, \dots, X_k, Y_1, \dots, Y_k, Z\}$ denote the usual basis of the Heisenberg group $\mathbb{H}_k$. Consider the operator

$$P = \sum_{j=1}^k -X_j^2 - Y_j^2 + i\mu Z \quad \text{with } \mu \in \mathbb{C}.$$

All osculating groups are isomorphic to $\mathbb{H}_k$. Recall that by Kirillov's orbit method there is a bijection between $\widehat{\mathbb{H}}_k$ and the coadjoint orbit space $\mathfrak{h}_k^*/\mathbb{H}_k$. Writing an element of $\mathfrak{h}_k^*$ as $cZ^* + \sum_{j=1}^k a_j X_j^* + b_j Y_j^*$ with $a_j, b_j, c \in \mathbb{R}$, there are two different types of orbits:

1. $c = 0$. In this case the orbit just consists of a single point and the polarizing subalgebra is $\mathfrak{h}_k$. The corresponding representation of $\mathbb{H}_k$ is one-dimensional and given by

$$\chi_{(a,b)}(x, y, z) = e^{i(\langle a,x\rangle + \langle b,y\rangle)} \quad \text{for } (x, y, z) \in \mathbb{H}_k.$$

One computes $d\chi_{(a,b)}(X_j) = a_j$, $d\chi_{(a,b)}(Y_j) = b_j$ and $d\chi_{(a,b)}(Z) = 0$. Hence, $d\chi_{(a,b)}(P_m) = |a|^2 + |b|^2 \neq 0$ for all $m \in \mathbb{H}_k$ and $(a, b) \neq (0, 0)$. Note that $(a, b) = (0, 0)$ corresponds to the trivial representation.

2. $c \neq 0$. Here, the orbits under the coadjoint action are $2k$-dimensional and of the form

$$\left\{ cZ^* + \sum_{j=1}^{k} a_j X_j^* + b_j Y_j^* \mid (a, b) \in \mathbb{R}^{2k} \right\}.$$

A polarizing subalgebra is given by $\mathbb{R}Z \oplus \mathbb{R}X_1 \oplus \ldots \mathbb{R}X_k$. One can show that the resulting infinite-dimensional representation is unitarily equivalent to the representation $\pi_c \colon \mathbb{H}_k \to L^2(\mathbb{R}^k)$ given by

$$\pi_c(x, y, z)h(u) = e^{ic(z+\frac{1}{2}\langle x,y\rangle)} e^{\pm i\sqrt{|c|}\langle y,u\rangle} h(u + \sqrt{|c|}x)$$

for $(x, y, z) \in \mathbb{H}_k$, $h \in L^2(\mathbb{R}^k)$ and $u \in \mathbb{R}^k$. Here, the $\pm$ corresponds to the sign of c. Computing the infinitesimal representations yields

$$\begin{aligned} d\pi_c(X_j) &= \sqrt{|c|}\frac{\partial}{\partial u_j}, \\ d\pi_c(Y_j) &= \pm\sqrt{|c|} i u_j \\ d\pi_c(Z) &= ic. \end{aligned}$$

The Rockland condition is satisfied if and only if

$$d\pi_\gamma(P_m) = c(-\Delta + |u|^2 \pm \mu I)$$

is injective on $\mathcal{S}(\mathbb{R}^k)$. The operator $-\Delta + |u|^2$ on $\mathbb{R}^k$ is the quantum harmonic oscillator. It has pure point spectrum and its spectrum is given by $\{k + 2l \mid l \in \mathbb{N}_0\}$. The corresponding eigenfunctions are Schwartz. Therefore, the operator is injective if and only if $\mu \notin \{k + 2l \mid l \in \mathbb{N}_0\} \cup \{-k - 2l \mid l \in \mathbb{N}_0\}$.

In summary, the operator P is H-elliptic if and only if μ is not contained in this set.

16 Index theory for contact manifolds

Rockland operators on graded nilpotent Lie groups satisfy certain a priori estimates. For example, on the Heisenberg group $\mathbb{H}_k$ for the Sublaplacian $\mathcal{L} = \sum_{j=1}^{k} -X_j^2 - Y_j^2$ and a left-invariant operator A of Heisenberg order ≤ 2, there is a $C > 0$ such that

$$\|Au\| \leq C(\|\mathcal{L}u\| + \|u\|) \quad \text{for all } u \in \mathcal{S}(G). \tag{38}$$

On a contact manifold M there are certain coordinates called *Darboux coordinates* that identify M locally with a open subset of the Heisenberg group. These allow to derive similar estimates for H-elliptic operators on M.

For a compact contact manifold M of dimension $2k + 1$, let $\{\varphi \colon U_i \to \mathbb{H}_k\}_{i=1}^{n}$ be an atlas of Darboux coordinates. Using a subordinate partition of unity $\{\chi_i\}$, one can define Sobolev spaces H^s on M by setting

$$\|u\|_{H^s}^2 := \sum_{i=1}^{n} \|(\chi_i \cdot u) \circ \varphi_i^{-1}\|_{H^s}^2.$$

Here, we use the Sobolev norm on $\mathbb{H}_k$ with respect to the Sublaplacian. Using the Darboux coordinates and the estimates for the model operators from (38) one can show

Proposition 16.1 ([van10]). *Let P be an H-elliptic operator P on a compact contact manifold M. For $s \in \mathbb{R}$, there is a $C > 0$ such that*

$$\|u\|_{H^s} \leq C(\|Pu\|_2 + \|u\|_2) \quad \textit{for all } u \in C^\infty(M).$$

Using these estimates one can show:

Theorem 16.2 ([van10]). *Let P be an H-elliptic operator P on a compact contact manifold M. Then P is hypoelliptic. Moreover, P is closable and $\overline{P}$ is Fredholm.*

Furthermore, one can show that the index just depends on the model operators.

Remark 16.1. The same is also true for H-elliptic operators on arbitrary compact filtered manifolds. This follows from the pseudo-differential calculus developed by van Erp and Yuncken in [vEY17].

Remark 16.2. In the contact case, one can show again that the index just depends on a K-theory class in $K_0(\mathrm{C}^*(T_H M))$ which one can build from the model operators. By iterating the Connes-Thom isomorphism in K-theory, one obtains an isomorphism $\psi\colon K_0(\mathrm{C}^*(T_H M))) \to K_0(C_0(T^*M))$. Using the inverse isomorphism, van Erp proves in [van10] an Atiyah-Singer index theorem for contact manifolds. In [BvE14] Baum and van Erp give a different, more explicit description of the cohomological form of the index.

A Computations in Heisenberg groups

The following is a simplified summary of the computations performed in [CG90, Example 1.1.2, Example 1.2.4, Example 1.3.9, Example 2.2.6, Example 4.3.11], with the rest of examples can be found in the sequel of these examples.

In this section we list the necessary results of computations of $\mathbb{H}^n(\mathbb{R})$ for $n = 1$, i.e., the simplest case of Heisenberg group with Lie algebra $\mathfrak{h}_n$ of the following form:

$$\mathfrak{h}_1 := \left\{ \begin{pmatrix} 0 & x & z \\ & 0 & y \\ & & 0 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\} \longleftrightarrow \mathbb{H}^1 := \left\{ \begin{pmatrix} 0 & x & z + \frac{1}{2}x \cdot y \\ & 0 & y \\ & & 0 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\}$$

Now fixes the element $W = (a, b, c) = aX + bY + cZ$ and $w = (x, y, z)$ corresponding to the following elements in $\mathfrak{h}_1$ and $\mathbb{H}^1$ respectively:

$$\begin{pmatrix} 0 & a & c \\ & 0 & b \\ & & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}$$

then the adjoint actions gives $\operatorname{Ad}_w W = \begin{pmatrix} 0 & a & c + x \cdot b - y \cdot a \\ & 0 & b \\ & & 0 \end{pmatrix}$.

Now for X, Y, Z we fix dualized basis X^*, Y^*, Z^* with $l = (\alpha, \beta, \gamma) = \alpha X^* + \beta Y^* + \gamma Z^* \in \mathfrak{g}$, then for $w = (x, y, z)$:

$$\operatorname{Ad}^*_w : (\alpha, \beta, \gamma) \mapsto (\alpha + \gamma \cdot y, \beta - \gamma \cdot x, \gamma)$$

Hence we have two types of orbits:

1. $\gamma = 0$**:** In this case $\operatorname{Ad}^*(G) \cdot l = l$ and we have 0-dimensional orbits;
2. $\gamma \neq 0$**:** In this case we have the G-orbits are $2n$-dimensional affine planes:

$$\operatorname{Ad}^*(G) \cdot (\alpha, \beta, \gamma) = \left\{ (\alpha', \beta', \gamma) \,\middle|\, \alpha', \beta' \in \mathbb{R}^1 \right\} \quad \text{for } (\alpha, \beta, \gamma) \in \mathfrak{g}^*$$

Writing 2. as $\gamma Z^* + Z^\perp := \gamma Z^* + \{l \in \mathfrak{g}^* \mid l(Z) = 0\}$ we can parametrize the polarizing subalgebra as the following cases respectively:

1. $\mathfrak{r}_l = \mathfrak{g}$ the only radical;
2. $\mathfrak{r}_l = \mathbb{R}Z$ are the radical for $l \in Z^\perp$ and $\mathfrak{m}$ are $2n$-dimensional subalgebras of the form $\mathbb{R}Z \oplus \mathbb{R}Y$ or $\mathbb{R}Z \oplus \mathbb{R}X$.

Now to parametrize $\widehat{G}$, we need some auxiliary lemma to separate the measure:

Theorem A.1 ([CG90, Theorem 1.2.12])**.** *Let $\mathfrak{h}$ be a k-dimensional subalgebra of a nilpotent algebra $\mathfrak{g}$, with respective Lie group $H \subseteq G$. Choose a weak Malcev basis $\{X_1, \cdots, X_n\}$ for $\mathfrak{g}$ through $\mathfrak{h}$. Then the following ϕ defines a analytic diffeomorphism:*

$$\phi : \mathbb{R}^{n-k} \to H\backslash G \qquad (t_1, \cdots, t_{n-k}) \mapsto H \cdots \exp(t_1 X_{k+1}) \cdots \exp(t_{n-k} X_n)$$

which takes the Lebesgue measure on $\mathbb{R}^{n-k}$ to a G-invariant measure on $H\backslash G$.

Remark A.1. This again use the fact that the Campbell-Baker-Hausdorff formula is a polynomial diffeomorphism with a polynomial inverse, which only applies when there is choice of Malcev basis.

Now $\widehat{G}$ (as a set) can be fully described based on γ respectively:

1. For $l \in Z^\perp$, as $\mathfrak{m} = \mathfrak{g}$ the only polarizing subalgebra for l, hence the inductions are trivial and:
$$\pi_l = \chi_l \qquad \text{where } \chi_l(\exp W) = e^{2\pi i l(W)} \text{ for } W \in \mathfrak{g}$$
There are one-dimensional representations which corresponding to $G_{\text{ab}} := G/[G,G] = G/G^{(1)}$. Note this corresponds to the Pontryagin Duality of all the characters;
2. For $l = \lambda Z^* + Z^\perp$ with $\lambda \neq 0$. Since $\pi_{l,M}$ are independent of choice of l and M within the same orbit we can choose for convenience the representative $l = \lambda Z^*$ with $\mathfrak{m} = \mathbb{R}Z \oplus \mathbb{R}Y$. On such, the characters are of the form: $\chi_l(\exp(zZ + yY)) = e^{2\pi i \lambda z}$. To describe the behaviour of $\pi_l := \operatorname{ind}_M^G(\chi_l)$, we describe the action of π_l on $L^2(M\backslash G, \mathbb{C})$. Take $S = \exp(\mathbb{R}X)$. Our choice of basis is clearly a Strong Malcev basis, so by Theorem A.1 we have the basis dt on $\mathbb{R}$ transfers to a right invariant measure on S. Hence the restriction map $f \mapsto f|_S$ gives an isometry between H_{π_l} and $L^2(\mathbb{R}^n) = L^2(\mathbb{R})$. So now it suffices to describe the action of $(x, y, z) \in \mathfrak{g}$ on $f(t, 0, 0)$ with $(x, y, z) := \exp(xX + yY + zZ) \in G$. The key is to use CBH-formula to write $S \cdot G$ to the form of $M \cdot S$:
$$(t, 0, 0) \cdot (x, y, z) = \left(0, y, z + t \cdot y + \frac{1}{2} x \cdot y\right) \cdot (t + x, 0, 0)$$
Hence the action of $(x, y, z) \in G$ on $f|_S \in L^2(\mathbb{R})$ is via:
$$\pi_l(x, y, z) f(0, 0, t) = \exp^{2\pi i \lambda (z + t \cdot y + \frac{1}{2} x \cdot y)} f(0, 0, t + x)$$
for non-zero λ. And one can see the infinite-dimensional representations are precisely those those on which the center $\exp(\mathbb{R}Z)$ acts nontrivially. This embodies the proof of general cases.

Now to parametrize $\widehat{\mathbb{H}^1}$ we see S in Lemma 5.3 (and its following theorems) can be chosen to be $S = \{2, 3, \ldots, 2n+1\}$ since every nontrivial orbit has dimension $2n$ and $V = \mathfrak{g}^*$ has dimension $2n + 1$. Hence U, V_S, V_T in this case admits a particular nice form:

$$V_S = \mathbb{R}Y^* \oplus \mathbb{R}X^* \qquad U = \{l \in \mathfrak{g}^* \mid l(Z) \neq 0\}$$
$$V_T = \mathbb{R}Z^* \qquad U \cap V_T = \mathbb{R}^\times Z^*$$

The adjoint action above is naturally expressed in the polynomial form, hence we have, for $u \in U$ parametrized by

$$\begin{cases} -x_j \gamma & \text{if } 1 \leq j \leq n \\ y_{j-n} \gamma & \text{if } n+1 \leq j \leq 2n \end{cases}$$

Hence when $n = 1$, the local trivialization $\varphi : (U \cap V_T) \times V_S \to U$ is given by:

$$\phi(\gamma Z^*, u) = \gamma Z^* + u_1 Y^* + u_2 X^*$$

Having the orbits parametrized, we see the symplectic form B appears in **Plancherel inversion theorem** associated with $l \in \mathfrak{g}^*$ admits the form:

$$\begin{pmatrix} 0 & l(Z) \cdot \mathrm{id}_n \\ -l(Z) \cdot \mathrm{id}_n & 0 \end{pmatrix} \in M_{2n}(\mathbb{R})$$

whose determinant is $l(Z)^{2n}$ and hence $\mathrm{Pf}(l) = l(Z)^n$. Hence the Fourier inversion theorem reads:

$$f(e) = \int_{\mathbb{R}} \mathrm{tr}\, \pi_{\lambda Z^*}(f) |\lambda|^n d\lambda$$

with $|\lambda|^n d\lambda$ the Plancherel measure, under the identification $V_T = \{\lambda Z^* : \lambda \in \mathbb{R}\} \cong \mathbb{R}$.

To end this computation, we are satisfied with laying down the kernel function $K_{l,\phi}(t,t) \in \mathcal{S}(\mathbb{R} \times \mathbb{R})$ for fixed $\phi \in \mathcal{S}(G)$ and $l = \lambda Z^*$ for $\lambda \neq 0$. Then the parametrization α, β, γ appears in Equation (Kernel of Trace) are:

$$\begin{aligned} \gamma(zZ, yY, xX) &= \exp(zZ)\exp(yY)\exp(xX) \\ \alpha(zZ, yY) &= \exp(zZ)\exp(yY) \\ \beta(xX) &= \exp(xX) \end{aligned}$$

Hence the integrand by CBH-formula gives:

$$\beta(xX)^{-1} \exp(yY + zZ)\beta(xX) = (-x, 0, 0) * (0, y, z) * (x, 0, 0) = (0, y, z - yx)$$

Now $K_{\phi,l}(x,x) \in \mathcal{S}(\mathbb{R}\times\mathbb{R})$ in Theorem 6.2 by identifying $dH = dYdZ$, $du = dx$ and $l(zZ+yY) = \lambda z$, that:

$$K_{\phi,l}(x,x) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{2\pi i \lambda z} \phi \begin{pmatrix} 1 & 0 & z - xy \\ & 1 & y \\ & & 1 \end{pmatrix} dydz$$

whereas the general form we used the 'straightened' form:

$$K_{\phi,l}^{(x_1,y_1,z_1)}(t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(t - x, y, z) e^{2\pi\lambda i(z - z_1 + x_1 y - x y_1)} dydz$$

where this kernel corresponding the trace corresponding to the function f evaluated at $\begin{pmatrix} 1 & x_1 & z_1 \\ & 1 & y_1 \\ & & 1 \end{pmatrix}$. Note here the exponential coefficients encodes the right action of G on the induced representation.

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