What is Hypoellipticity?

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I meet the word "Hypoellipticity" when I was the first year PhD student at University of Göttingen. Before, I only know elliptic, hyperbolic and parabolic operators and associated equations. So I wrote this note to clarify them. Some materials are from [1, 3].

We first recall the two top classical partial differential equations (PDEs):

- Laplace's equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$
- Wave equation
$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

In fact, they all come from the usual differential operator are of the form

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha.$$

Here $D^\alpha = \partial^\alpha$ denotes a general mixed partial derivative depending on α. For Laplace's equation, the degree $m = 2$, the relevant α are $(2, 0)$ and $(0, 2)$, and the associated coefficients $a_\alpha(x)$ are constant 1 and 1. For the wave equation, m, α are the same as the Laplacian, but the associated coefficients are 1 and -1.

As we have known, although the above two equations look similar, their solutions have different smoothness properties. Indeed, the solution to Laplace's equation is called harmonic function, are infinitely differentiable, while general solutions to the wave equation need not be smooth or even continuous. The only reason is the different sign for coefficients.

In the theory of partial differential equations, elliptic operators are differential operators that generalize the Laplace operator. They are defined by the condition that the coefficients of the highest-order derivatives satisfy,

$$\sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha \neq 0$$

for every x and every non-zero ξ in $\mathbb{R}^n$, which implies the key property that the principal symbol is invertible, or equivalently that there are no real characteristic directions. What's more, when we consider the elliptic equation:

$$Pu = f$$

in the sense of distribution. A basic problem is : what can we say about solution u for some given f? We often resorts to understanding what sort of properties u inherits from f. For example, if f is smooth, does it follow that u is smooth? As we know, the positive coefficients can guarantee the smoothness by elliptic regularity theorem. However, ellipticity is sufficient but not necessary for smoothness. For example, the solution to the heat equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$$

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is smooth even though the equation is not elliptic. Just now, there is no known nice condition that both necessary and sufficient for smoothness. Therefore, we need the definition of hypoelliptic, which can be seen as the weaker versions of ellipticity.

Definition 0.1 *If all solutions are smooth, the PDE is called hypoelliptic.*

Definition 0.2 *A partial differential operator P is hypoelliptic if on every open subset: if Pu is smooth, then u is smooth in the sense of distribution.*

Example 0.1 *On $\mathbb{R}$, $P = \frac{d}{dx}$ is hypoelliptic: if the derivative is smooth, then the primitive is also smooth. This follows from the fundamental theorem of calculus: $u(x) = \int_0^x Pu(t)dt + C$.*

Example 0.2 *On $\mathbb{R}^2$ with coordinates (x, y), $P = \frac{\partial}{dx}$ is not hypoelliptic. In fact, $Pu = 0$ for any u that depends only on y, and thus u need not to be smooth.*

We can formulate a more quantitative version of hypoellipticity using Sobolev spaces. For $p \in [1, \infty]$, $s \in \mathbb{R}$, let $W^{s,p}$ denote the Sobolev space of order s on $\mathbb{R}^n$. A simple consequence of the Sobolev embedding theorem is the following lemma.

Lemma 0.1 *Suppose $\forall s \in R$, $\exists r(s) \in R$ such that*

$$Pu \in H^{r(s)} \Rightarrow u \in H^s \Longleftrightarrow \|u\|_{H^s} \leq C\|Pu\|_{H^{r(s)}}.$$

Then P is hypoelliptic.

In fact, when P is a differential elliptic operator of order m, we usually have that

$$Pu \in W^{s-m,p} \Longrightarrow u \in W^{s,p}.$$

Thus u is smoother than Pu by m derivatives. This is the best result could possibly hope for just now. However, hypoellipticity becomes more subtle when P is not elliptic.

Definition 0.3 *We say P is subelliptic if $\exists \varepsilon > 0$ such that the conditions of Lemma 0.1 hold with $r(s) = s - \varepsilon$:*

$$\|u\|_{H^{s-\varepsilon}} \leq C\|Pu\|_{H^{r(s)}}.$$

In this case, u is smoother than Pu by ε derivatives, which can be much weaker than ellipticity ($\varepsilon = m$).

Let $X_1, \cdots, X_j$ be smooth vector fields on $\mathbb{R}^n$. Define $L = X_1^* X_1 + \cdots + X_j^* X_j$, where X_j^* denotes the formal L^2 adjoint of X_j. (When $X_j = \frac{\partial}{\partial x_j}, j = 1, \cdots, n$, then $L = \Delta$.)

Example 0.3 *If the vector fields $X_1, \cdots, X_j$ are tangent to a lower dimensional submanifold of $\mathbb{R}^n$, then L is not hypoelliptic. This generalized Example 0.2.*

Hörmander dealt with the opposite situdation.

Definition 0.4 *We say $X_1, \cdots, X_j$ satisfy Hörmander's condition if the Lie algebra generated by $X_1, \cdots, X_j$ spans the tangent space.*

Hörmander proved that if $X_1, \cdots, X_j$ satisfy this Hörmander's condition, then L is subelliptic operator, which is called the Hörmander sub-Laplacian.

Example 0.4 *On $\mathbb{R}^2$, let $X_1 = \frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial y}$. Then $[X_1, X_2] = \frac{\partial}{\partial y}$ so that X_1, X_2 satisfy Hörmander's condition at $\mathbb{R}^2$. Thus $L = X_1^* X_1 + X_2^* X_2 = -\partial_x^2 - x^2\partial_y^2$ is subelliptic at $\mathbb{R}^2$. L is not elliptic.*

In fact, Hörmander's sub-Laplacian is one of the most fundamental examples of a far-reaching generalization of ellipticity, known as maximal hypoellipticity. Let

$$Q(y) = \sum_{|\alpha| \leq m} b_\alpha(x) y^\alpha$$

be a polynomial of degree m in noncommuting indeterminates $y_1, \cdots, y_j$ with smooth coefficients $b_\alpha \in C^\infty(\mathbb{R}^n)$: here we have used ordered multi-index notation to deal with the noncommuting indeterminates. If $X_1, \cdots, X_j$ are vector fields, then it makes sense to consider $P = Q(X_1, \cdots, X_j)$ as a partial differential operator of degree at most m.

Definition 0.5 *Suppose $X_1, \cdots, X_j$ satisfy Hörmander's condition. We say $P = Q(X_1, \cdots, X_j)$ is maximal hypoellipticity if*

$$Pu \in L^2 \Longrightarrow X^\alpha u \in L^2$$

for all ordered multi-indicies α with $|\alpha| \leq m$.

A general method of a priori estimates developed by Kohn can be used to show that maximally hypoelliptic operators are subelliptic. Beyond subellipticity, Kohns method does not give a complete understanding of maximal hypoellipticity, and unlike the case of elliptic operators, the Fourier transform is not a decisive tool. Finally, we mention a delicate phenomenon which is far from being understood: hypoelliptic operators that are not subelliptic. For example, Kohn [2] found the complex analog of Hörmander sub-Laplacians might be subelliptic, might be hypoelliptic but not subelliptic, or might not be hypoelliptic at all. Besides these and some other intriguing results, many aspects of hypoellipticity without subellipticity remain uncharted territory.

In conclusion, we have:

$$\textit{Ellipticity} \Longrightarrow Maximal \;\; \textit{Hypoellipticity} \Longrightarrow \textit{Subellipticity} \Longrightarrow Hypoellipticity,$$

and none of the reverse implications hold.

References

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[3] Brian Street. What else about ... hypoellipticity? *Notices Amer. Math. Soc.*, 65(4):424–425, 2018.