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Three solutions for a fractional Schrödinger equation with vanishing potentials☆

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ABSTRACT

In this paper, we study the following fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = K(x)f(u) + \lambda W(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

where $\lambda > 0$ is a parameter, $(-\Delta)^s$ denotes the fractional Laplacian of order $s \in (0,1)$, $N > 2s$, $W \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$, $1 < p < 2$, V, K are nonnegative continuous functions and f is a continuous function with a quasicritical growth. Under some mild assumptions, we prove that the above equation has three solutions.

1. Introduction and the main results

This paper is concerned with the following fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = K(x)f(u) + \lambda W(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $(-\Delta)^s$ denotes the fractional Laplacian of order $s \in (0,1)$, $N > 2s$, $\lambda > 0$, $V, K, f \in C(\mathbb{R}^N, \mathbb{R})$, $W \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$ and $1 < p < 2$.

In the last few years, the study of elliptic equation involving fractional Laplace operator appears widely in optimization, finance, phase transitions, stratified materials, crystal dislocation, flame propagation, conservation laws, materials science and water waves (see [1]). A basic motivation for the study of Eq. (1.1) arises in looking for the standing wave solutions of the type $\Psi(x,t) = e^{-iEt/\varepsilon}u(x)$ for the following time-dependent fractional Schrödinger equation

$$i\varepsilon\frac{\partial\Psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s\Psi + (V(x)+E)\Psi - f(x,\Psi) \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{1.2}$$

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Eq. (1.2) was introduced by Laskin [2,3], which describes how the wave function of a physical system evolves over time. Over the past decades, problem (1.1) and problems similar as (1.1) have captured a lot of interest, many authors have shown their interest in elliptic equation and system both in bounded domains and unbounded domains, see [4–8].

Most of those results need to assume that the potential V admits a positive bounded from below. However, we point out that when $s = 1$, Ambrosetti, Felli and Malchiodi in [9] considered the zero mass case (i.e. $\lim_{|x|\to\infty} V(x) = 0$) for the problem

$$-\Delta u + V(x)u = K(x)|u|^p \ (1 < p < 2^* - 1),$$

where $V, K : \mathbb{R}^N \to \mathbb{R}$ are smooth functions and there exist $a_1, a_2, a_3, A, k_1 > 0$ such that

$$\frac{a_3}{1 + |x|^{a_1}} \leq V(x) \leq A \ \text{ and } \ 0 < K(x) \leq \frac{k_1}{1 + |x|^{a_2}}, \quad \forall\, x \in \mathbb{R}^N.$$

Later, in [10], Alves and Souto consider a more general condition on V and K, from which the working space can be embedded into the weighted space. Using the idea in [10], authors in [11] and [12] obtain a positive solution for a class of critical fractional Schrödinger equation, respectively.

Inspired by the above papers, in the present paper, we shall study the fractional problem with mixed nonlinearity and the potential V vanishing at infinity. To the best of our knowledge, few works concerning on this case up to now. To state our main results, we introduce the notion: we say that $(V, K) \in \mathcal{K}$ if the following conditions hold:

(V) $K \in L^\infty(\mathbb{R}^N)$, $V(x), K(x) > 0$, $\forall x \in \mathbb{R}^N$ and $\lim_{|x|\to+\infty} V(x) = 0$ (shortly $V(\infty) = 0$).

(K_1) If $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borel sets such that $|A_n| \leq R$ for all n and some $R > 0$, then

$$\lim_{r\to+\infty} \int_{A_n \cap B_r^c(0)} K(x)dx = 0, \quad \text{uniformly in } n \in \mathbb{N}.$$

Moreover, one of the following conditions occurs:

(K_2) $\frac{K}{V} \in L^\infty(\mathbb{R}^N)$.

(K_3) there is $\sigma \in (2, 2_s^*)$ such that

$$\lim_{|x|\to+\infty} \frac{K(x)}{V(x)^{\frac{2_s^*-\sigma}{2_s^*-2}}} = 0, \ \text{ where } 2_s^* = \frac{2N}{N - 2s}.$$

The hypotheses on functions $V(x)$ and $K(x)$ were firstly introduced in [10]. Moreover, for the function f, we assume the following conditions:

(f_1) $\lim_{|t|\to0} \frac{f(t)}{|t|} = 0$ if (K_2) holds, or $\lim_{|t|\to0} \frac{f(t)}{|t|^{\sigma-1}} < +\infty$ if (K_3) holds.

(f_2) f has a quasicritical growth, that is, $\lim_{|t|\to+\infty} \frac{f(t)}{|t|^{2_s^*-1}} = 0$.

(f_3) there exists $\theta > 2$ such that $0 \leq \theta F(t) \leq tf(t)$ for all $t \in \mathbb{R}$, where $F(u) = \int_0^u f(t)dt$.

Our main result is the following:

Theorem 1.1. *Assume that $(V, K) \in \mathcal{K}$, $(f_1) - (f_3)$ are satisfied and $W \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$ $(1 < p < 2)$. Then there exists a positive constant λ_0 such that for every $0 < \lambda < \lambda_0$, problem (1.1) has at least three solutions.*

Remark 1.1. From our condition $(f_1) - (f_3)$, it is easy to see that 0 is a trivial solution of problem (1.1), therefore we will prove that problem (1.1) has at least two nontrivial solutions, which is different from the results in [11] and [12].

Throughout this paper, we denote $\|\cdot\|_r$ the usual norm of the space $L^r(\mathbb{R}^N)$, $1 \leq r < \infty$, $B_r(x)$ denotes the open ball with center at x and radius r, C or $C_i (i = 1, 2, \ldots)$ denote some positive constants may change from line to line.

2. Preliminary results and proof of Theorem 1.1

In the sequel, we always assume that the hypotheses of Theorem 1.1 are satisfied. A complete introduction to fractional Sobolev space $H^s(\mathbb{R}^N)$ can be found in [13], we only recall that the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for any $r \in [2, 2_s^*]$ and is locally compact whenever $r \in [2, 2_s^*)$. We introduce the subspace

$$E = \left\{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty\right\},$$

which is a Hilbert space equipped with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \int_{\mathbb{R}^N} V(x)u^2 dx.$$

Denote by $L_K^r(\mathbb{R}^N)$ the weighted Lebesgue space

$$L_K^r(\mathbb{R}^N) = \{u \in H^s(\mathbb{R}^N) : u \text{ is measurable and } \int_{\mathbb{R}^N} K(x)|u|^r dx < +\infty\}$$

and owed with the norm

$$\|u\|_{L_K^r(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} K(x)|u|^r dx\Big)^{\frac{1}{r}}.$$

The energy functional associated with (1.1) is

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} K(x)F(u)dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} W(x)|u|^p dx.$$

It is easy to prove that I_λ is well defined on E and $I_\lambda \in C^1(E, \mathbb{R})$.

In a standard way (see e.g. [14]), one can check that the functional I_λ satisfies the Mountain Pass geometry.

Lemma 2.1. *The functional I_λ satisfies the following conditions:*

(i) *There exist $\beta, \rho > 0$ such that $I_\varepsilon(u) \geq \beta$ for $\|u\| = \rho$ and $\lambda \in (0, \lambda_0)$;*
(ii) *There exists an $e \in E$ satisfying $\|e\| > \rho$ such that $I_\lambda(e) < 0$.*

As a consequence of Mountain Pass Theorem [15] and Lemma 2.1, there exists a (PS) sequence $\{u_n\}$ at the Mountain Pass level c, i.e.,

$$I_\lambda(u_n) \to c \text{ and } I'_\lambda(u_n) \to 0. \tag{2.1}$$

Lemma 2.2. *Let $\{u_n\}$ be a (PS) sequence $\{u_n\}$ of I_λ. Then $\{u_n\}$ is bounded in E.*

Proof. As in [12, Lemma 2.2], we can get

$$\begin{aligned} 1 + c + \|u_n\| &\leq I_\lambda(u_n) - \frac{1}{\theta}\langle I'_\lambda(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 + \int_{\mathbb{R}^N} K(x)\Big(\frac{1}{\theta}u_n f(u_n) - F(u_n)\Big)dx + \left(\frac{1}{\theta} - \frac{1}{p}\right)\lambda\int_{\mathbb{R}^N} W(x)|u_n|^p dx. \end{aligned}$$

Since

$$\left(\frac{1}{p}-\frac{1}{\theta}\right)\lambda\int_{\mathbb{R}^N}W(x)|u_n|^p dx \le \left(\frac{1}{p}-\frac{1}{\theta}\right)\lambda\Big(\int_{\mathbb{R}^N}|W(x)|^{\frac{2}{2-p}}dx\Big)^{\frac{2-p}{2}}\Big(\int_{\mathbb{R}^N}|u_n|^2dx\Big)^{\frac{p}{2}}$$
$$\le \left(\frac{1}{p}-\frac{1}{\theta}\right)C\|W\|_{\frac{2}{2-p}}\|u_n\|^p.$$

Hence,

$$1+c+\|u_n\|+\left(\frac{1}{p}-\frac{1}{\theta}\right)\lambda\int_{\mathbb{R}^N}W(x)|u_n|^pdx \ge \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2,$$

which implies that $\{u_n\}$ is bounded in E. ■

To prove the main result, we need to establish the following compactness result.

Lemma 2.3. *Any (PS) sequence $\{u_n\}$ of I_λ has a convergent subsequence.*

Proof. By Lemma 2.2, $\{u_n\}$ is bounded in E. Up to a subsequence, we may assume that $u_n \rightharpoonup u$ in E. From $\langle I'_\lambda(u_n), u_n\rangle = o_n(1)$, we have

$$\lim_{n\to+\infty}\|u_n\|^2 = \lim_{n\to+\infty}\Big(\int_{\mathbb{R}^N}K(x)f(u_n)u_ndx+\lambda\int_{\mathbb{R}^N}W(x)|u_n|^pdx\Big). \tag{2.2}$$

From [10,12], there hold

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}K(x)f(u_n)u_ndx = \int_{\mathbb{R}^N}K(x)f(u)udx, \tag{2.3}$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}K(x)f(u_n)udx = \int_{\mathbb{R}^N}K(x)f(u)udx, \tag{2.4}$$

$$\lim_{n\to+\infty}\lambda\int_{\mathbb{R}^N}W(x)|u_n|^pdx = \lambda\int_{\mathbb{R}^N}W(x)|u|^pdx, \tag{2.5}$$

and

$$\lim_{n\to+\infty}\lambda\int_{\mathbb{R}^N}W(x)|u_n|^{p-2}u_nudx = \lambda\int_{\mathbb{R}^N}W(x)|u|^pdx. \tag{2.6}$$

Combining (2.2), (2.3) and (2.5), we obtain

$$\lim_{n\to+\infty}\|u_n\|^2 = \int_{\mathbb{R}^N}K(x)f(u)udx+\lambda\int_{\mathbb{R}^N}W(x)|u|^pdx. \tag{2.7}$$

On the other hand, it follows from $\langle I'_\lambda(u_n), u\rangle = o_n(1)$ that

$$(u_n,u)-\int_{\mathbb{R}^N}K(x)f(u_n)udx-\lambda\int_{\mathbb{R}^N}W(x)|u_n|^{p-2}u_nudx = o_n(1). \tag{2.8}$$

By the definition of weak convergence, we obtain $(u_n,u)\to(u,u)$. Using (2.4), (2.6) and taking the limit in (2.8), we get

$$\|u\|^2 = \int_{\mathbb{R}^N}K(x)f(u)udx+\lambda\int_{\mathbb{R}^N}W(x)|u|^pdx,$$

which together with (2.2) yields that $\|u_n\|^2\to\|u\|^2$. So $u_n\to u$ in E. This completes the proof. ■

Define

$$c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{0 \le t \le 1} I_\lambda(\gamma(t)),$$

where

$$\Gamma_\lambda = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}.$$

Proof of Theorem 1.1. By Mountain Pass Theorem and Lemma 2.1, we obtain that, for each $0 < \lambda < \lambda_0$, there exists a (PS) sequence $\{u_n\} \subset E$ for I_λ in E. Then, by Lemma 2.3, we can conclude that there exist a subsequence $\{u_n\} \subset E$ and $u^* \in E$ such that $u_n \to u^*$ in E. Moreover, $I'_\lambda(u^*) = 0$ and $I_\lambda(u^*) = c_\lambda \ge \beta > 0$.

The second solution of the problem (1.1) will be constructed through the local minimization. Since $W \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$, we can choose a function $\phi \in E$ such that $\int_{\mathbb{R}^N} W(x)|\phi|^p dx > 0$. Thus, by (V) and (f_3) we have

$$\begin{aligned} I_\lambda(t\phi) &= \frac{t^2}{2}\|\phi\|^2 - \int_{\mathbb{R}^N} K(x)F(t\phi)dx - \frac{\lambda t^p}{p}\int_{\mathbb{R}^N} W(x)|\phi|^p dx \\ &\le \frac{t^2}{2}\|\phi\|^2 - \frac{\lambda t^p}{p}\int_{\mathbb{R}^N} W(x)|\phi|^p dx \\ &< 0, \end{aligned}$$

for $t > 0$ large enough. Hence, let $\rho > 0$ be given in Lemma 2.1, we have $-\infty < \inf_{u \in \bar{B}_\rho} I_\lambda(u) < 0$. By the Ekeland's variational principle [14,16], there exists a minimizing sequence $v_n \in \bar{B}_\rho$ such that $I_\lambda(v_n) \to \inf_{u \in \bar{B}_\rho} I_\lambda(u)$ and $I'_\lambda(v_n) \to 0$ as $n \to \infty$. Hence, Lemma 2.3 implies that there exists a nontrivial solution u^{**} of problem (1.1) satisfying $I_\lambda(u^{**}) < 0$ and $\|u^{**}\| \le \rho$. Therefore, we can conclude that$I_\lambda(u^{**}) < 0 = I_\lambda(0) < I_\lambda(u^*)$ for all $0 < \lambda < \lambda_0$. This completes the proof of Theorem 1.1.

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