The concentration behavior of ground state solutions for a critical fractional Schrödinger–Poisson system

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Abstract
In this paper, we study the following critical fractional Schrödinger–Poisson system

\[
\begin{align*}
\varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u &= P(x)f(u) + Q(x)|u|^{2^*_{s}-2}u, \quad \text{in } \mathbb{R}^3, \\
\varepsilon^{2t}(-\Delta)^t \phi &= u^2, \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

where \(\varepsilon > 0\) is a small parameter, \(s \in \left(\frac{3}{4}, 1\right), t \in (0, 1)\) and \(2s + 2t > 3\), \(2^*_s := \frac{6}{3 - 2s}\) is the fractional critical exponent for 3-dimension, \(V(x) \in C(\mathbb{R}^3)\) has a positive global minimum, and \(P(x), Q(x) \in C(\mathbb{R}^3)\) are positive and have global maximums. We obtain the existence of a positive ground state solution by using variational methods, and we determine a concrete set related to the potentials \(V, P\) and \(Q\) as the concentration position of these ground state solutions as \(\varepsilon \to 0^+\). Moreover, we consider some properties of these ground state solutions, such as convergence and decay estimate.

KEYWORDS
concentration, critical growth, fractional Schrödinger–Poisson system, ground state solution

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35J50, 35Q40, 58E05

1 | INTRODUCTION AND MAIN RESULTS

In this paper, we study the existence and concentration of solutions for the following critical fractional Schrödinger–Poisson system

\[
\begin{align*}
\varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u &= P(x)f(u) + Q(x)|u|^{2^*_{s}-2}u, \quad \text{in } \mathbb{R}^3, \\
\varepsilon^{2t}(-\Delta)^t \phi &= u^2, \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

where \(\varepsilon > 0\) is a small parameter, \(s \in \left(\frac{3}{4}, 1\right), t \in (0, 1)\), \(2s + 2t > 3\) and \((-\Delta)^a\) is the fractional Laplacian operator, which can be defined by the Fourier transform \((-\Delta)^a u = P^{-1}(|\xi|^{2a} Fu)\). In (1.1), the first equation is a nonlinear fractional Schrödinger equation in which the potential \(\phi\) satisfies the second equation which is a fractional Poisson equation. For this reason, (1.1) is referred to as a fractional nonlinear Schrödinger–Poisson system (also called Schrödinger–Maxwell system). When \(s = \frac{1}{2}\) and \(t = 1\), such a system becomes more interesting in Physics. It comes from the semi-relativistic theory in the repulsive (plasma...
physics) Coulomb case (see e.g. [29]). If one put the second equation into the first equation, such a system reduces to the semi-relativistic Hartree equation which arises from the quantum theory of boson stars ([21]).

If \( \phi(x) = 0 \) in the first equation, (1.1) becomes the fractional Schrödinger equation like

\[
\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N. \tag{1.2}
\]

Equation (1.2) is related to standing wave solutions of the fractional time-dependent Schrödinger equation of the form

\[
i \varepsilon \frac{\partial \psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s \psi + V(x)\psi - f(x,|\psi|), \quad x \in \mathbb{R}^N,
\]

which is a fundamental equation in fractional quantum mechanics (see [20]). It is well known that, different to the classical Laplacian operator, the usual analysis tools for elliptic PDEs can not be directly applied to (1.2) since \((-\Delta)^s\) is a nonlocal operator. To overcome this difficulty, Caffarelli and Silvestre [5] developed a powerful extension method which transfer the nonlocal Equation (1.2) into a local one settled on the half-space \( \mathbb{R}^{N+1}_+ \).

In the local case that \( s = t = 1 \), (1.1) reduces to the following system

\[
\begin{cases}
-\varepsilon^2\Delta u + V(x)u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\
-\varepsilon^2\Delta \phi = u^2, & \text{in } \mathbb{R}^3. 
\end{cases} \tag{1.3}
\]

From the point of Quantum Mechanics, the system (1.3) describes mutual interactions of many particles (see [36]) and also arises in Abelian Gauge Theories. These theories consist of field equations that provide a model to describe the interaction of a nonlinear Schrödinger field with the electromagnetic field (see [7,8]). In the past decades, the system likes or similars to (1.3) has been studied extensively by means of variational tools. See [2,10,16,19,27,40,45] and the references therein for the existence of ground state solution which concentrates around the global minimum of \( V \). The following semiclassical Schrödinger–Poisson system has also attracted a lot of attention

\[
\begin{cases}
-\varepsilon^2\Delta u + V(x)u + \phi u = f(u), & \text{in } \mathbb{R}^3, \\
-\Delta \phi = u^2, & \text{in } \mathbb{R}^3. 
\end{cases} \tag{1.4}
\]

D’Aprile and Wei [9] constructed a family of positive radially symmetric bound states and showed the concentration around a sphere in \( \mathbb{R}^3 \) as \( \varepsilon \to 0 \) for (1.4) with \( f(u) = |u|^{q-2}u, 1 < q < \frac{N}{2} \).

Recently, there is an increasing interest in the existence of solutions to the fractional Schrödinger–Poisson system. A fractional Schrödinger–Poisson system with \( V = 0 \) and a general nonlinearity in the subcritical and critical case was considered in [44], where a positive solution was obtained by using a perturbation approach, and the asymptotic behavior of solutions for a vanishing parameter was also given. In [34] and [35], Teng adapted the monotonicity trick (see e.g. [18]) to obtain the existence of ground state solutions to the critical and subcritical cases, respectively. In [24], the authors considered the following system

\[
\begin{cases}
\varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = g(u), & \text{in } \mathbb{R}^3, \\
\varepsilon^2(-\Delta)^s \phi = \gamma u^2, & \text{in } \mathbb{R}^3,
\end{cases}
\]

where \( \gamma \) is a constant, and they established the multiplicity of solutions for small \( \varepsilon \) via the Ljusternik–Schnirelmann category theory, where \( g \) is subcritical at infinity. However, the concentration behavior of solutions was almost not considered before in literatures. To the best of our knowledge, the only result was due to Liu and Zhang [22], where the authors considered the following system

\[
\begin{cases}
\varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = f(u) + |u|^{2^*_{s}-2}u, & \text{in } \mathbb{R}^3, \\
\varepsilon^{2}(-\Delta)^s \phi = u^2, & \text{in } \mathbb{R}^3,
\end{cases}
\]
Under the assumptions

(C) \( f : \mathbb{R} \to \mathbb{R} \) is a function of \( C^1 \)-class,

(M) \( \frac{f(t)}{t^2} \) is strictly increasing for \( t > 0 \),

and other suitable conditions, Liu and Zhang obtained the multiplicity of positive solutions which concentrate on the minima of \( V(x) \) by the minimax theorems and Ljusternik–Schnirelmann category theory. When \( f \) is not a function of \( C^1 \)-class, the multiplicity of solutions was established in [43]. The concentration behavior of ground state solutions for a subcritical case with two competing potentials was studied in [42].

In this paper, we are concerned with the existence and concentration behavior of ground state solutions for (1.1). We note that (1.1) involves three different potentials which make our problem more complicated than that one in [22]. This brings a competition between the potentials \( V \), \( P \) and \( Q \); each one would like to attract ground states to their minimum or maximum points, respectively. It makes difficulties in determining the concentration position of solutions. This kind of problem can be traced back to [38,39] and [6] for the semilinear Schrödinger equation. In [11], the authors found new concentration phenomena for Dirac equations with competing potentials and subcritical or critical nonlinearities, respectively. See also [12,37] and [41] for other related results.

To state our main results, we need the following assumptions

\( (f_1) \) \( f \in C(\mathbb{R}, \mathbb{R}) \), \( f(t) = o(t^3) \) as \( t \to 0 \) and \( f(t) = 0 \) for all \( t \leq 0 \);

\( (f_2) \) There exists \( 4 < p < 2^* \) such that

\[
|f(t)| \leq c_1 (1 + |t|^{p-1})
\]

for all \( t \in \mathbb{R} \) and some \( c_1 > 0 \);

\( (f_3) \) \( tf(t) - 4F(t) \geq stf(st) - 4F(st), \forall t \geq 0, \forall s \in [0, 1] \), where \( F(t) = \int_0^t f(\tau) d\tau; \)

\( (f_4) \) There exists \( 4 \leq \sigma < 2^* \) such that

\[
F(t) \geq c_2 t^\sigma
\]

for all \( t > 0 \) and some \( c_2 > 0 \).

Remark 1.1. If the nonlinearity is differentiable, then it is easy to see that \( (f_3) \) is equivalent to the condition \( (M) \) by the derivative rules. In the present paper, we only need \( f \in C(\mathbb{R}, \mathbb{R}) \). At the time, the condition \( (f_3) \) is weaker than \( (M) \). In fact, for any \( s \in [0, 1], t > 0 \), let \( k(s) = s^4 tf(t) - 4F(st) \), then \( k'(s) = 4s^3 tf(t) - 4 tf(st) = 4s^3 tf(t) - 4 \frac{f(t)}{t^5} (st)^3 \). If \( (M) \) holds, then

\[
k'(s) \geq 4s^3 tf(t) - 4 \frac{f(t)}{t^5} (st)^3 = 0, \quad \text{for all} \quad t \in \mathbb{R}.
\]

Therefore, \( k(s) \) is increasing on \([0,1]\). Consequently, \( k(1) \geq k(s) \), for all \( s \in [0,1] \). Thus, for any \( s \in [0,1] \), by \( (M) \) we have

\[
 tf(t) - 4F(t) \geq s^4 tf(t) - 4F(st) = s^4 t^4 \frac{f(t)}{t^5} - 4F(st) \geq stf(st) - 4F(st).
\]

Here is an example of nonlinearity which satisfies \( (f_3) \) but does not satisfy the condition \( (M) \). Define

\[
F(t) = \begin{cases} 
  t^4 \int_0^1 \frac{\sin^5 \left( \frac{\pi}{2} \tau \right)}{\tau^3} \, d\tau, & t \in [0, 1], \\
  t^4 \left( \int_0^1 \frac{\sin^5 \left( \frac{\pi}{2} \tau \right)}{\tau^3} \, d\tau + \int_1^t \frac{1}{\tau^3} \, d\tau \right), & t \geq 1.
\end{cases}
\]
By a direct computation, one has

\[
    f(t) = \begin{cases} 
        4t^3 \int_0^t \frac{\sin^5\left(\frac{\pi}{2} \tau\right)}{\tau^5} \, d\tau + \frac{\sin^5\left(\frac{\pi}{2} t\right)}{t}, & t \in [0, 1], \\
        4t^3 \left( \int_0^1 \frac{\sin^5\left(\frac{\pi}{2} \tau\right)}{\tau^5} \, d\tau + \int_1^t \frac{1}{\tau^5} \, d\tau \right) + \frac{1}{t}, & t \geq 1.
    \end{cases}
\]

Thereby, for \( t \in [0, +\infty) \) and \( s \in [0, 1] \), as \( 0 \leq t \leq 1 \), one has

\[
    st f(st) - 4F(st) = \sin^5\left(\frac{\pi}{2} st\right) \leq \sin^5\left(\frac{\pi}{2} t\right) = tf(t) - 4F(t).
\]

As \( t \geq 1 \), if \( 0 \leq st \leq 1 \), then

\[
    st f(st) - 4F(st) = \sin^5\left(\frac{\pi}{2} st\right) \leq 1 = tf(t) - 4F(t),
\]

and if \( st \geq 1 \), then

\[
    st f(st) - 4F(st) = 1 = tf(t) - 4F(t).
\]

All in all, \( st f(st) - 4F(st) \leq tf(t) - 4F(t) \) for any \( s \in [0, 1], t \in [0, +\infty) \).

By the definition of \( f \), we have

\[
    f(t) = \begin{cases} 
        4t^3 \int_0^t \frac{\sin^5\left(\frac{\pi}{2} \tau\right)}{\tau^5} \, d\tau + \frac{\sin^5\left(\frac{\pi}{2} t\right)}{t^4}, & t \in [0, 1], \\
        4t^3 \left( \int_0^1 \frac{\sin^5\left(\frac{\pi}{2} \tau\right)}{\tau^5} \, d\tau + \int_1^t \frac{1}{\tau^5} \, d\tau \right) + \frac{1}{t^4}, & t \geq 1.
    \end{cases}
\]

Notice that \( \int_0^1 \frac{\sin^5\left(\frac{\pi}{2} \tau\right)}{\tau^5} \, d\tau < +\infty \). Thus, if \( t \geq 1 \), one has

\[
    \frac{f(t)}{t^3} \equiv 4 \int_0^1 \frac{\sin^5\left(\frac{\pi}{2} \tau\right)}{\tau^5} \, d\tau + 1.
\]

So \( \frac{f(t)}{t^3} \) is not strictly increasing for \( t > 0 \), that is, \( f \) does not satisfy condition \((M)\).

We need some notations to help us to determine the concentration set of solutions. Set

\[
    V_{\min} := \min_{x \in \mathbb{R}^3} V(x), \quad V_{\max} := \sup_{x \in \mathbb{R}^3} V(x), \quad V := \{ x \in \mathbb{R}^3 : V(x) = V_{\min}\}, \quad V_{\infty} := \liminf_{|x| \to \infty} V(x),
\]

\[
    P_{\min} := \inf_{x \in \mathbb{R}^3} P(x), \quad P_{\max} := \max_{x \in \mathbb{R}^3} P(x), \quad P := \{ x \in \mathbb{R}^3 : P(x) = P_{\max}\}, \quad P_{\infty} := \limsup_{|x| \to \infty} P(x),
\]
Moreover, we assume that $V$, $P$, and $Q$ satisfy the following conditions:

(A$_0$) $V, P, Q$ are three continuous and bounded functions with $V_{\text{min}} > 0$, $P_{\text{min}} > 0$ and $Q_{\text{min}} > 0$; either

(A$_1$) $P > P_{\text{∞}}$ and there exists $x_P \in C_P$ such that $V(x_P) \leq V(x)$ for $|x| \geq R$ with $R > 0$ sufficiently large, where $C_P := \{x \in Q : P(x) = P_Q\}$

or

(A$_2$) $Q < Q_{\text{∞}}$ and there exists $x_Q \in C_Q$ such that $P(x_Q) \geq P(x)$ for $|x| \geq R$ with $R > 0$ sufficiently large, where $C_Q := \{x \in Q : V(x) = V_Q\}$.

If (A$_1$) holds, we set

$$H_P = \{x \in C_P : V(x) \leq V(x_P)\} \cup \{x \in Q \setminus C_P : V(x) < V(x_P)\} \cup \{x \notin Q : P(x) > P_Q \text{ or } V(x) < V(x_P)\}.$$  

If (A$_2$) holds, we set

$$H_V = \{x \in C_V : P(x) \geq P(x_V)\} \cup \{x \in Q \setminus C_V : P(x) > P(x_V)\} \cup \{x \notin Q : V(x) > V_Q \text{ or } P(x) > P(x_V)\}.$$  

Clearly, $H_P$ and $H_V$ are bounded sets. Moreover, if $\forall \cap P \cap Q \neq \emptyset$, then $H_P = H_V = \forall \cap P \cap Q$.

Now we state our main results as follows.

**Theorem 1.2.** Assume that (f$_1$)-(f$_4$), $s \in (\frac{3}{2}, 1)$, $t \in (0, 1)$, $2s + 2t > 3$, (A$_0$) and (A$_1$) hold, then for all small $\varepsilon > 0$:

(i) The system (1.1) has a positive ground state solution $(\omega_{\varepsilon}, \phi_{\omega_{\varepsilon}})$:

(ii) $\omega_{\varepsilon}$ possesses a global maximum point $x_{\varepsilon}$ such that, up to a subsequence, $x_{\varepsilon} \to x_0$ as $\varepsilon \to 0$, $\lim_{\varepsilon \to 0} \text{dist}(x_{\varepsilon}, H_P) = 0$, and $(v_{\varepsilon}(x), \psi_{\varepsilon}(x)) := (\omega_{\varepsilon}(\varepsilon x + x_{\varepsilon}), \phi_{\varepsilon}(\varepsilon x + x_{\varepsilon}))$ converges in $H^1(\mathbb{R}^3)$ to a positive ground state solution of

$$\begin{aligned}
(-\Delta)u + V(x_0)u + \phi u &= P(x_0)f(u) + Q(x_0)|u|^{2^*_s - 2}u, & \text{in } \mathbb{R}^3, \\
(-\Delta)\phi &= u^2, & \text{in } \mathbb{R}^3.
\end{aligned}$$

In particular if $\forall \cap P \cap Q \neq \emptyset$, then $\lim_{\varepsilon \to 0} \text{dist}(x_{\varepsilon}, \forall \cap P \cap Q) = 0$, and up to a subsequence, $(v_{\varepsilon}, \psi_{\varepsilon})$ converges in $H^1(\mathbb{R}^3)$ to a positive ground state solution of

$$\begin{aligned}
(-\Delta)u + V_{\text{min}}u + \phi u &= P_{\text{max}}f(u) + Q_{\text{max}}|u|^{2^*_s - 2}u, & \text{in } \mathbb{R}^3, \\
(-\Delta)\phi &= u^2, & \text{in } \mathbb{R}^3.
\end{aligned}$$

(iii) There exists a constant $C > 0$ such that

$$\omega_{\varepsilon}(x) \leq \frac{C\varepsilon^{3/2}}{\varepsilon^{3/2 + 2s} + |x - x_{\varepsilon}|^{3/2 + 2s}}, \quad \text{for all } x \in \mathbb{R}^3.$$

**Theorem 1.3.** Assume that (f$_1$)-(f$_4$), $s \in (\frac{3}{2}, 1)$, $t \in (0, 1)$, $2s + 2t > 3$, (A$_0$) and (A$_2$) hold, and we replace $(H_P)$ by $(H_V)$, then all the conclusions of Theorem 1.2 remain true.

**Remark 1.4.** Comparing to [22], there are some different points in our paper. First, we do not need $f$ satisfies the smooth condition $(S)$, and this prevents us using the Nehari manifold in a standard way. Second, we do not assume $f$ satisfies the monotonicity condition $(M)$ which plays an important role in [22].

In the sequel, we only give the detailed proof for Theorem 1.2 because the argument for Theorem 1.3 is similar to that for Theorem 1.2.

This paper is organized as follows. In Section 2, we provide some preliminary lemmas which will be used later. In Section 3, we consider the autonomous problem of the system (1.1) and prove the existence of positive ground state solutions. In Section 4,
we prove the existence of positive ground state solutions of the system (1.1) for small $\epsilon > 0$. In Section 5, we study the concentration phenomenon and convergence of ground state solutions. In Section 6, we obtain the decay estimate of solution, which is polynomial instead of exponential form. Finally, we give the proof of Theorem 1.2.

Notation. In this paper we make use of the following notations.

- For any $R > 0$ and for any $x \in \mathbb{R}^3$, $B_R(x)$ denotes the ball of radius $R$ centered at $x$;
- $L^p(\mathbb{R}^3)$, $1 \leq p \leq +\infty$, denotes the Lebesgue space with the following norm
  \[
  \|u\|_p = \begin{cases} 
  \left( \int_{\mathbb{R}^3} |u|^p \, dx \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\
  \text{ess sup}_{x \in \mathbb{R}^3} |u(x)|, & \text{if } p = \infty.
  \end{cases}
  \]
- $C$ or $C_i$ ($i = 1, 2, \ldots$) denote some positive constants could change from line to line.

2 | PRELIMINARIES

First, we collect some preliminary results for the fractional Laplacian from [3]. We define the homogeneous fractional Sobolev space $\dot{H}^s(\mathbb{R}^3)$ as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

\[
\|u\|_{\dot{H}^s} = \left( \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dxdy \right)^{\frac{1}{2}} = [u]_{\dot{H}^s}.
\]

We denote by $H^s(\mathbb{R}^3)$ the standard fractional Sobolev space, defined as the set of $u \in \dot{H}^{s,2}(\mathbb{R}^3)$ satisfying $u \in L^2(\mathbb{R}^3)$ with the norm

\[
\|u\|_{H^s}^2 = \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi = \frac{1}{2} C(s) \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dxdy,
\]

also, in light of [3] and [25, Proposition 3.4 and Proposition 3.6], we have

\[
\left\|(-\Delta)^{\frac{s}{2}} u\right\|_{L^2}^2 = \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi = \frac{1}{2} C(s) \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dxdy,
\]

where $\hat{u}$ stands for the Fourier transform of $u$ and

\[
C(s) = \left( \int_{\mathbb{R}^3} \frac{1 - \cos \xi_1}{|\xi|^{3+2s}} \, d\xi \right)^{-1}, \quad \xi = (\xi_1, \xi_2, \xi_3).
\]

As a consequence, the norms on $H^s(\mathbb{R}^3)$ defined below

\[
\begin{align*}
\|u\|_2 = & \left( \int_{\mathbb{R}^3} u^2 \, dx \right)^{\frac{1}{2}}, \\
\|u\|_{H^s} = & \left( \int_{\mathbb{R}^3} u^2 \, dx + \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dxdy \right)^{\frac{1}{2}}, \\
\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2 = & \left( \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u \, dx \right)^{\frac{1}{2}}, \\
\left\|(-\Delta)^{\frac{s}{2}} u\right\|_{L^2} = & \left( \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u \, dx \right)^{\frac{1}{2}}.
\end{align*}
\]
are all equivalent. Furthermore, it is well known that \( H^s(\mathbb{R}^3) \) is continuously embedded into \( L^r(\mathbb{R}^3) \) for any \( 2 \leq r \leq 2^* \) and compactly embedding into \( L^{r}_{loc}(\mathbb{R}^3) \) for any \( 1 \leq r < 2^* \) and there exists a constant \( S_s > 0 \) such that

\[
S_s = \inf_{u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |(\Delta)_{\frac{s}{2}} u|^2 \, dx}{\left( \int_{\mathbb{R}^3} |u|^{2^*_s} \, dx \right)^{\frac{s}{2^*_s}}}.
\]

Moreover, \((-\Delta)^s u\) can be equivalently represented as (see [25, Lemma 3.2])

\[
(-\Delta)^s u(x) = \frac{C(s)}{2} \int_{\mathbb{R}^3} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{3+2s}} \, dy,
\]

for all \( x \in \mathbb{R}^3 \).

We denote \( \| \cdot \|_{H^s} \) by \( \| \cdot \| \) in the sequel for convenience.

Recall that by the Lax–Milgram theorem, we know that for every \( u \in H^s(\mathbb{R}^3) \), there exists a unique \( \phi^s_u \in D^{s,2}(\mathbb{R}^3) \) such that \((-\Delta)^s \phi^s_u = u^2 \) and \( \phi^s_u \) can be expressed by

\[
\phi^s_u(x) = C_s \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|^{3-2t}} \, dy,
\]

which is called \( t \)-Riesz potential, where

\[
C_s = \frac{\Gamma(\frac{3}{2} - t)}{\pi^\frac{3}{2} 2^s \Gamma(t)}.
\]

Making the change of variables \( x \mapsto \epsilon x \), we can rewrite the system (1.1) as the following equivalent system

\[
\begin{cases}
(-\Delta)^s u + V(\epsilon x) u + \phi u = P(\epsilon x) f(u) + Q(\epsilon x)|u|^{2^*_s - 2} u, & \text{in } \mathbb{R}^3, \\
(-\Delta)^s \phi = u^2, & \text{in } \mathbb{R}^3.
\end{cases}
\]  

If \( u \) is a solution of the system (2.2), then \( \omega(x) := u^2(x) \) is a solution of the system (1.1). Thus, to study the system (1.1), it suffices to study the system (2.2). In view of the presence of potential \( V(x) \), we introduce the subspace

\[
H_x = \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\epsilon x) u^2 \, dx < \infty \right\},
\]

which is a Hilbert space equipped with the inner product

\[
(u, v)_x = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v \, dx + \int_{\mathbb{R}^3} V(\epsilon x) uv \, dx,
\]

and the equivalent norm

\[
\| u \|_x^2 = (u, u)_x = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u^2 \, dx + \int_{\mathbb{R}^3} V(\epsilon x) u^2 \, dx.
\]

Moreover, it can be proved that \( (u, \phi^s_u) \in H_x \times D^{s,2}(\mathbb{R}^3) \) is a solution of (2.2) if and only if \( u \in H_x \) is a critical point of the functional \( I_x : H_x \to \mathbb{R} \) defined as

\[
I_x(u) = \frac{1}{2} \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} V(\epsilon x) u^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi^s_u u^2 \, dx - \int_{\mathbb{R}^3} P(\epsilon x) F(u) \, dx - \frac{1}{2^s} \int_{\mathbb{R}^3} Q(\epsilon x)|u|^{2^*_s} \, dx,
\]  

(2.3)

where \( \phi^s_u \) is the unique solution of the second equation in (2.2). Note that \( 2 \leq \frac{12}{3+2d} \leq 2^*_s \) if \( 4s + 2r \geq 3 \), then by the Hölder inequality and the Sobolev inequality, we have

\[
\int_{\mathbb{R}^3} \phi^s_u u^2 \, dx \leq \left( \int_{\mathbb{R}^3} |u|^{\frac{12}{3+2d}} \, dx \right)^{\frac{3+2d}{6}} \left( \int_{\mathbb{R}^3} |\phi^s_u|^{2^*_s} \, dx \right)^{\frac{1}{2^*_s}} \leq \frac{1}{S_s^\frac{1}{2}} \left( \int_{\mathbb{R}^3} |u|^{\frac{12}{3+2d}} \, dx \right)^{\frac{3+2d}{6}} \| \phi^s_u \|_{L^{2^*_s}} \leq C \| u \|_x^2 \| \phi^s_u \|_{L^{2^*_s}} < \infty.
\]
Therefore, the functional $I_\varepsilon$ is well-defined for every $u \in H_\varepsilon$ and belongs to $C^1(H_\varepsilon, \mathbb{R})$. Moreover, for any $u, v \in H_\varepsilon$, we have

$$
(I'_\varepsilon(u), v) = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} v \, dx + \int_{\mathbb{R}^3} V(\varepsilon x)uv \, dx + \int_{\mathbb{R}^3} \phi' u v \, dx
- \int_{\mathbb{R}^3} P(\varepsilon x)f(u)v \, dx - \int_{\mathbb{R}^3} Q(\varepsilon x)|u|^{2^*_s-2}uv \, dx.
$$

(2.4)

The properties of the function $\phi'_u$ are given in the following lemma (see [35, Lemma 2.3]).

**Lemma 2.1.** If $4s + 2t \geq 3$, then for any $u \in H^s(\mathbb{R}^3)$, we have

(i) $\phi'_u \geq 0$;

(ii) $\phi'_u : H^s(\mathbb{R}^3) \to D^{s,2}(\mathbb{R}^3)$ is continuous and maps bounded sets into bounded sets;

(iii) $\int_{\mathbb{R}^3} \phi'_u u^2 \, dx \leq C\|u\|_{2}_4^4 \leq C\|u\|^4$;

(iv) If $u_n \to u$ in $H^s(\mathbb{R}^3)$, then $\phi'_u u_n \to \phi'_u u$ in $D^{s,2}(\mathbb{R}^3)$;

(v) If $u_n \to u$ in $H^s(\mathbb{R}^3)$, then $\phi'_u u_n \to \phi'_u u$ in $D^{s,2}(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} \phi'_u u_n^2 \, dx \to \int_{\mathbb{R}^3} \phi'_u u^2 \, dx$.

Define $N : H^s(\mathbb{R}^3) \to \mathbb{R}$ by

$$
N(u) = \int_{\mathbb{R}^3} \phi'_u u^2 \, dx.
$$

The next lemma shows that the functional $N$ and $N'$ possesses $BL$-splitting property which is similar to the well-known Brezis–Lieb lemma ([4]).

**Lemma 2.2.** ([35, Lemma 2.4]) Assume that $2s + 2t > 3$. Let $u_n \to u$ in $H^s(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$. Then

(i) $N(u_n - u) = N(u_n) - N(u) + o(1)$;

(ii) $N'(u_n - u) = N'(u_n) - N'(u) + o(1)$, in $(H^s(\mathbb{R}^3))^*$.

The following vanishing lemma is a version of the concentration-compactness principle proved by P. L. Lions. We can consult [15, Lemma 2.2] and [30, Lemma 2.4].

**Lemma 2.3.** If $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$ and it satisfies

$$
\sup_{x \in \mathbb{R}^3} \int_{B_R(x)} |u_n|^2 \, dx \to 0 \text{ as } n \to \infty,
$$

for some $R > 0$. Then $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for any $2 \leq r < 2^*_s$.

In order to find critical points for $I_\varepsilon$, we will use the Nehari methods. The Nehari manifold corresponding to $I_\varepsilon$ is defined by

$$
\mathcal{N}_\varepsilon = \{ u \in H_\varepsilon \setminus \{0\} : (I'_\varepsilon(u), u) = 0 \}.
$$

Thus, for any $u \in \mathcal{N}_\varepsilon$, we have that

$$
\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx + \int_{\mathbb{R}^3} V(\varepsilon x)u^2 \, dx + \int_{\mathbb{R}^3} \phi'_u u^2 \, dx = \int_{\mathbb{R}^3} P(\varepsilon x)f(u)u \, dx + \int_{\mathbb{R}^3} Q(\varepsilon x)|u|^{2^*_s} \, dx.
$$

Since $f$ is only continuous but not belongs to $C^1$-class, $\mathcal{N}_\varepsilon$ need not be of class $C^1$ in our case, so we cannot use standard arguments on the Nehari manifold in the standard way. To overcome the nondifferentiability of the Nehari manifold, we shall use the reduction method developed by Szulkin and Weth in [33].

First, $(f_1)$ and $(f_2)$ imply that for each $\tau > 0$, there is $C_\tau > 0$ such that

$$
|f(u)| \leq \tau |u|^3 + C_\tau |u|^{p-1} \quad \text{and} \quad |F(u)| \leq \frac{\tau}{4} |u|^4 + \frac{C_\tau}{q} |u|^p
$$

(2.5)
for all $u \in H^\varepsilon(\mathbb{R}^3)$. By ($f_1$) and ($f_3$), we deduce that
\begin{equation}
F(u) \geq 0 \quad \text{and} \quad \frac{1}{4}f(u)u - F(u) \geq 0.
\end{equation}

In the following we shall show some properties for $\mathcal{N}_\varepsilon$.

**Lemma 2.4.** For any $u \in H_\varepsilon \setminus \{0\}$, we have

(i) There exists a unique $\theta_u$ such that $\theta_u u \in \mathcal{N}_\varepsilon$. Moreover, $I_\varepsilon(\theta_u u) = \max_{\theta \geq 0} I_\varepsilon(\theta u)$.

(ii) There exist $T_2 > T_1 > 0$ independent of $\varepsilon > 0$ such that $T_1 \leq \theta_u \leq T_2$.

**Proof.** (i) For $\theta > 0$, let
\[
g(\theta) = I_\varepsilon(\theta u) = \frac{\theta^2}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{\varepsilon}{2}} u \right|^2 dx + \frac{\theta^2}{2} \int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx
+ \frac{\theta^4}{4} \int_{\mathbb{R}^3} \phi(x) u^2 dx - \int_{\mathbb{R}^3} P(\varepsilon x)F(\theta u) \, dx - \frac{\theta^2}{2\varepsilon} \int_{\mathbb{R}^3} Q(\varepsilon x) |u|^2 \, dx.
\]

Then, by (2.5) and Sobolev embedding inequality, we have
\[
g(\theta) \geq \frac{1}{2} \theta^2 \| u \|_2^2 - C \theta^4 \int_{\mathbb{R}^3} |u|^4 \, dx - C \theta^6 \int_{\mathbb{R}^3} |u|^6 \, dx - \frac{\theta^2}{2\varepsilon} \theta^2 \max_{\theta \geq 0} \int_{\mathbb{R}^3} |u|^{2\varepsilon} \, dx
\geq \frac{\theta^2}{2} \| u \|_2^2 - C \theta^4 \| u \|_4^4 - C \theta^6 \| u \|_6^6 - C \theta^2 \| u \|_2^{2\varepsilon}.
\]

and
\[
g'(\theta) \geq \theta \| u \|_2^2 - C \theta^3 \int_{\mathbb{R}^3} |u|^4 \, dx - C \theta^{p-1} \int_{\mathbb{R}^3} |u|^p \, dx - \theta^2 \max_{\theta \geq 0} \int_{\mathbb{R}^3} |u|^{2\varepsilon} \, dx
\geq \theta \| u \|_2^2 - C \theta^3 \| u \|_4^4 - C \theta^{p-1} \| u \|_6^6 - C \theta^2 \| u \|_2^{2\varepsilon}.
\]

Since $4 < p < 2\varepsilon$, $g(\theta) > 0$ and $g'(\theta) > 0$ for small $\theta > 0$. Moreover, by Lemma 2.1(iii), we get
\[
g(\theta) \leq \frac{\theta^2}{2} \| u \|_2^2 + C \theta^4 \| u \|_4^4 - \frac{\theta^2}{2\varepsilon} \theta^2 \int_{\mathbb{R}^3} |u|^{2\varepsilon} \, dx.
\]

Hence, $g(\theta) \to -\infty$ as $\theta \to \infty$ and $g$ has a positive maximum and there exist $\theta_u > 0$ such that $g'(\theta_u) = 0$, $g'(\theta) > 0$ for $0 < \theta < \theta_u$.

Next we claim that $g'(\theta) \neq 0$ for all $\theta > \theta_u$. Indeed, if the conclusion is false, then, from the above arguments, there exists a $\theta'_u < \theta_u$ such that $g'(\theta'_u) = 0$ and $g(u \theta_u) \geq g(u \theta'_u)$. However, ($f_3$) implies that
\[
g(u \theta'_u) = g(u \theta'_u) - \frac{\theta'_u^2}{4} g'(u \theta'_u)
= \frac{\theta'_u^2}{4} \| u \|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3} P(\varepsilon x) \left[ f(u \theta'_u) u - 4F(\theta'_u u) \right] \, dx + \frac{4s - 3}{12} \theta'_u^2 \max_{\theta \geq 0} \int_{\mathbb{R}^3} Q(\varepsilon x) |u|^{2\varepsilon} \, dx
\geq \frac{\theta'_u^2}{4} \| u \|_2^2 + \frac{1}{4} \int_{\mathbb{R}^3} P(\varepsilon x) \left[ f(u \theta'_u) u - 4F(\theta'_u u) \right] \, dx + \frac{4s - 3}{12} \theta'_u^2 \max_{\theta \geq 0} \int_{\mathbb{R}^3} Q(\varepsilon x) |u|^{2\varepsilon} \, dx
= g(\theta_u) - \frac{\theta_u}{4} g'(\theta_u)
= g(\theta_u),
\]
here we use $s > \frac{3}{4}$, this is a contradiction. This claim is proved and then $g$ has a unique maximum at $\theta_u$. Moreover, notice that $g'(\theta) = \theta^{-1} I'_\varepsilon(\theta u), \theta u)$, then $g'(\theta_u) = 0$ implies $\theta_u u \in \mathcal{N}_\varepsilon$. Thus (i) holds.
(ii) By \( \theta_u \mu \in \mathcal{N}_\varepsilon \) and Lemma 2.1(iii), we have

\[
C_1 \theta_u^2 \|u\|^2 + C_2 \theta_u^4 \|u\|^4 \geq \theta_u^2 \|u\|^2 + \theta_u^4 \int_{\mathbb{R}^3} \phi_u^2 \, dx = \theta_u \int_{\mathbb{R}^3} P(\varepsilon x) f(\theta_u u) \, dx + \theta_u^{2\gamma} \int_{\mathbb{R}^3} Q(\varepsilon x) |u|^{2\gamma} \, dx \\
\geq C_3 \theta_u^{2\gamma} \int_{\mathbb{R}^3} |u|^{2\gamma} \, dx.
\]

Thus, there exists a \( T_3 > 0 \) independent of \( \varepsilon \) such that \( \theta_u \leq T_2 \).

On the other hand, using \( \theta_u \mu \in \mathcal{N}_\varepsilon \) again and Lemma 2.1(i), we have

\[
C_4 \theta_u^2 \|u\|^2 \leq \theta_u^2 \|u\|^2 \leq C_5 \theta_u^2 \int_{\mathbb{R}^3} |u|^\sigma \, dx + C_6 \theta_u^{2\gamma} \int_{\mathbb{R}^3} |u|^{2\gamma} \, dx \leq C \theta_u^\sigma \|u\|^\sigma + C \theta_u^{2\gamma} \|u\|^{2\gamma},
\]

which yields that there exists a \( T_1 > 0 \) independent of \( \varepsilon \) such that \( \theta_u \geq T_1 \). \( \square \)

**Lemma 2.5.** For any fixed \( \varepsilon > 0 \), we have the following facts:

(i) There exist \( \rho > 0 \) such that \( c_\varepsilon = \inf_{\mathcal{N}_\varepsilon} I_\varepsilon \geq \inf_{S_\rho} I_\varepsilon > 0 \), where \( S_\rho = \{ u \in H_\varepsilon : \| u \|_\varepsilon = \rho \} \).

(ii) There exists \( r^* > 0 \) such that

\[
\| u \|_\varepsilon \geq r^*, \quad \text{for all} \quad u \in \mathcal{N}_\varepsilon.
\]

**Proof.**

(i) For any \( u \in H_\varepsilon \setminus \{0\} \), then by Lemma 2.1(i) and (2.5), we have

\[
I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} V(\varepsilon x) u^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^2 \, dx - \int_{\mathbb{R}^3} \Delta u^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} \phi_u^2 \, dx + \int_{\mathbb{R}^3} |u|^{2\gamma} \, dx \\
\geq \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} V(\varepsilon x) u^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^3} |u|^{2\gamma} \, dx - C \left( \int_{\mathbb{R}^3} u^2 \, dx + \int_{\mathbb{R}^3} |u|^p \, dx \right) \\
\geq \frac{1}{2} \| u \|_{\varepsilon}^2 - C \left( \| u \|_{\varepsilon}^4 + \| u \|_{\varepsilon}^p + \| u \|_{\varepsilon}^{2\gamma} \right).
\]

Hence, \( \inf_{S_\rho} I_\varepsilon > 0 \) for sufficiently small \( \rho \). Moreover, for any \( u \in \mathcal{N}_\varepsilon \), Lemma 2.4 implies that \( I_\varepsilon(u) = \max_{\theta \geq 0} I_\varepsilon(\theta u) \).

Taking a \( t_0 > 0 \) with \( t_0 u \in S_\rho \). Then

\[
I_\varepsilon(u) \geq I_\varepsilon(t_0 u) \geq \inf_{v \in S_\rho} I_\varepsilon(v).
\]

This completes the proof of (i).

(ii) For any \( u \in \mathcal{N}_\varepsilon \), similar to (i), we have

\[
0 = \langle I_\varepsilon'(u), u \rangle \geq \| u \|_{\varepsilon}^2 - C \left( \| u \|_{\varepsilon}^4 + \| u \|_{\varepsilon}^p + \| u \|_{\varepsilon}^{2\gamma} \right),
\]

from which we obtain that

\[
\| u \|_{\varepsilon} \geq r^* > 0
\]

for some \( r^* > 0 \) in view of \( u \in \mathcal{N}_\varepsilon \subset H_\varepsilon \setminus \{0\} \). \( \square \)

**Lemma 2.6.** If \( W \) is a compact subset of \( H_\varepsilon \setminus \{0\} \), then there exists \( R > 0 \) such that \( I_\varepsilon(u) \leq 0 \) on \( (\mathbb{R}^+ W) \setminus B_R(0) \) for each \( u \in W \).
Lemma 2.7. \(I_\varepsilon\) is coercive on \(N_\varepsilon\), i.e., \(I_\varepsilon(u) \to \infty\) as \(\|u\|_\varepsilon \to \infty\), \(u \in N_\varepsilon\).

Proof. Since \(u \in N_\varepsilon\), we have

\[
I_\varepsilon(u) = I_\varepsilon(u) - \frac{1}{4}\langle I'_\varepsilon(u), u \rangle
\]

\[
= \frac{1}{4} \int_{\mathbb{R}^3} \left(-\Delta\right) u_n^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \psi'(u_n) u_n^2 dx + \left(\frac{1}{4} - \frac{1}{2^*}\right) \int_{\mathbb{R}^3} Q(|u_n|) |u_n|^{2^*} dx
\]

\[
+ \frac{1}{4} \int_{\mathbb{R}^3} P(\varepsilon x)[f(u_n)] u_n - 4F(u_n)] dx
\]

\[
\geq \frac{1}{4} \|u\|_\varepsilon^2.
\]

Thus, \(I_\varepsilon\) is coercive on \(N_\varepsilon\).

Define the mapping \(\tilde{m}_\varepsilon : H_\varepsilon \setminus \{0\} \to N_\varepsilon\) and \(m_\varepsilon : S_\varepsilon \to N_\varepsilon\) by setting

\[
\tilde{m}_\varepsilon(u) = \theta_u u \quad \text{and} \quad m_\varepsilon = \tilde{m}_\varepsilon|_{S_\varepsilon},
\]

where \(\theta_u\) is as in Lemma 2.4, \(S_\varepsilon = \{u \in H_\varepsilon : \|u\|_\varepsilon = 1\}\).

We also consider the functionals \(\bar{Y}_\varepsilon : H_\varepsilon \setminus \{0\} \to \mathbb{R}\) and \(Y_\varepsilon : S_\varepsilon \to \mathbb{R}\) defined by

\[
\bar{Y}_\varepsilon(u) = I_\varepsilon(\tilde{m}_\varepsilon(u)) \quad \text{and} \quad Y_\varepsilon = \bar{Y}_\varepsilon|_{S_\varepsilon}. \tag{2.7}
\]

Since \(H_\varepsilon\) is a Hilbert space, Lemma 2.4, Lemma 2.5(ii) and Lemma 2.6 imply that the hypotheses \((A_1), (A_2)\) and \(A_3\) in [33] (see, Chapter 3) are satisfied. Hence, we have the following Lemmas 2.8–2.9.

Lemma 2.8. (See[33]) The mapping \(\tilde{m}_\varepsilon : H_\varepsilon \setminus \{0\} \to N_\varepsilon\) is continuous and \(m_\varepsilon\) is a homeomorphism between \(S_\varepsilon\) and \(N_\varepsilon\), and the inverse of \(m_\varepsilon\) is given by \(m_\varepsilon^{-1}(u) = u/\|u\|_\varepsilon\).

Lemma 2.9. (See[33]) For each \(\varepsilon > 0\), we have

(i) \(Y_\varepsilon \in C^1(S_\varepsilon, \mathbb{R})\) and for each \(w \in S_\varepsilon\), one has

\[
\langle Y'_\varepsilon(w), z \rangle = \|m_\varepsilon(w)\|_\varepsilon \langle I'_\varepsilon(m_\varepsilon(w)), z \rangle
\]

for all \(z \in T_w(S_\varepsilon) = \{v \in H_\varepsilon : \langle w, v \rangle = 0\}\).

(ii) If \(\{u_n\}\) is a (PS) sequence for \(Y_\varepsilon\), then \(\{m_\varepsilon(u_n)\}\) is a (PS) sequence for \(I_\varepsilon\). If \(\{u_n\} \subset N_\varepsilon\) is a bounded (PS) sequence for \(I_\varepsilon\), then \(\{m_\varepsilon^{-1}(u_n)\}\) is a (PS) sequence for \(I_\varepsilon\).
In the section, we shall prove some properties of the least energy solutions of the autonomous problem. Precisely, for any \(a, b, c > 0\), we define the mapping \(m_{abc} : H^s(\mathbb{R}^3) \setminus \{0\} \rightarrow \mathcal{N}_{abc}\) by \(m_{abc}(u) = \theta_a u\), and the inverse of \(m_{abc}\) is given by \(m^{-1}_{abc}(u) = \frac{u}{\|u\|_a}\). Let the functional \(\gamma_{abc} : H^s(\mathbb{R}^3) \setminus \{0\} \rightarrow \mathbb{R}\) be

\[
\gamma_{abc}(u) = I_{abc}(m_{abc}(u)) \quad \text{and} \quad Y_{abc} = \gamma_{abc}|_{S_a}.
\]

Moreover, we also have

\[
\gamma_{abc} = \inf_{u \in \mathcal{N}_{abc}} I_{abc}(u) = \inf_{u \in H^s(\mathbb{R}^3) \setminus \{0\}} \max_{\theta \geq 0} I_{abc}(\theta u) = \inf_{u \in S_a} \max_{\theta \geq 0} I_{abc}(\theta u) > 0.
\]

**Lemma 3.1.** For any \(a, b, c > 0\), the following inequality holds:

\[
0 < \gamma_{abc} < \frac{s}{3c} \frac{S_{abc}}{2s}. \tag*{3.1}
\]

**Proof.** The proof is similar to the proof of Lemma 3.3 in [35]. For the sake of completeness, we give the details here.

We define

\[
u_s(x) = \psi(x)U_s(x), \quad x \in \mathbb{R}^3,
\]

\(\psi \) is a critical point of \(\mathcal{Y}_s\) if and only if \(m_{s}(\psi)\) is a nontrivial critical point of \(I_s\). Moreover, the corresponding values of \(\mathcal{Y}_s\) and \(I_s\) coincide and \(\inf_{s} \mathcal{Y}_s = \inf_{\mathcal{N}_s} I_s\). Moreover, we also have

**Lemma 2.10.**

\[
c_{\varepsilon} = \inf_{u \in \mathcal{N}_s} I_s(u) = \inf_{u \in H_s(\mathbb{R}^3) \setminus \{0\}} \max_{\theta \geq 0} I_s(\theta u) = \inf_{u \in S_s} \max_{\theta \geq 0} I_s(\theta u) > 0.
\]

**3 | THE AUTONOMOUS PROBLEM**

In the section, we shall prove some properties of the least energy solutions of the autonomous problem. Precisely, for any \(a, b, c > 0\), we consider the following constant coefficient problem

\[
\begin{aligned}
(-\Delta)^su + au + \phi u &= bf(u) + c|u|^{2s-2}u, & & \text{in } \mathbb{R}^3, \\
(-\Delta)^s\phi &= u^2, & & \text{in } \mathbb{R}^3,
\end{aligned}
\]

and the corresponding energy functional

\[
I_{abc}(u) = \frac{1}{2}\|u\|^2_a + \frac{1}{4} \int_{\mathbb{R}^3} \phi' u^2 dx - b \int_{\mathbb{R}^3} F(u) dx - \frac{c}{2s} \int_{\mathbb{R}^3} |u|^{2s} dx,
\]

defined for \(u \in H^s(\mathbb{R}^3)\), where \(\|u\|_a = \left( \int_{\mathbb{R}^3} \left| (-\Delta)^s u \right| dx + a \int_{\mathbb{R}^3} u^2 dx \right)^{\frac{1}{2}}\). The Nehari manifold corresponding to \(I_{abc}\) is defined by

\[
\mathcal{N}_{abc} = \{ u \in H^s(\mathbb{R}^3) \setminus \{0\} : \langle I'_{abc}(u), u \rangle = 0 \}.
\]

We define the least energy associated with (3.1) by

\[
\gamma_{abc} = \inf_{u \in \mathcal{N}_{abc}} I_{abc}(u).
\]

The number \(\gamma_{abc}\) and the manifold \(\mathcal{N}_{abc}\) have properties similar to those of \(c_s\) and \(\mathcal{N}_s\) stated in Lemmas 2.4–2.7. Hence, for each \(u \in H^s(\mathbb{R}^3) \setminus \{0\}\), there exists a unique \(\theta_u > 0\) such that \(\theta_u u \in \mathcal{N}_{abc}\). Recall that \(S_a = \{ u \in H^s(\mathbb{R}^3) : \|u\|_a = 1 \}\) and define the mapping \(m_{abc} : H^s(\mathbb{R}^3) \setminus \{0\} \rightarrow \mathcal{N}_{abc}\) by \(m_{abc}(u) = \theta_u u\), and \(m_{abc} = m_{abc}|_{S_a}\). Moreover, the inverse of \(m_{abc}\) is given by \(m^{-1}_{abc}(u) = \frac{u}{\|u\|_a}.\) Let the functional \(\gamma_{abc} : H^s(\mathbb{R}^3) \setminus \{0\} \rightarrow \mathbb{R}\) be

\[
\gamma_{abc}(u) = I_{abc}(m_{abc}(u)) \quad \text{and} \quad Y_{abc} = \gamma_{abc}|_{S_a}.
\]

Moreover, we also have

\[
\gamma_{abc} = \inf_{u \in \mathcal{N}_{abc}} I_{abc}(u) = \inf_{u \in H^s(\mathbb{R}^3) \setminus \{0\}} \max_{\theta \geq 0} I_{abc}(\theta u) = \inf_{u \in S_a} \max_{\theta \geq 0} I_{abc}(\theta u) > 0.
\]
Moreover, by (3.2)–(3.3), using the elementary inequality 
\[ 3 \leq S_3 \frac{2}{\kappa} + O(\varepsilon^{3-2s}), \]
and
\[ \int_{\mathbb{R}^3} |u_\varepsilon(x)|^{2s} dx = S_3^{\frac{2}{s}} + O(\varepsilon^3), \]

From (3.4), we have
\[ \mathcal{I}_{abc}(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx + \frac{a}{2} \int_{\mathbb{R}^3} u^2 dx + \frac{b}{4} \int_{\mathbb{R}^3} \phi_0 u^2 dx - \int_{\mathbb{R}^3} F(u) dx - \frac{c}{2^*} \int_{\mathbb{R}^3} |u|^{2^*} dx \]
\[ \leq \frac{1}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx + \frac{a}{2} \int_{\mathbb{R}^3} u^2 dx + \frac{b}{4} \int_{\mathbb{R}^3} \phi_0 u^2 dx - c_2 b \int_{\mathbb{R}^3} |u|^s dx - \frac{c}{2^*} \int_{\mathbb{R}^3} |u|^{2^*} dx := \Psi_{abc}(u). \]

By a direct calculation, we have
\[ \Psi_{abc}(\theta u) = \theta^2 \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx + \frac{\theta^2 - 1}{2} \int_{\mathbb{R}^3} u^2 dx + \theta_0^4 \int_{\mathbb{R}^3} \phi_0 u^2 dx - c \theta_0^2 b \int_{\mathbb{R}^3} |u|^s dx - \frac{\theta_0^2 c}{2} \int_{\mathbb{R}^3} |u|^{2^*} dx. \]

Define \( g(\theta) = \frac{\theta^2}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx - \frac{\theta^2 c}{2} \int_{\mathbb{R}^3} |u|^s dx \) for \( \theta \geq 0 \). We note that \( g(\theta) \) attains its maximum at
\[ \theta_0 = \left( \frac{\int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx}{c \int_{\mathbb{R}^3} |u|^s dx} \right)^{\frac{1}{2}}. \]

Moreover, by (3.2)–(3.3), using the elementary inequality \( (a + b)^q \leq a^q + b \) which holds for \( q \geq 1 \) and \( a, b \geq 0 \), we deduce that
\begin{align*}
\max_{\theta \geq 0} g(\theta) &= g(\theta_0) = \frac{1}{2} \left( \frac{\int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx}{c \int_{\mathbb{R}^3} |u|^s dx} \right)^{\frac{1}{2}} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx - \frac{1}{2} \left( \frac{\int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx}{c \int_{\mathbb{R}^3} |u|^s dx} \right)^{\frac{1}{2}} c \int_{\mathbb{R}^3} |u|^s dx \\
&= \frac{s}{3} \left( 3 \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx \right)^{\frac{1}{3}} \left( \frac{\int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx}{c \int_{\mathbb{R}^3} |u|^s dx} \right)^{\frac{1}{2}}. 
\end{align*}
Moreover, we deduce that
\[ s > \frac{s}{3c} \left( S_s^{\frac{3}{2s}} + O(\epsilon^{3-2s}) \right) \]
\[ \leq \frac{s}{3c} \left( \frac{3}{2s} S_s^{\frac{3}{2s}} + O(\epsilon^{3-2s}) \right) \]
\[ \leq \frac{s}{3c} \left( \frac{3}{2s} S_s^{\frac{3}{2s}} + O(\epsilon^{3-2s}) \right) \]
\[ \leq \frac{s}{3c} \left( \frac{3}{2s} S_s^{\frac{3}{2s}} + O(\epsilon^{3-2s}) \right). \] (3.5)

Since \( I_{abc}(\theta u_\epsilon) \to -\infty \) as \( \theta \to \infty \), by standard argument, there exists \( \theta_\epsilon > 0 \) such that
\[ 0 < \gamma_{abc} \leq \max_{\theta \geq 0} I_{abc}(\theta u_\epsilon) = I_{abc}(\theta_\epsilon u_\epsilon) \leq \Psi_{abc}(\theta_\epsilon u_\epsilon), \] (3.6)
which implies that \( \theta_\epsilon \geq A_1 > 0 \) for some constant \( A_1 \). On the other hand, from (3.2)–(3.4), for any \( \epsilon > 0 \), we have that
\[ 0 < \gamma_{abc} \leq \Psi_{abc}(\theta_\epsilon u_\epsilon) \leq C_1 \theta_\epsilon^2 + C_2 \theta_\epsilon^4 - C_3 \theta_\epsilon^{2s}, \]
which implies that there exists \( A_2 > 0 \) such that \( \theta_\epsilon \leq A_2 \) and thus \( 0 < A_1 < \theta_\epsilon \leq A_2 \) for any \( \epsilon > 0 \).

Now, by (3.2)–(3.6), we deduce that
\[ \Psi_{abc}(\theta_\epsilon u_\epsilon) \leq \frac{s}{3c} \left( \frac{3}{2s} S_s^{\frac{3}{2s}} + O(\epsilon^{3-2s}) \right) + \frac{\theta_\epsilon^2}{2} \int_{\mathbb{R}^3} u_\epsilon^2 dx + \frac{\theta_\epsilon^4}{4} \int_{\mathbb{R}^3} \phi u_\epsilon^2 dx - c_2 b \theta_\epsilon^\sigma \int_{\mathbb{R}^3} |u_\epsilon|^\sigma dx \]
\[ \leq \frac{s}{3c} \left( \frac{3}{2s} S_s^{\frac{3}{2s}} + O(\epsilon^{3-2s}) \right) + \frac{A_2^2}{2} \int_{\mathbb{R}^3} u_\epsilon^2 dx + \frac{A_2^4}{4} \int_{\mathbb{R}^3} \phi u_\epsilon^2 dx - c_2 b A_1^\sigma \int_{\mathbb{R}^3} |u_\epsilon|^\sigma dx \]
\[ \leq \frac{s}{3c} \left( \frac{3}{2s} S_s^{\frac{3}{2s}} + O(\epsilon^{3-2s}) \right) + \frac{A_2^2}{2} \int_{\mathbb{R}^3} u_\epsilon^2 dx + C A_2^2 \left( \int_{\mathbb{R}^3} |u_\epsilon|^\frac{12}{3+2s} dx \right)^{\frac{3+2s}{3}} - A_1^\sigma \int_{\mathbb{R}^3} |u_\epsilon|^\sigma dx. \]

Since \( s > \frac{3}{4} \), then \( \frac{3}{3+2s} > 2 \) and
\[ \int_{\mathbb{R}^3} u_\epsilon^2 dx = O(\epsilon^{3-2s}). \]

Therefore,
\[ \Psi_{abc}(\theta_\epsilon u_\epsilon) \leq \frac{s}{3c} \left( \frac{3}{2s} S_s^{\frac{3}{2s}} + O(\epsilon^{3-2s}) \right) + C \left( \int_{\mathbb{R}^3} |u_\epsilon|^\frac{12}{3+2s} dx \right)^{\frac{3+2s}{3}} - C \int_{\mathbb{R}^3} |u_\epsilon|^\sigma dx. \]

Moreover, we deduce that
\[ \lim_{\epsilon \to 0} \left( \frac{\int_{\mathbb{R}^3} |u_\epsilon(x)|^{\frac{12}{3+2s}} dx}{\epsilon^{3-2s}} \right)^{\frac{3+2s}{3}} = \begin{cases} 
\lim_{\epsilon \to 0} \frac{O(\epsilon^{4s+2r-3})}{\epsilon^{3-2s}} = 0, & \frac{12}{3+2r} > \frac{3}{3-2s}, \\
\lim_{\epsilon \to 0} \frac{O(\epsilon^{4s+2r-3} \log \epsilon^{\frac{3+2s}{3}})}{\epsilon^{3-2s}} = 0, & \frac{12}{3+2r} = \frac{3}{3-2s}, \\
\lim_{\epsilon \to 0} \frac{O(\epsilon^{6-4s})}{\epsilon^{3-2s}} = 0, & \frac{12}{3+2r} < \frac{3}{3-2s}. \end{cases} \]
and we also have \( \frac{3}{3-2s} < \frac{4s}{3-2s} < 4 \leq \sigma < 2^s \), then we deduce that
\[
\lim_{\varepsilon \to 0} \int_{Q_\varepsilon} |u_\varepsilon(x)|^\sigma dx = \lim_{\varepsilon \to 0} \frac{O\left(\varepsilon^{\frac{3-2s}{2}}\right)}{\varepsilon^{3-2s}} = +\infty.
\]
Therefore, the above arguments imply that
\[
0 < r_{abc} \leq I_{abc}(\theta_\varepsilon u_\varepsilon) \leq \Psi_{abc}(\theta_\varepsilon u_\varepsilon) < \frac{s}{3-2s} S_{\frac{3}{2}}^\frac{3}{2}.
\]
Thus we complete the proof. \(\square\)

**Lemma 3.2.** For any \( a, b, c > 0 \), system (3.1) has a positive ground state solution in \( H^s(\mathbb{R}^3) \).

**Proof.** If \( u \in \mathcal{N}_{\text{abc}} \) satisfies \( I_{\text{abc}}(u) = r_{\text{abc}} \), then
\[
Y_{\text{abc}}(m_{\text{abc}}^{-1}(u)) = I_{\text{abc}}(m_{\text{abc}}^{-1}(u)) = I_{\text{abc}}(u) = r_{\text{abc}} = \underset{S_a}{\inf} Y_{\text{abc}}(u).
\]
That is, \( m_{\text{abc}}^{-1}(u) \) is a minimizer of \( Y_{\text{abc}} \), and hence a critical point of \( Y_{\text{abc}} \). Therefore, similar to Lemma 2.9, we see that \( u \) is a critical point of \( I_{\text{abc}} \). It remains to show that there exists a minimizer \( u \) of \( I_{\text{abc}} \mid_{\mathcal{N}_{\text{abc}}} \). By Ekeland’s variational principle in [14], there exists a sequence \( \{u_n\} \subset S_a \) with \( Y_{\text{abc}}(u_n) \to r_{\text{abc}}, Y_{\text{abc}}'(u_n) \to 0 \) as \( n \to \infty \). In fact, set
\[
g_{\frac{3}{2}}(u) = \|u\|_{\frac{3}{2}}^2 - 1, \quad \text{for all } u \in H^s(\mathbb{R}^3).
\]
Notice that \( S_a = \{ u \in H^s(\mathbb{R}^3) : g_{\frac{3}{2}}(u) = 0 \} \) and for each \( u \in S_a \), one has
\[
\langle g_{\frac{3}{2}}'(u), u \rangle = 2\|u\|_{\frac{3}{2}}^2 = 2 > 0.
\]
By Proposition 9 in [33], we know that \( \tilde{Y}_{\text{abc}} \in C^1(\mathcal{N}_{\text{abc}}) \setminus \{0, \mathbb{R}\} \) and
\[
\langle \tilde{Y}_{\text{abc}}'(u), v \rangle = \langle \tilde{m}(u), v \rangle \|u\|_{\frac{3}{2}}^2 - \langle \tilde{I}_{\text{abc}}(\tilde{m}_{\text{abc}}(u)), v \rangle, \quad \text{for all } \ 0 \neq u, v \in H^s(\mathbb{R}^3).
\]
Hence, by Corollary 3.4 in [14] there exists a sequence \( \{w_n\} \subset S_a \) such that \( Y_{\text{abc}}(w_n) \to r_{\text{abc}} \) and there exists \( \alpha_n \in \mathbb{R} \) such that
\[
\|Y_{\text{abc}}'(w_n) - \alpha_n g_{\frac{3}{2}}'(w_n)\|_{\frac{3}{2}}^2 \to 0. \text{ It implies}
\]
\[
\alpha_n = \frac{\langle Y_{\text{abc}}'(w_n), g_{\frac{3}{2}}'(w_n) \rangle}{\|g_{\frac{3}{2}}'(w_n)\|_{\frac{3}{2}}^2} + o(1).
\]
Hence, \( Y_{\text{abc}}'(w_n) - \frac{\langle Y_{\text{abc}}'(w_n), g_{\frac{3}{2}}'(w_n) \rangle}{\|g_{\frac{3}{2}}'(w_n)\|_{\frac{3}{2}}^2} g_{\frac{3}{2}}'(w_n) = o(1) \), i.e., \( Y_{\text{abc}}'(w_n) = o(1) \). Let \( u_n = m_{\text{abc}}^{-1}(w_n) \), by the definition of \( m_{\text{abc}} \), we know \( u_n \in \mathcal{N}_{\text{abc}} \) for all \( n \in \mathbb{N} \). Similar to Lemma 2.9, one has \( I_{\text{abc}}(u_n) \to r_{\text{abc}}, I_{\text{abc}}'(u_n) \to 0 \) as \( n \to \infty \). Similar to Lemma 2.7, we know that \( \{u_n\} \) is bounded in \( H^s(\mathbb{R}^3) \).

Next, we claim that there exists a sequence \( \{y_n\} \subset \mathbb{R}^3 \) and \( R, \delta > 0 \) such that
\[
\int_{B_R(y_n)} |u_n|^2 \, dx \geq \delta, \quad n \in \mathbb{N}.
\]
Otherwise, by Lemma 2.3, we have
\[
u_n \to 0 \text{ in } L^r(\mathbb{R}^3) \text{ for } 2 < r < 2^s.
\]
Thus, by (2.5), we have
\[
\int_{\mathbb{R}^3} F(u_n) \, dx \to 0, \quad \int_{\mathbb{R}^3} f(u_n) u_n \, dx \to 0 \text{ as } n \to \infty.
\]
Moreover, by Lemma 2.1 (iii), we can obtain
\[
\int_{\mathbb{R}^3} \phi_{u_n}^i u_n^2 \, dx \to 0 \text{ as } n \to \infty.
\] (3.9)

Notice that
\[
I_{abc}(u_n) - \frac{1}{2s} \left\langle I'_{abc}(u_n), u_n \right\rangle = \frac{s}{3} \|u_n\|^2 + \frac{4s-3}{12} \int_{\mathbb{R}^3} \phi_{u_n}^i u_n^2 \, dx - b \int_{\mathbb{R}^3} F(u_n) \, dx + \frac{b}{2s} \int_{\mathbb{R}^3} f(u_n) u_n \, dx.
\]

Therefore, (3.8)–(3.9) imply that
\[
\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx \leq \frac{3}{s} \gamma_{abc} + o(1).
\]

Similarly, we have
\[
\int_{\mathbb{R}^3} |u_n|^2 \, dx = \frac{3}{s} \gamma_{abc} + o(1).
\]

Moreover,
\[
\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \, dx - c \int_{\mathbb{R}^3} |u_n|^2 \, dx \leq o(1),
\]

which implies
\[
\gamma_{abc} \geq \frac{s}{3c - 2s} S_{\frac{3}{2}}^2,
\]

which is a contradiction with Lemma 3.1. Let \( u_n(x) = u_n(x + y_n) \), then \( \{u_n\} \) is bounded in \( H^s(\mathbb{R}^3) \) by the boundedness of \( \{u_n\} \) and, up to a subsequence, we assume that \( u_n \rightharpoonup v \) in \( H^s(\mathbb{R}^3) \). By (3.7), we see that \( v \neq 0 \) and it is easy check that \( I_{abc}(v) = \gamma_{abc} \).

Moreover, by Lemma 2.2(ii) and Lemma 2.3, we can obtain \( I'_{abc}(v) = 0 \)

Next we only need to prove that \( v \) is positive. Put \( v^\pm = \max(\pm v, 0) \), the positive (negative) part of \( v \). We note that all the calculations above can be repeated word by word, replacing \( I'_{abc}(u) \) with the functional
\[
I_{abc}^+(v) = \frac{1}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} v \right|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} a v^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{v}^i v^2 \, dx - b \int_{\mathbb{R}^3} F(v^+) \, dx - \frac{c}{2s} \int_{\mathbb{R}^3} |v^+|^{2^*} \, dx.
\]

In this way we get a ground state solution \( v \) of the equation
\[
(-\Delta)^{\frac{s}{2}} v + av + \phi_{v}^i v = b f(v^+) + c |v^+|^{2^*-2} v^+, \quad \text{in} \ \mathbb{R}^3.
\] (3.10)

Using \( v^- \) as a test function in (3.10) we obtain
\[
\int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} v^- \right|^2 \, dx + \int_{\mathbb{R}^3} a |v^-|^2 \, dx + \int_{\mathbb{R}^3} \phi_{v}^i (v^-)^2 \, dx = 0.
\] (3.11)

On the other hand,
\[
\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} v \cdot (-\Delta)^{\frac{s}{2}} v^- \, dx = \frac{1}{2} C(s) \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\nu(x) - \nu(y))(\nu^-(x) - \nu^-(y)) \frac{1}{|x - y|^{3+2s}} \, dx \, dy
\]
\[
\geq \frac{1}{2} C(s) \left[ \int_{\nu^+ > 0} \frac{(\nu(x) - \nu(y))(\nu^-(x) - \nu^-(y))}{|x - y|^{3+2s}} \, dx \, dy \right.
\]
\[
+ \int_{\nu^- > 0} \frac{(\nu^-(x) - \nu^-(y))^2}{|x - y|^{3+2s}} \, dx \, dy + \int_{\nu^- < 0} \frac{(\nu(x) - \nu(y))\nu^-(x)}{|x - y|^{3+2s}} \, dx \, dy \right]
\]
\[
\geq 0.
\]
Thus, it follows from (3.11) and Lemma 2.1(i), we have \( \nu^{-} = 0 \) and \( \nu \geq 0 \). Moreover, if \( \nu(y_{0}) = 0 \) for some \( y_{0} \in \mathbb{R}^{3} \), then \( (-\Delta)^{s} \nu(y_{0}) = 0 \) and by (2.1), we have

\[
(-\Delta)^{s} \nu(y_{0}) = -\frac{C(s)}{2} \int_{\mathbb{R}^{3}} \frac{\nu(y_{0} + y) + \nu(y_{0} - y) - 2\nu(y_{0})}{|y|^{3+2s}} \, dy,
\]

therefore,

\[
\int_{\mathbb{R}^{3}} \frac{\nu(y_{0} + y) + \nu(y_{0} - y)}{|y|^{3+2s}} \, dy = 0,
\]

yielding \( \nu \equiv 0 \), a contradiction. \( \square \)

The following lemma describes a comparison between the mountain pass values for different parameters \( a, b, c > 0 \), which will play an important role in proving the existence results in Section 4.

**Lemma 3.3.** Let \( a_{j} > 0 \) and \( b_{j} > 0 \), \( j = 1, 2 \), with \( a_{1} \leq a_{2}, b_{1} \geq b_{2} \) and \( c_{1} \geq c_{2} \). Then \( \gamma_{a_{1}b_{1}c_{1}} \leq \gamma_{a_{2}b_{2}c_{2}} \). In particular, if one of inequalities is strict, then \( \gamma_{a_{1}b_{1}c_{1}} < \gamma_{a_{2}b_{2}c_{2}} \).

**Proof.** Let \( u \in \mathcal{N}_{a_{2}b_{2}c_{2}} \) be such that

\[
\gamma_{a_{2}b_{2}c_{2}} = I_{a_{2}b_{2}c_{2}} (u) = \max_{\theta \geq 0} I_{a_{2}b_{2}c_{2}} (\theta u).
\]

Let \( u_{0} = \theta_{1} u \) be such that \( I_{a_{1}b_{1}c_{1}} (u_{0}) = \max_{\theta \geq 0} I_{a_{1}b_{1}c_{1}} (\theta u) \). One has

\[
\gamma_{a_{2}b_{2}c_{2}} = I_{a_{2}b_{2}c_{2}} (u) \geq I_{a_{2}b_{2}c_{2}} (u_{0})
\]

\[
= I_{a_{1}b_{1}c_{1}} (u_{0}) + \frac{1}{2} (a_{2} - a_{1}) \int_{\mathbb{R}^{3}} |u_{0}|^{2} \, dx + (b_{1} - b_{2}) \int_{\mathbb{R}^{3}} F(u_{0}) \, dx + \frac{1}{2s} (c_{1} - c_{2}) \int_{\mathbb{R}^{3}} |u_{0}|^{2s} \, dx
\]

\[
\geq \gamma_{a_{1}b_{1}c_{1}}.
\]

The second part can be obtained similarly. Thus, we complete the proof. \( \square \)

## 4 | Existence of Ground State Solutions

In the section, we will prove the existence of ground state solutions to the system (2.2). Observing that for any \( x_{p} \in C_{p} \), we set \( \tilde{V}(x) = V(x + x_{p}) \), \( \tilde{P}(x) = P(x + x_{p}) \) and \( \tilde{Q}(x) = Q(x + x_{p}) \). Clearly, if \( \tilde{u}(x) \) is a solution of

\[
\begin{cases}
(-\Delta)^{s} \tilde{u} + \tilde{V}(\epsilon x) \tilde{u} + \phi \tilde{u} = \tilde{P}(\epsilon x) f(\tilde{u}) + \tilde{Q}(\epsilon x) |\tilde{u}|^{2\gamma-2} \tilde{u}, & \text{in } \mathbb{R}^{3},
\end{cases}
\]

then \( u(x) = \tilde{u}(x - x_{p}) \) solves (2.2). Thus, without loss of generality, we may assume that

\[
x_{p} = 0 \in C_{p},
\]

so

\[
Q(0) = Q_{\max}, \quad P(0) = P_{\tilde{Q}} \quad \text{and} \quad \nu := V(0) \leq V(x) \text{ for all } |x| \geq R.
\]

**Lemma 4.1.** \( \limsup_{\epsilon \to 0} c_{\epsilon} \leq \gamma_{\nu \tilde{P}_{\tilde{Q}} Q_{\max}} \).

**Proof.** Denote \( V_{\epsilon}^{a}(x) = \max \{a, V(\epsilon x)\} \), \( P_{\epsilon}(x) = \min \{b, P(\epsilon x)\} \) and \( Q_{\epsilon}(x) = \min \{c, Q(\epsilon x)\} \), where \( a, b, c \) are three positive constants. Define the auxiliary functional as follows:

\[
I_{\epsilon}^{abc}(u) := \frac{1}{2} \int_{\mathbb{R}^{3}} (-\Delta)^{s} u^{2} \, dx + \frac{1}{2} \int_{\mathbb{R}^{3}} V_{\epsilon}(x) u^{2} \, dx + \frac{1}{4} \int_{\mathbb{R}^{3}} \phi_{u}^{2} u^{2} \, dx - \int_{\mathbb{R}^{3}} P_{\epsilon}(x) F(u) \, dx - \frac{1}{2s} \int_{\mathbb{R}^{3}} Q_{\epsilon}(x) |u|^{2s} \, dx.
\]
for any \( u \in H^1(\mathbb{R}^3) \), which implies that \( I_{abc}(u) \leq I_{abc}^{\text{ab}}(u) \), and thus \( y_{abc} \leq c_{\epsilon}^{abc} \), where \( c_{\epsilon}^{abc} \) is the least energy of \( I_{abc}^{\text{ab}} \). By the definition of \( V_{\min}^\epsilon \), \( P_{\max}^\epsilon \) and \( Q_{\max}^\epsilon \), we get \( V_{\min}^\epsilon(x) = V(x) \), \( P_{\max}^\epsilon(x) = P(x) \) and \( Q_{\max}^\epsilon(x) = Q(x) \). Therefore, we have

\[
I_{\epsilon}^\epsilon V_{\min}^\epsilon P_{\max}^\epsilon Q_{\max}^\epsilon(u) = I_{\epsilon}(u),
\]

(4.2)

and \( V_{\min}^\epsilon(x) \rightarrow V(0) = \nu \), \( P_{\max}^\epsilon(x) \rightarrow P(0) = P \). \( Q_{\max}^\epsilon(x) \rightarrow Q(0) = Q \) on bounded sets of \( x \) as \( \epsilon \rightarrow 0 \).

Now, we claim \( \limsup_{\epsilon \rightarrow 0} c_{\epsilon}^\epsilon V_{\min}^\epsilon P_{\max}^\epsilon Q_{\max}^\epsilon \leq \gamma_{\nu} P Q \).

Indeed, let \( u \) be a ground state solution of \( I_{\epsilon}^\epsilon V_{\min}^\epsilon P_{\max}^\epsilon Q_{\max}^\epsilon \) by Lemma 3.2, that is, \( I_{\epsilon}^\epsilon V_{\min}^\epsilon P_{\max}^\epsilon Q_{\max}^\epsilon(u) = \gamma_{\nu} P Q \), then there exists \( \theta_{\epsilon} > 0 \) such that \( \theta_{\epsilon} u \in N_{\epsilon}^\epsilon V_{\min}^\epsilon P_{\max}^\epsilon Q_{\max}^\epsilon \), where \( N_{\epsilon}^\epsilon V_{\min}^\epsilon P_{\max}^\epsilon Q_{\max}^\epsilon \) is the Nehari manifold of the functional \( J_{\epsilon}^\epsilon V_{\min}^\epsilon P_{\max}^\epsilon Q_{\max}^\epsilon \). Thus

\[
c_{\epsilon}^\epsilon V_{\min}^\epsilon P_{\max}^\epsilon Q_{\max}^\epsilon \leq I_{\epsilon}^\epsilon V_{\min}^\epsilon P_{\max}^\epsilon Q_{\max}^\epsilon(\theta_{\epsilon} u) = \max_{\theta_{\epsilon} \geq 0} I_{\epsilon}^\epsilon V_{\min}^\epsilon P_{\max}^\epsilon Q_{\max}^\epsilon(\theta u).
\]

One has

\[
I_{\epsilon}^\epsilon V_{\min}^\epsilon P_{\max}^\epsilon Q_{\max}^\epsilon(\theta u) = I_{\epsilon}^\epsilon_{\nu} P_{\nu} Q_{\nu}(\theta u) + \frac{1}{2} \int_{\mathbb{R}^3} \left( V_{\min}^\epsilon(x) - \nu \right) |\theta u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} \left( P_{\nu} - P_{\nu}^{\max}(x) \right) F(\theta u) dx + \frac{1}{2} \int_{\mathbb{R}^3} \left( Q_{\nu} - Q_{\nu}^{\max}(x) \right) |\theta u|^{2} dx.
\]

(4.3)

By Lemma 2.4(ii), we can assume that \( \theta_{\epsilon} \rightarrow \theta_{0} \) as \( \epsilon \rightarrow 0 \). Since \( u \in L^2(\mathbb{R}^3) \), for any \( \tau > 0 \), there exists a \( R > 0 \) such that

\[
\int_{\mathbb{R}^3 \setminus B_R(0)} |u|^2 dx < \tau.
\]

Therefore,

\[
\int_{\mathbb{R}^3} \left( V_{\min}^\epsilon(x) - \nu \right) |\theta_{\epsilon} u|^2 dx = \int_{\mathbb{R}^3} \left( V_{\min}^\epsilon(x) - \nu \right) |\theta_{0} u|^2 dx + o(1)
\]

\[
= \int_{\mathbb{R}^3 \setminus B_R(0)} \left( V_{\min}^\epsilon(x) - \nu \right) |\theta_{0} u|^2 dx + \int_{B_R(0)} \left( V_{\min}^\epsilon(x) - \nu \right) |\theta_{0} u|^2 dx + o(1)
\]

\[
\leq C \theta_{0}^2 \tau + o(1),
\]

where we use the fact that \( V_{\min}^\epsilon(x) \rightarrow \nu \) in \( x \in B_R(0) \). Thus, we obtain

\[
\int_{\mathbb{R}^3} \left( V_{\min}^\epsilon(x) - \nu \right) |\theta_{\epsilon} u|^2 dx = o(1).
\]

Similarly, we have

\[
\int_{\mathbb{R}^3} \left( P_{\nu} - P_{\nu}^{\max}(x) \right) F(\theta u) dx = o(1), \quad \int_{\mathbb{R}^3} \left( Q_{\nu} - Q_{\nu}^{\max}(x) \right) |\theta u|^{2} dx = o(1).
\]

Thus, by (4.3), we have

\[
I_{\epsilon}^\epsilon V_{\min}^\epsilon P_{\max}^\epsilon Q_{\max}^\epsilon(\theta_{\epsilon} u) = I_{\epsilon}^\epsilon_{\nu} P_{\nu} Q_{\nu}(\theta_{\epsilon} u) + o(1) \rightarrow I_{\nu}^\nu P_{\nu} Q_{\nu}(\theta_{0} u) \quad \text{as} \quad \epsilon \rightarrow 0.
\]

(4.4)

Consequently

\[
c_{\epsilon}^\epsilon V_{\min}^\epsilon P_{\max}^\epsilon Q_{\max}^\epsilon \leq I_{\epsilon}^\epsilon V_{\min}^\epsilon P_{\max}^\epsilon Q_{\max}^\epsilon(\theta_{\epsilon} u) \rightarrow I_{\nu}^\nu P_{\nu} Q_{\nu}(\theta_{0} u) \leq \max_{\theta_{\epsilon} \geq 0} I_{\epsilon}^\epsilon_{\nu} P_{\nu} Q_{\nu}(\theta u) = I_{\nu}^\nu P_{\nu} Q_{\nu}(\theta u) = \gamma_{\nu} P Q.
\]

From (4.2), we obtain \( c_{\epsilon}^\epsilon V_{\min}^\epsilon P_{\max}^\epsilon Q_{\max}^\epsilon = c_{\epsilon} \). This completes the proof. \( \square \)

Next we only truncate the functional \( V(x) \) and \( P(x) \) with \( a = \nu \) and \( b \in (P_{\infty}, P_{\nu}) \) and consider the truncated energy functional

\[
\tilde{I}_{\epsilon}^{ab}(u) = \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta) u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V^{\epsilon}(x) u^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} |\phi|^2 u^2 dx - \int_{\mathbb{R}^3} P^{\epsilon}(x) F(u) dx - \frac{1}{2 \epsilon} \int_{\mathbb{R}^3} Q(\epsilon x) |u|^{2 \epsilon} dx.
\]
The corresponding Nehari manifold and least energy are $\mathcal{N}^{vb}_{\epsilon}$ and $\tilde{c}^{vb}_{\epsilon}$, respectively.

We have an important lower bound for the least energy $\tilde{c}^{vb}_{\epsilon}$.

**Lemma 4.2.** $\tilde{c}^{vb}_{\epsilon} \geq V_{vb_{\max}}$.  

*Proof.* Since $V^{\prime}_{\epsilon}(x) \geq v$, $P^{b}_{\epsilon}(x) \leq b, Q_{\epsilon}(x) \leq Q_{\max}$, from the characterization of the value $\gamma_{vb_{\max}}$, we know that

$$\inf_{u \in H_{\epsilon} \backslash \{0\}} \max_{\theta \geq 0} \tilde{T}^{vb}_{\epsilon}(\theta u) \geq \inf_{u \in H_{\epsilon} \backslash \{0\}} \max_{\theta \geq 0} I_{vb_{\max}}(\theta u),$$

which gives

$$\tilde{c}^{vb}_{\epsilon} \geq V_{vb_{\max}}.$$

This completes the proof. $\square$

**Lemma 4.3.** $c_{\epsilon}$ is attained at some positive $u_{\epsilon}$ for small $\epsilon > 0$.

*Proof.* Similar to the arguments of Lemma 3.2, there exists a sequence $\{w_{n}\} \subseteq S_{\epsilon}$ with $Y_{\epsilon}(w_{n}) \to c_{\epsilon}, Y^{\prime}_{\epsilon}(w_{n}) \to 0$ as $n \to \infty$. Let $u_{n} = m_{\epsilon}(w_{n})$, by the definition of $m_{\epsilon}$, we know $u_{n} \in \mathcal{N}_{\epsilon}$ for all $n \in \mathbb{N}$. By Lemma 2.9, one has $I_{\epsilon}(u_{n}) \to c_{\epsilon}, I^{\prime}_{\epsilon}(u_{n}) \to 0$ as $n \to \infty$. Moreover, we know that $\{u_{n}\}$ is bounded in $H_{\epsilon}$ by Lemma 2.7. Assume that $u_{n} \rightharpoonup u_{\epsilon} \in H_{\epsilon}$, then by Lemma 2.2(ii) and Lemma 2.3, we have $I^{\prime}_{\epsilon}(u_{\epsilon}) = 0$. If $u_{\epsilon} \neq 0$, it is easy to check that $I_{\epsilon}(u_{\epsilon}) = c_{\epsilon}$. Next we show that $u_{\epsilon} \neq 0$ for small $\epsilon > 0$. Assume by contradiction that there exists a sequence $\epsilon_{j} \to 0$ such that $u_{\epsilon_{j}} = 0$, then $u_{\epsilon_{j}} \to 0$ in $H_{\epsilon}$, and thus $u_{\epsilon_{j}} \to 0$ in $L_{loc}^{2}(\mathbb{R}^{3})$ for $r \in [1, 2^{\ast})$ and $u_{\epsilon_{j}}(x) \to 0$ a.e. in $x \in \mathbb{R}^{3}$.

By $(A_{1})$, choose $b \in (P_{\infty}, P_{Q})$ and consider the functional $\tilde{T}^{vb}_{\epsilon}$, where $v$ is defined in (4.1). Let $\theta_{n} > 0$ be such that $\theta_{n}u_{n} \in \mathcal{N}^{vb}_{\epsilon_{j}}$, from Lemma 2.4(ii), $(\theta_{n})$ is bounded. Assume $\theta_{n} \to \theta$ as $n \to \infty$. By $(A_{1})$ again, the set $\{x \in \mathbb{R}^{3} : V^{\prime}_{\epsilon}(x) < v\}$ is bounded. Thus,

$$\int_{\mathbb{R}^{3}} (V^{\prime}_{\epsilon}(x) - V(\epsilon_{j}x))|\theta_{n}u_{n}|^{2} \, dx = \int_{\{V_{\epsilon}(x) < v\}} (v - V(\epsilon_{j}x))|\theta_{n}u_{n}|^{2} \, dx = o(1).$$

(4.5)

Similarly, since $b > P_{\infty}$ implies $\{x \in \mathbb{R}^{3} : P_{\epsilon}(x) \geq b\}$ is bounded and $f$ is subcritical growth, we have

$$\int_{\mathbb{R}^{3}} (P(\epsilon_{j}x) - P^{b}_{\epsilon}(x)) F(\theta_{n}u_{n}) \, dx = o(1).$$

(4.6)

Therefore, by (4.5)–(4.6) and $I_{\epsilon_{j}}(\theta_{n}u_{n}) \leq I_{\epsilon_{j}}(u_{n})$, we have

$$\tilde{c}^{vb}_{\epsilon_{j}} \leq \tilde{T}^{vb}_{\epsilon_{j}}(\theta_{n}u_{n}) = I_{\epsilon_{j}}(\theta_{n}u_{n}) + \frac{1}{2} \int_{\mathbb{R}^{3}} (V^{\prime}_{\epsilon}(x) - V(\epsilon_{j}x))|\theta_{n}u_{n}|^{2} \, dx + \int_{\mathbb{R}^{3}} (P(\epsilon_{j}x) - P^{b}_{\epsilon}(x)) F(\theta_{n}u_{n}) \, dx$$

$$= I_{\epsilon_{j}}(\theta_{n}u_{n}) + o(1) = I_{\epsilon_{j}}(u_{n}) + o(1),$$

which implies that $\tilde{c}^{vb}_{\epsilon_{j}} \leq c_{\epsilon_{j}}$ as $n \to \infty$. Notice that $\tilde{c}^{vb}_{\epsilon_{j}} \geq \gamma_{vb_{\max}}$ by Lemma 4.2. Thus, we have

$$\gamma_{vb_{\max}} \leq c_{\epsilon_{j}}.$$

In virtue of Lemma 4.1, letting $\epsilon_{j} \to 0$ yields

$$\gamma_{vb_{\max}} \leq \gamma_{v^{2}P_{\max}}.$$

Applying Lemma 3.3 and the fact that $b < P_{Q}$ yield a contradiction. Therefore, $c_{\epsilon}$ is attained at some $u_{\epsilon} \neq 0$ for small $\epsilon > 0$. Moreover, similar to Lemma 3.2, $u_{\epsilon}$ is a positive solution of the system (2.2) and the proof is completed. $\square$
5 | CONCENTRATION AND CONVERGENCE OF GROUND STATE SOLUTIONS

In this section, we are devoted to the concentration behavior of the ground state solutions $u_\varepsilon$ as $\varepsilon \to 0$. We will prove the following results.

**Theorem 5.1.** Let $u_\varepsilon$ be a solution of the system (2.2) given by Lemma 4.3, then $u_\varepsilon$ possesses a global maximum point $y_\varepsilon$ such that, up to a subsequence, $\varepsilon y_\varepsilon \to x_0$ as $\varepsilon \to 0$, and $v_\varepsilon(x) := u_\varepsilon(x + y_\varepsilon)$ converges in $H^4(\mathbb{R}^3)$ to a positive ground state solution of

$$
\begin{cases}
(-\Delta)^s u + V(x_0)u + \phi u = P(x_0) f(u) + Q(x_0) |u|^{2^*_s - 2}u, & \text{in } \mathbb{R}^3, \\
(-\Delta)^s \phi = u^2, & \text{in } \mathbb{R}^3.
\end{cases}
$$

In particular, if $V \cap P \cap Q \neq \emptyset$, then $\lim_{\varepsilon \to 0} \text{dist}(\varepsilon y_\varepsilon, \Psi \cap \mathbb{R}^3) = 0$, and up to a subsequence, $v_\varepsilon$ converges in $H^4(\mathbb{R}^3)$ to a positive ground state solution of

$$
\begin{cases}
(-\Delta)^s u + V_{\text{min}} u + \phi u = P_{\text{max}} f(u) + Q_{\text{max}} |u|^{2^*_s - 2}u, & \text{in } \mathbb{R}^3, \\
(-\Delta)^s \phi = u^2, & \text{in } \mathbb{R}^3.
\end{cases}
$$

**Lemma 5.2.** There exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*)$, there exist $\{y_\varepsilon\} \subset \mathbb{R}^3$ and $\tilde{R}, \delta > 0$ such that

$$
\int_{B_{\tilde{R}}(y_\varepsilon)} u_\varepsilon^2 \, dx \geq \delta.
$$

**Proof.** Assume by contradiction that there exists a sequence $\varepsilon_j \to 0$ as $j \to \infty$, such that for any $R > 0$,

$$
\lim_{j \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} u_{\varepsilon_j}^2 \, dx = 0.
$$

Thus, by Lemma 2.3, we have

$$
u_{\varepsilon_j} \to 0 \text{ in } L^r(\mathbb{R}^3) \text{ for } 2 < r < 2^*_s.
$$

Thus, since the potential function $P$ is bounded and (2.5), we have

$$
\int_{\mathbb{R}^3} P(\varepsilon_j x) F(u_{\varepsilon_j}) \, dx \to 0, \quad \int_{\mathbb{R}^3} P(\varepsilon_j x) f(u_{\varepsilon_j}) u_{\varepsilon_j} \, dx \to 0 \quad \text{as } j \to \infty.
$$

Moreover, since $4s + 2t > 3$, we have that $2 < \frac{12}{3 + 2t} < 2^*_s$, and by Lemma 2.1 (iii), we have

$$
\int_{\mathbb{R}^3} \phi_t' u_{\varepsilon_j}^2 \, dx \to 0 \quad \text{as } j \to \infty.
$$

Notice that

$$
I_{\varepsilon_j}(u_{\varepsilon_j}) - \frac{1}{2^*_s} \left( I'_{\varepsilon_j}(u_{\varepsilon_j}), u_{\varepsilon_j} \right) = \frac{s}{3} \|u_{\varepsilon_j}\|_{s_j}^2 + \frac{4s - 3}{12} \int_{\mathbb{R}^3} \phi_t u_{\varepsilon_j}^2 \, dx - \int_{\mathbb{R}^3} P(\varepsilon_j x) F(u_{\varepsilon_j}) \, dx + \frac{1}{2^*_s} \int_{\mathbb{R}^3} P(\varepsilon_j x) f(u_{\varepsilon_j}) u_{\varepsilon_j} \, dx.
$$

Thus, by (5.1)–(5.2), we have

$$
\int_{\mathbb{R}^3} (-\Delta)^s u_{\varepsilon_j}^2 \, dx \leq \frac{3}{s} c_{\varepsilon_j} + o(1).
$$

Similarly, we have

$$
\int_{\mathbb{R}^3} Q(\varepsilon_j x) |u_{\varepsilon_j}|^{2^*_s} \, dx = \frac{3}{s} c_{\varepsilon_j} + o(1).
$$
Moreover,
\[ \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u_{\varepsilon} \right|^2 \, dx - \int_{\mathbb{R}^3} Q(\varepsilon x) |u_{\varepsilon}|^{2^*} \, dx \leq o(1). \]

Thus, by the best constant of the Sobolev imbedding, we get
\[
\int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u_{\varepsilon} \right|^2 \, dx \leq \int_{\mathbb{R}^3} Q(\varepsilon x) \left| u_{\varepsilon} \right|^{2^*} \, dx + o(1)
\]
\[
= \left( \int_{\mathbb{R}^3} Q(\varepsilon x) \left| u_{\varepsilon} \right|^{2^*} \, dx \right)^{\frac{2}{2^*}} \left( \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u_{\varepsilon} \right|^2 \, dx \right)^{\frac{2^*}{2}} + o(1)
\]
\[
= \left( \int_{\mathbb{R}^3} \left| \nabla u_{\varepsilon} \right|^2 \, dx \right)^{\frac{2}{2^*}} \left( \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u_{\varepsilon} \right|^2 \, dx \right)^{\frac{2^*}{2}} + o(1)
\]
\[
\leq \frac{Q_{\max}}{S_s} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u_{\varepsilon} \right|^2 \, dx \left( \int_{\mathbb{R}^3} Q(\varepsilon x) \left| u_{\varepsilon} \right|^{2^*} \, dx \right)^{\frac{2^*}{2}} + o(1),
\]
which implies
\[
\liminf_{j \to \infty} c_{\varepsilon_j} \geq \frac{s}{3\cdot 2^{\frac{2^*}{s}}} S_s^{\frac{3^*}{2}}
\]
a contradiction with Lemma 3.1 and Lemma 4.1.

Set \( v_{\varepsilon}(x) := u_{\varepsilon}(x + y_\varepsilon) \), then \( v_{\varepsilon} \) satisfies
\[
(-\Delta)^{\frac{s}{2}} v_{\varepsilon} + V(x + y_\varepsilon) v_{\varepsilon} + \phi_{v_{\varepsilon}} v_{\varepsilon} = P(\varepsilon(x + y_\varepsilon)) f(v_{\varepsilon}) + Q(\varepsilon(x + y_\varepsilon)) |v_{\varepsilon}|^{2^* - 2} v_{\varepsilon},
\]
with energy
\[
J_s(v_{\varepsilon}) = \frac{1}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} v_{\varepsilon} \right|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x + y_\varepsilon) |v_{\varepsilon}|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{v_{\varepsilon}} v_{\varepsilon}^2 \, dx
\]
\[
- \int_{\mathbb{R}^3} P(\varepsilon(x + y_\varepsilon)) F(v_{\varepsilon}) \, dx - \frac{1}{2s} \int_{\mathbb{R}^3} Q(\varepsilon(x + y_\varepsilon)) |v_{\varepsilon}|^{2^*} \, dx
\]
\[
= J_s(v_{\varepsilon}) - \frac{1}{4} \langle J'_s(v_{\varepsilon}), v_{\varepsilon} \rangle
\]
\[
= \frac{1}{4} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} v_{\varepsilon} \right|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} V(x + y_\varepsilon) |v_{\varepsilon}|^2 \, dx
\]
\[
+ \frac{1}{4} \int_{\mathbb{R}^3} P(\varepsilon(x + y_\varepsilon)) [f(v_{\varepsilon}) v_{\varepsilon} - 4 F(v_{\varepsilon})] \, dx + \frac{4s - 3}{12} \int_{\mathbb{R}^3} Q(\varepsilon(x + y_\varepsilon)) |v_{\varepsilon}|^{2^*} \, dx
\]
\[
= I_s(u_{\varepsilon}) - \frac{1}{4} \langle I'_s(u_{\varepsilon}), u_{\varepsilon} \rangle = I_s(u_{\varepsilon}) = c_{\varepsilon}.
\]
We may assume \( v_{\varepsilon} \to u \) in \( H_s \), and \( v_{\varepsilon} \to u \) in \( L^r_{loc}(\mathbb{R}^3) \) for \( r \in [1, 2^*_s) \) with \( u \neq 0 \).

By condition \( (A_0) \), without loss of generality, we may assume that \( V(\varepsilon y_\varepsilon) \to V_0, P(\varepsilon y_\varepsilon) \to P_0 \) and \( Q(\varepsilon y_\varepsilon) \to Q_0 \) as \( \varepsilon \to 0 \).

**Lemma 5.3.** \( u \) is a positive ground state solution of
\[
(-\Delta)^{\frac{s}{2}} u + V_0 u + \phi'_u u = P_0 f(u) + Q_0 |u|^{2^* - 2} u \text{ in } \mathbb{R}^3.
\]

**Proof.** By (5.3), for any \( \varphi \in C_0^{\infty}(\mathbb{R}^3) \), there holds that
\[
0 = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} \left( (-\Delta)^{\frac{s}{2}} v_{\varepsilon} + V(x + y_\varepsilon) v_{\varepsilon} + \phi'_{v_{\varepsilon}} v_{\varepsilon} - P(\varepsilon(x + y_\varepsilon)) f(v_{\varepsilon}) - Q(\varepsilon(x + y_\varepsilon)) |v_{\varepsilon}|^{2^* - 2} v_{\varepsilon} \right) \varphi \, dx.
\]

(5.5)
Since \( V, P, Q \) are all continuous and bounded, we have
\[
\int_{\mathbb{R}^3} V(\epsilon(x + y_\epsilon)) v_\epsilon \psi \, dx \to V_0 \int_{\mathbb{R}^3} u \psi \, dx, \quad \int_{\mathbb{R}^3} P(\epsilon(x + y_\epsilon)) f(v_\epsilon) \psi \, dx \to P_0 \int_{\mathbb{R}^3} f(u) \psi \, dx
\]
and
\[
\int_{\mathbb{R}^3} Q(\epsilon(x + y_\epsilon)) |v_\epsilon|^{2^*_s - 2} v_\epsilon \psi \, dx \to Q_0 \int_{\mathbb{R}^3} |u|^{2^*_s - 2} u \psi \, dx,
\]
which combined with (5.5) implies that
\[
(-\Delta)^s u + V_0 u + \phi_\epsilon^s u = P_0 f(u) + Q_0 |u|^{2^*_s - 2} u, \quad \text{in } \mathbb{R}^3,
\]
that is, \( u \) solves (5.4) with energy
\[
I_{V_0, P_0, Q_0}(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^s u \right|^2 \, dx + \frac{1}{4} V_0 \int_{\mathbb{R}^3} u^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_\epsilon^s u^2 \, dx - P_0 \int_{\mathbb{R}^3} F(u) \, dx - \frac{1}{2^*_s} Q_0 \int_{\mathbb{R}^3} |u|^{2^*_s} \, dx
\]
\[
= I_{V_0, P_0, Q_0}(u) - \frac{1}{4} \langle I_{V_0, P_0, Q_0}(u), u \rangle
\]
\[
= \frac{1}{4} \int_{\mathbb{R}^3} \left| (-\Delta)^s u \right|^2 \, dx + \frac{1}{4} V_0 \int_{\mathbb{R}^3} u^2 \, dx + \frac{1}{4} P_0 \int_{\mathbb{R}^3} (f(u) - 4 F(u)) \, dx + \frac{4s - 3}{12} Q_0 \int_{\mathbb{R}^3} |u|^{2^*_s} \, dx
\]
\[
\geq \gamma_{V_0, P_0, Q_0}.
\]
By Fatou’s lemma and the proof of Lemma 4.1, we have
\[
\gamma_{V_0, P_0, Q_0} \leq \frac{1}{4} \int_{\mathbb{R}^3} \left| (-\Delta)^s u \right|^2 \, dx + \frac{1}{4} V_0 \int_{\mathbb{R}^3} u^2 \, dx + \frac{1}{4} P_0 \int_{\mathbb{R}^3} (f(u) - 4 F(u)) \, dx + \frac{4s - 3}{12} Q_0 \int_{\mathbb{R}^3} |u|^{2^*_s} \, dx
\]
\[
\leq \lim \inf_{\epsilon \to 0} \left[ \frac{1}{4} \int_{\mathbb{R}^3} \left| (-\Delta)^s v_\epsilon \right|^2 \, dx + \frac{1}{4} V_0 \int_{\mathbb{R}^3} v_\epsilon^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} P(\epsilon(x + y_\epsilon)) (f(v_\epsilon) v_\epsilon - 4 F(v_\epsilon)) \, dx \right]
\]
\[
+ \frac{4s - 3}{12} \int_{\mathbb{R}^3} Q(\epsilon(x + y_\epsilon)) |v_\epsilon|^{2^*_s} \, dx
\]
\[
= \lim \inf_{\epsilon \to 0} J_\epsilon(v_\epsilon)
\]
\[
\leq \lim \sup_{\epsilon \to 0} I_\epsilon(u_\epsilon)
\]
\[
\leq \gamma_{V_0, P_0, Q_0}.
\]
Consequently,
\[
\lim_{\epsilon \to 0} J_\epsilon(v_\epsilon) = \lim_{\epsilon \to 0} c_\epsilon = I_{V_0, P_0, Q_0}(u) = \gamma_{V_0, P_0, Q_0}.
\]  \hspace{1cm} (5.6)

Thus, \( u \) is a ground state solution of Equation (5.4). As in the proof of Lemma 3.2, \( u \) is positive. \( \square \)

**Lemma 5.4.** \( \{\epsilon y_\epsilon\} \) is bounded.

**Proof.** Suppose to the contrary that, after passing to a subsequence, \( |\epsilon y_\epsilon| \to \infty \). Since \( P(0) = P \) and \( v = V(0) \leq V(x) \) for all \( |x| \geq R \), we deduce that \( P_0 < P \) and \( v \leq V_0 \). So it follows from Lemma 3.3 that \( \gamma_{V_0, P_0, Q_0} > \gamma_{v, P_0 Q_{\text{max}}} \). However, by (5.6) and Lemma 4.1, \( c_\epsilon \to \gamma_{V_0, P_0, Q_0} \leq \gamma_{v, P_0 Q_{\text{max}}} \), which is a contradiction. Therefore, \( \{\epsilon y_\epsilon\} \) is bounded. \( \square \)

After extracting a subsequence, we may assume \( \epsilon y_\epsilon \to x_0 \) as \( \epsilon \to 0 \), then \( V_0 = V(x_0) \), \( P_0 = P(x_0) \) and \( Q_0 = Q(x_0) \).

**Lemma 5.5.** \( \lim_{\epsilon \to 0} \text{dist}(\epsilon y_\epsilon, H_P) = 0 \).
Proof. It suffices to show that \( x_0 \in H_p \). We argue by contradiction, if \( x_0 \not\in H_p \), then it is easy to check that 
\[ YV(x_0)P(x_0)Q(x_0) > Y_r P_2 Q_{\max} \text{ by (A1) and Lemma 3.3.} \] 
Therefore, by Lemma 4.1, we have 
\[ \lim_{\epsilon \to 0} c_\epsilon = YV(x_0)P(x_0)Q(x_0) > Y_r P_2 Q_{\max} \geq \lim_{\epsilon \to 0} c_\epsilon, \]
which is absurd. \( \square \)

Lemma 5.6. \( v_\epsilon \to u \) in \( H^s(\mathbb{R}^3) \).

Proof. Recall that \( u \) is a ground state solution of (5.4), we have 
\[ \frac{1}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx \leq \liminf_{\epsilon \to 0} \frac{1}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_\epsilon|^2 \, dx \]
\[ \leq \limsup_{\epsilon \to 0} \frac{1}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_\epsilon|^2 \, dx \]
\[ \leq \limsup_{\epsilon \to 0} \frac{1}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_\epsilon|^2 \, dx + \liminf_{\epsilon \to 0} \frac{1}{4} \int_{\mathbb{R}^3} V(\epsilon(x + y_\epsilon))v_\epsilon^2 \, dx - \frac{1}{4} V_0 \int_{\mathbb{R}^3} u^2 \, dx \]
\[ + \liminf_{\epsilon \to 0} \frac{1}{4} \int_{\mathbb{R}^3} P(\epsilon(x + y_\epsilon))[f(v_\epsilon)v_\epsilon - 4 F(v_\epsilon)] \, dx - \frac{1}{4} P_0 \int_{\mathbb{R}^3} [f(u)u - 4 F(u)] \, dx \]
\[ + \liminf_{\epsilon \to 0} \frac{4s - 3}{12} \int_{\mathbb{R}^3} Q(\epsilon(x + y_\epsilon))[v_\epsilon]^{2^*_s} \, dx - \frac{4s - 3}{12} Q_0 \int_{\mathbb{R}^3} |u|^{2^*_s} \, dx \]
\[ = \frac{1}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx. \]
Consequently, 
\[ \lim_{\epsilon \to 0} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_\epsilon|^2 \, dx = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx. \]
Similarly, we have 
\[ \lim_{\epsilon \to 0} \int_{\mathbb{R}^3} V(\epsilon(x + y_\epsilon))v_\epsilon^2 \, dx = V_0 \int_{\mathbb{R}^3} u^2 \, dx. \]
Notice that 
\[ \lim_{\epsilon \to 0} \left( \int_{\mathbb{R}^3} V(\epsilon(x + y_\epsilon))v_\epsilon^2 \, dx - V_0 \int_{\mathbb{R}^3} u^2 \, dx \right) = 0. \]
Thus 
\[ \lim_{\epsilon \to 0} \left\{ \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_\epsilon|^2 \, dx + V_0 \int_{\mathbb{R}^3} v_\epsilon^2 \, dx \right\} = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx + V_0 \int_{\mathbb{R}^3} u^2 \, dx. \]
Together with \( v_\epsilon \to u \) in \( H^s(\mathbb{R}^3) \), we have \( v_\epsilon \to u \) in \( H^s(\mathbb{R}^3) \). \( \square \)

To establish the \( L^\infty \)-estimate of ground state solutions, we first recall the following result which can be found in [13, (5.1.1) and (5.1.2)]. (See [42] for the proof.)
Lemma 5.7. Suppose that \( g : \mathbb{R} \to \mathbb{R} \) is convex and Lipschitz continuous with the Lipschitz constant \( L \), \( g(0) = 0 \). Then for each \( u \in H^1(\mathbb{R}^3) \), \( g(u) \in H^s(\mathbb{R}^3) \) and
\[
(-\Delta)^s g(u) \leq g'(u)(-\Delta)^s u
\] (5.7)
in the weak sense.

Remark 5.8. In fact, from the above arguments, one can see that (5.7) holds for a.e. \( x \in \mathbb{R}^3 \). Moreover, Lemma 5.7 is true for general dimension \( N \).

The following lemma plays a fundamental role in the study of behavior of the maximum points of the solutions, whose proof is related to the Moser iterative method [23].

Lemma 5.9. Let \( \varepsilon_n \to 0 \) and \( v_{\varepsilon_n} \) be a solution of the following equation
\[
(-\Delta)^s v_{\varepsilon_n} + V(\varepsilon_n(x + y_{\varepsilon_n})) v_{\varepsilon_n} + \phi_{\varepsilon_n}' v_{\varepsilon_n} = P(\varepsilon_n(x + y_{\varepsilon_n})) f(v_{\varepsilon_n}) + Q(\varepsilon_n(x + y_{\varepsilon_n})) |v_{\varepsilon_n}|^{2^*_s} v_{\varepsilon_n}, \quad \text{in } \mathbb{R}^3,
\] (5.8)
where \( y_{\varepsilon_n} \) is given in Lemma 5.2. Then \( v_{\varepsilon_n} \in L^\infty(\mathbb{R}^3) \) and there exists \( C > 0 \) such that
\[
\|v_{\varepsilon_n}\|_\infty \leq C, \quad \text{uniformly in } n \in \mathbb{N}.
\]

Moreover, \( v_{\varepsilon_n} \to u \) in \( L^q(\mathbb{R}^3) \), for all \( q \in [2, +\infty) \).

Proof. For simplicity of notations, we denote \( v_{\varepsilon_n} \) and \( y_{\varepsilon_n} \) by \( v_n \) and \( y_n \), respectively. Define
\[
h(x, v_n) := P(\varepsilon_n(x + y_n)) f(v_n) + Q(\varepsilon_n(x + y_n)) |v_n|^{2^*_s} v_n - V(\varepsilon_n(x + y_n)) v_n - \phi_n'(v_n).
\]
From Lemma 5.6, \( \{v_n\} \) is bounded in \( H^s(\mathbb{R}^3) \), and hence in \( L^q(\mathbb{R}^3) \) for any \( q \in [2, 2^*_s] \). So there exists some \( C > 0 \) such that
\[
\|v_n\|_q \leq C,
\]
uniformly in \( n \). Since \( v_n \) is a solution of (5.8), then
\[
\phi_n'(x) = \int_{\mathbb{R}^3} \frac{v_n^2(y)}{|x - y|^{3-2t}} \, dy = \int_{|x - y| \leq 1} \frac{v_n^2(y)}{|x - y|^{3-2t}} \, dy + \int_{|x - y| > 1} \frac{v_n^2(y)}{|x - y|^{3-2t}} \, dy
\]
\[
\leq \int_{|x - y| \leq 1} \frac{v_n^2(y)}{|x - y|^{3-2t}} \, dy + \int_{|x - y| > 1} v_n^2(y) \, dy
\]
\[
\leq \left( \int_{|x - y| \leq 1} \frac{1}{|x - y|^{(3-2t)r'}} \, dy \right)^{\frac{1}{r'}} \left( \int_{|x - y| \leq 1} v_n^{2r}(y) \, dy \right)^{\frac{1}{r}} + C
\]
\[
\leq C,
\]
where \( r'(3-2t) < 3 \), \( 2r \in \left[2, 2^*_s\right] \), \( \frac{1}{r'} + \frac{1}{r} = 1 \) since \( 2s + 2t > 3 \). Therefore, we have
\[
|h(x, v_n)| \leq C(\|v_n\| + |v_n|^{p-1}) \leq C \left(1 + |v_n|^{2^*_s-1}\right).
\] (5.9)

Let \( T > 0 \), we follow [13] and define
\[
H(\theta) = \begin{cases} 0, & \text{if } \theta \leq 0, \\ \theta^\beta, & \text{if } 0 < \theta < T, \\ \beta T^{\theta-1}(\theta - T) + T^\beta, & \text{if } \theta \geq T, \end{cases}
\]
with $\beta > 1$ to be determined later. Since $H$ is convex and Lipschitz with constant $L_0 = \beta T^{\beta - 1}$ and $H(0) = 0$, by Lemma 5.7, we have $H(v_n) \in D^{2,2}(\mathbb{R}^3)$ and

$$(-\Delta)\beta H(v_n) \leq H'(v_n)(-\Delta)^\beta v_n$$

(5.10)

in the weak sense. Thus, from $H(v_n) \in D^{2,2}(\mathbb{R}^3)$, the self-adjointness of the operator $(-\Delta)^{\beta/2}$ and (5.9)–(5.10), we have

$$\|H(v_n)\|_{L^2} \leq C \int_{\mathbb{R}^3} \left|(-\Delta)^{\beta/2} H(v_n)\right|^2 dx = C \int_{\mathbb{R}^3} H(v_n)(-\Delta)^\beta H(v_n) dx$$

$$\leq C \int_{\mathbb{R}^3} H(v_n)H'(v_n)(-\Delta)^\beta v_n dx = C \int_{\mathbb{R}^3} H(v_n)H'(v_n)h(x,v_n) dx$$

$$\leq C \int_{\mathbb{R}^3} H(v_n)H'(v_n) dx + C \int_{\mathbb{R}^3} H(v_n)H'(v_n)v_n^{2\gamma-1} dx.$$  

Using the fact that $H(v_n)H'(v_n) \leq \beta^2 v_n^{2\beta-1}$ and $v_n H'(v_n) \leq \beta H(v_n)$, we have

$$\left(\int_{\mathbb{R}^3} (H(v_n))^{2\gamma} dx\right)^{\frac{1}{2\gamma}} \leq C \beta^2 \left(\int_{\mathbb{R}^3} v_n^{2\beta-1} dx + \int_{\mathbb{R}^3} (H(v_n))^{2\gamma} v_n^{3\gamma-2} dx\right),$$

(5.11)

where $C$ is a positive constant that does not depend on $\beta$. Notice that the last integral is well defined for $T$ in the definition of $H$. Indeed

$$\int_{\mathbb{R}^3} (H(v_n))^{2\gamma} v_n^{\gamma-2} dx = \int_{\{v_n \leq \hat{T}\}} (H(v_n))^{2\gamma} v_n^{\gamma-2} dx + \int_{\{v_n > \hat{T}\}} (H(v_n))^{2\gamma} v_n^{\gamma-2} dx$$

$$\leq \hat{T}^{2\beta-2} \int_{\mathbb{R}^3} v_n^{\gamma} dx + C \int_{\mathbb{R}^3} v_n^{\gamma} dx < \infty.$$  

We choose now $\beta$ in (5.11) such that $2\beta - 1 = 2_s^\ast$, and we name it $\beta_1$, that is

$$\beta_1 := \frac{2^s + 1}{2}.$$  

(5.12)

Let $\overset{\sim}{R} > 0$ to be fixed later. Attending to the last integral in (5.11) and applying Holder’s inequality with exponents $\gamma := \frac{2^s}{2}$ and $\gamma' := \frac{2^s}{3\gamma - 2}$,

$$\int_{\mathbb{R}^3} (H(v_n))^{2} v_n^{2\gamma-2} dx = \int_{\{v_n \leq \overset{\sim}{R}\}} (H(v_n))^{2} v_n^{2\gamma-2} dx + \int_{\{v_n > \overset{\sim}{R}\}} (H(v_n))^{2} v_n^{2\gamma-2} dx$$

$$\leq \int_{\{v_n \leq \overset{\sim}{R}\}} \frac{(H(v_n))^2}{v_n} \overset{\sim}{R}^{2\gamma-1} dx + \left(\int_{\mathbb{R}^3} (H(v_n))^{2\gamma} dx\right)^{\frac{2^s}{2}} \left(\int_{\{v_n > \overset{\sim}{R}\}} v_n^{\gamma} dx\right)^{\frac{2^s}{3\gamma - 2}}.$$  

(5.13)

By the monotone convergence theorem, we can choose $\overset{\sim}{R}$ large enough so that

$$\left(\int_{\{v_n > \overset{\sim}{R}\}} v_n^{\gamma} dx\right)^{\frac{2^s}{2}} \leq \frac{1}{2C \beta_1^2},$$

where $C$ is the constant appearing in (5.11). Therefore, we can absorb the last term in (5.13) by the left hand side of (5.11) to get

$$\left(\int_{\mathbb{R}^3} (H(v_n))^{2^s} dx\right)^{\frac{2^s}{2}} \leq 2C \beta_1^2 \left(\int_{\mathbb{R}^3} v_n^{2^s} dx + \overset{\sim}{R}^{2\gamma-1} \int_{\mathbb{R}^3} \frac{(H(v_n))^2}{v_n} dx\right).$$
Now we use the fact that \( H(v_n) \leq v_n^{\beta_1} \) and (5.12) once again in the right hand side and we take \( T \to \infty \) we obtain

\[
\left( \int_{\mathbb{R}^3} v_n^{2\beta_1} \, dx \right)^{\frac{2}{\beta_1}} \leq 2C\beta_1^2 \left( \int_{\mathbb{R}^3} v_n^{2\beta_1 - 1} \, dx + \int_{\mathbb{R}^3} v_n^{2\beta_1 + 2\gamma_2 - 2} \, dx \right),
\]

and therefore

\[
v_n \in L^{2\beta_1} (\mathbb{R}^3), \quad \text{for all } n,
\]

(5.14)

and

\[
\|v_n\|_{2\beta_1} \leq C,
\]

(5.15)

uniformly in \( n \).

Let us suppose now \( \beta > \beta_1 \). Thus, using that \( H(v_n) \leq v_n^{\beta} \) in the right hand side of (5.11) and letting \( T \to \infty \) we get

\[
\left( \int_{\mathbb{R}^3} v_n^{2\beta} \, dx \right)^{\frac{2}{\beta}} \leq C \beta^2 \left( \int_{\mathbb{R}^3} v_n^{2\beta - 1} \, dx + \int_{\mathbb{R}^3} v_n^{2\beta + 2\gamma_2 - 2} \, dx \right).
\]

(5.16)

Set \( r_0 := \frac{2\gamma_2 - 1}{2(\beta_1 - 1)} \) and \( r_1 := 2\beta - 1 - r_0 \). Notice that, since \( \beta > \beta_1 \), then \( 0 < r_0 < 2^+ \), \( r_1 > 0 \). Hence, applying Young’s inequality with exponents \( \gamma := 2^+ / r_0 \) and \( \gamma' := 2^+/2^+ - r_0 \), we have

\[
\int_{\mathbb{R}^3} v_n^{2\beta - 1} \, dx \leq \int_{\mathbb{R}^3} v_n^{2\beta} \, dx + \frac{2^+}{2^+ - r_0} \int_{\mathbb{R}^3} v_n^{\frac{2\gamma_2 - 1}{2^+ - r_0}} \, dx
\]

\[
\leq \int_{\mathbb{R}^3} v_n^{2\beta} \, dx + \int_{\mathbb{R}^3} v_n^{2\beta + 2\gamma_2 - 2} \, dx
\]

\[
\leq C \left( 1 + \int_{\mathbb{R}^3} v_n^{2\beta + 2\gamma_2 - 2} \, dx \right),
\]

with \( C > 0 \) independent of \( \beta \). Plugging into (5.16).

\[
\left( \int_{\mathbb{R}^3} v_n^{2\beta} \, dx \right)^{\frac{2}{\beta}} \leq C \beta^2 \left( 1 + \int_{\mathbb{R}^3} v_n^{2\beta + 2\gamma_2 - 2} \, dx \right),
\]

with \( C \) changing from line to line, but remaining independent of \( \beta \). Therefore

\[
\left( 1 + \int_{\mathbb{R}^3} v_n^{2\beta} \, dx \right)^{\frac{1}{2(\beta - 1)}} \leq (C \beta^2)^{\frac{1}{2(\beta - 1)}} \left( 1 + \int_{\mathbb{R}^3} v_n^{2\beta + 2\gamma_2 - 2} \, dx \right)^{\frac{1}{2(\beta - 1)}}.
\]

(5.17)

Repeating this argument we will define a sequence \( \beta_m, m \geq 1 \) such that

\[
2\beta_{m+1} + 2^+ - 2 = 2^+ \beta_m.
\]

Thus,

\[
\beta_{m+1} - 1 = \left( \frac{2^+}{2} \right)^m (\beta_1 - 1).
\]

Replacing it in (5.17) one has

\[
\left( 1 + \int_{\mathbb{R}^3} v_n^{2\beta_{m+1}} \, dx \right)^{\frac{1}{2(\beta_{m+1} - 1)}} \leq (C \beta_m^2)^{\frac{1}{2(\beta_{m+1} - 1)}} \left( 1 + \int_{\mathbb{R}^3} v_n^{2\beta_{m+1} - 2} \, dx \right)^{\frac{1}{2(\beta_{m+1} - 1)}}.
\]
Defining $C_{m+1} := C\beta_{m+1}^2$ and

$$A_m := \left(1 + \int_{\mathbb{R}^3} v_n^{2^* \beta_m} \, dx\right)^{-\frac{1}{2^* \beta_m - 1}}.$$  

So

$$A_{m+1} \leq (C_{m+1})^{\frac{1}{2^* \beta_m - 1}} A_m, \quad m = 1, 2, \ldots.$$  

Now from an iterative procedure we conclude that there exists a constant $C_0 > 0$ independent of $m$, such that

$$A_m \leq \prod_{k=1}^m C_k \frac{1}{2^* \beta_k - 1} A_1 \leq C_0 A_1, \quad \text{for all } m.$$  

Thus, from (5.14),

$$\|v_n\|_{\infty} \leq C_0 A_1 < \infty,$$

and hence $v_n \in L^\infty(\mathbb{R}^3)$. By (5.15),

$$\|v_n\|_{\infty} \leq C,$$  

uniformly in $n \in \mathbb{N}$. Finally, by interpolation on the $L^q$-spaces and $v_n \to u$ in $L^2(\mathbb{R}^3)$, we have $v_n \to u$ in $L^q(\mathbb{R}^3)$, for all $q \in [2, +\infty)$. This finishes the proof of Lemma 5.9.

**Lemma 5.10.** $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n$.

**Proof.** Since $v_n$ satisfies the equation

$$(-\Delta)^{\frac{1}{2}} v_n + v_n = Y_n, \quad x \in \mathbb{R}^3,$$

where

$$Y_n(x) = v_n(x) - V(\epsilon_n(x + y_n)) v_n(x) - \phi^\prime(\epsilon_n(x + y_n)) v_n(x) + P(\epsilon_n(x + y_n)) f(v_n(x)) + Q(\epsilon_n(x + y_n)) v_n^{2^* - 1}(x), \quad x \in \mathbb{R}^3,$$

Putting $Y(x) = u(x) - V(x_0) u(x) - \phi^\prime(u(x)) u(x) + P(x_0) f(u(x)) + Q(x_0) u^{2^* - 1}(x)$, by Lemma 5.9, we see that

$$Y_n \to Y \text{ in } L^q(\mathbb{R}^3), \quad \text{for all } q \in [2, +\infty),$$

and there exists a $C_2 > 0$ such that

$$\|Y_n\|_{\infty} \leq C_2, \quad \text{for all } n \in \mathbb{N}.$$

From [15], we have that

$$v_n(x) = G \ast Y_n = \int_{\mathbb{R}^3} G(x - y) Y_n(y) \, dy,$$

where $G$ is the Bessel kernel

$$G(x) = F^{-1} \left( \frac{1}{1 + |\xi|^{2^*}} \right).$$

It is known from [15, Theorem 3.3] that, $G$ is positive, radially symmetric and smooth in $\mathbb{R}^3 \setminus \{0\}$; there is $C > 0$ such that $G(x) \leq C|x|^{3-2^*}$, and $G \in L^q(\mathbb{R}^3)$, for all $q \in \left[1, \frac{3}{3-2^*}\right)$. Now argue as in the proof of [1, Lemma 2.6], we conclude that

$$v_n(x) \to 0 \quad \text{as } |x| \to \infty,$$

uniformly in $n \in \mathbb{N}$.
Proof of Theorem 5.1. First we claim that there exists a $\rho_0 > 0$ such that $\|v_n\|_\infty \geq \rho_0$, for all $n \in \mathbb{N}$. In fact, suppose that $\|v_n\|_\infty \to 0$ as $n \to \infty$. Then, by (2.5), we have

$$\int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{2}{3}} v_n \right|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon_n(x + y_n)) v_n^2 dx$$

$$\leq \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{2}{3}} v_n \right|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon_n(x + y_n)) v_n^2 dx + \int_{\mathbb{R}^3} \phi_{\varepsilon_n} v_n^2 dx$$

$$= \int_{\mathbb{R}^3} P(\varepsilon_n(x + y_n)) f(v_n) v_n dx + \int_{\mathbb{R}^3} Q(\varepsilon_n(x + y_n)) |v_n|^{2^*_q} dx$$

$$\leq P_{\max} \int_{\mathbb{R}^3} \left( v^4_n + v^p_n \right) dx + Q_{\max} \int_{\mathbb{R}^3} v^2_n dx$$

$$\leq C P_{\max} \|v_n\|_{\infty}^2 \int_{\mathbb{R}^3} v^2_n dx + C P_{\max} \|v_n\|_{\infty}^{p-2} \int_{\mathbb{R}^3} v^2_n dx + Q_{\max} \|v_n\|_{\infty}^{2^*_q-2} \int_{\mathbb{R}^3} v^2_n dx$$

$$= \left( C P_{\max} \|v_n\|_{\infty}^2 + C P_{\max} \|v_n\|_{\infty}^{p-2} + Q_{\max} \|v_n\|_{\infty}^{2^*_q-2} \right) \int_{\mathbb{R}^3} v^2_n dx$$

$$\leq \frac{V_{\min}}{2} \int_{\mathbb{R}^3} v^2_n dx$$

for $n$ large enough. This implies that $\|v_n\| = 0$ for $n$ large enough, which is impossible because $v_n \to u$ in $H^s(\mathbb{R}^3)$ and $u \neq 0$. Then, the claim is true.

From [32, Proposition 2.9], we see that $v_n \in C^{1,\alpha}(\mathbb{R}^3)$ for any $\alpha < 2s - 1$. Thus, we know that $v_n$ has a global maximum point $p_n$ by (5.20) and the claim above, we also see that $p_n \in B_{R_0}(0)$ for some $R_0 > 0$. Hence, the global maximum point of $u_n$ given by $p_n + y_n$. Define $\psi_n(x) \mathrel{=:} u_{\varepsilon_n}(x + y_n + p_n)$, where $u_{\varepsilon_n}(x) = v_n(x + y_n)$. Since $\{p_n\} \subset B_{R_0}(0)$ is bounded, then we know that $\{\varepsilon_n(x + y_n + p_n)\}$ is bounded and $\varepsilon_n(x + y_n + p_n) \to x_0 \in H^s$. It follows from the boundedness of $\{u_{\varepsilon_n}\}$ that $\{\psi_n\}$ is bounded in $H^s(\mathbb{R}^3)$, and we assume that $\psi_n \to \psi$ in $H^s(\mathbb{R}^3)$, $\psi_n \to \psi$ in $L^q_{loc}(\mathbb{R}^3)$ for $q \in \left[1, 2s \right]$. On the other hand, by Lemma 4.1, we have

$$\int_{B_{\delta R_0}(0)} \psi^2_n(x) dx \geq \int_{\{x + p_n \subset R\}} \psi^2_n(x) dx = \int_{B_{\delta R}(y_n)} u^2_{\varepsilon_n}(x) dx \geq \sigma,$$

so we obtain $\psi \neq 0$. Moreover, similar to the argument above, we know that $\psi$ is a ground state solution of (5.4) and $\psi_n \to \psi$ in $H^s(\mathbb{R}^3)$. Therefore, $\psi$ possesses a same properties as $v_n$, and we can assume that $y_n$ is a global maximum point of $u_{\varepsilon_n}$. Then, by Lemma 5.2–5.6 above, one can obtain Theorem 5.1. \hfill \square

6 | DECAY ESTIMATES

In this section, we estimate the decay properties of $v_n$.

Lemma 6.1. There exist $C > 0$ such that

$$v_n(x) \leq \frac{C}{1 + |x|^{3+2s}}, \text{ for all } x \in \mathbb{R}^3.$$

Proof. According to [15, Lemma 4.2], there exists a continuous function $\tilde{\omega}$ such that

$$0 < \tilde{\omega}(x) \leq \frac{C}{1 + |x|^{3+2s}}, \quad (6.1)$$

and

$$(-\Delta)^s \tilde{\omega} + \frac{V_{\min}}{2} \tilde{\omega} = 0, \text{ in } \mathbb{R}^3 \setminus B_R(0) \quad (6.2)$$
for some suitable \( \bar{R} > 0 \). Thanks to (5.20), we have that \( v_n(x) \to 0 \) as \( |x| \to \infty \) uniformly in \( n \). Therefore, for some large \( R_1 > 0 \), we obtain

\[
(-\Delta)^s v_n + \frac{V_{\text{min}}}{2} v_n = (-\Delta)^s v_n + V(\varepsilon_n(x + y_n))v_n - \left(V(\varepsilon_n(x + y_n)) - \frac{V_{\text{min}}}{2}\right)v_n
\]

\[
= -\phi^{s}_{v_n} v_n + P(\varepsilon_n(x + y_n)) f(v_n) + Q(\varepsilon_n(x + y_n)) |v_n|^{2s-2}v_n - \left(V(\varepsilon_n(x + y_n)) - \frac{V_{\text{min}}}{2}\right)v_n
\]

\[
\leq \left( C_{\text{max}}(v_n^3 + v_n^{\rho-1}) + Q_{\text{max}} |v_n|^{2s-2}v_n - \frac{V_{\text{min}}}{2}\right)v_n
\]

\[
= \left( C_{\text{max}}(v_n^3 + v_n^{\rho-2}) + Q_{\text{max}} v_n^{2s-2} - \frac{V_{\text{min}}}{2}\right)v_n
\]

\[
\leq 0,
\]

(6.3)

for \( x \in \mathbb{R}^3 \setminus B_{R_1}(0) \). Now we take \( R_2 := \max \{ \bar{R}, R_1 \} \) and set

\[
z_n := (m + 1)\tilde{\omega} - bv_n,
\]

(6.4)

where \( m := \sup_{n \in \mathbb{N}} \|v_n\|_{\infty} < \infty \) and \( b := \min_{\bar{B}_{R_2}(0)} \tilde{\omega} > 0 \). We next show that \( z_n \geq 0 \) in \( \mathbb{R}^3 \). For this we suppose by contradiction that, there is a sequence \( \{x_j^j\} \) such that

\[
\inf_{x \in \mathbb{R}^3} z_n(x) = \lim_{j \to \infty} z_n(x_j^j) < 0.
\]

(6.5)

Notice that

\[
\lim_{|x| \to \infty} \tilde{\omega}(x) = 0.
\]

Jointly with (5.20), we obtain

\[
\lim_{|x| \to \infty} z_n(x) = 0,
\]

uniformly in \( n \in \mathbb{N} \). Consequently, the sequence \( \{x_j^j\} \) is bounded and therefore, up to a subsequence, we may assume that \( x_j^j \to x_n^* \) as \( j \to \infty \) for some \( x_n^* \in \mathbb{R}^3 \). Hence (6.5) becomes

\[
z_n(x_n^*) = \inf_{x \in \mathbb{R}^3} z_n(x) < 0.
\]

(6.6)

From (6.6) and (2.1), we have

\[
(-\Delta)^s z_n(x_n^*) = -\frac{C(s)}{2} \int_{\mathbb{R}^3} \frac{z_n(x_n^* + y) + z_n(x_n^* - y) - 2z_n(x_n^*)}{|y|^{3+2s}} dy \leq 0.
\]

(6.7)

By (6.4), we get

\[
z_n(x) \geq mb + \tilde{\omega} - mb > 0, \quad \text{in } B(0, R_2).
\]

Therefore, combining this with (6.6), we see that

\[
x_n^* \in \mathbb{R}^3 \setminus B_{R_2}(0).
\]

(6.8)

From (6.2)–(6.3), we conclude that

\[
(-\Delta)^s z_n + \frac{V_{\text{min}}}{2} z_n \geq 0, \quad \text{in } \mathbb{R}^3 \setminus B_{R_2}(0).
\]

(6.9)
Thinks to (6.8), we can evaluate (6.9) at the point $x^*_n$, and recall (6.6), (6.7), we conclude that

$$0 \leq (-\Delta)^s z_n(x^*_n) + \frac{V_{\min}}{2} z_n(x^*_n) < 0,$$

this is a contradiction, so $z_n(x) \geq 0$ in $\mathbb{R}^3$. That is to say, $v_n \leq (m + 1)b^{-1} \tilde{w}$, which together with (6.1), implies that

$$v_n(x) \leq \frac{C}{1 + |x|^{3+2s}}, \quad \text{for all } x \in \mathbb{R}^3.$$

Then the proof is completed. \hfill \Box

**Proof of Theorem 6.2.** Define $\omega_n(x) := u_n(\frac{x}{\varepsilon_n})$, then $\omega_n$ is a positive ground state solution of the system (1.1) and $x_{\varepsilon_n} := \varepsilon_n y_n$ is a maximum point of $\omega_n$, and by Theorem 5.1, we know that the Theorem 1.2(i), (ii) hold. Moreover, we have

$$\omega_n(x) = u_n\left(\frac{x}{\varepsilon_n}\right) = v_n\left(\frac{x}{\varepsilon_n} - y_n\right) \leq \frac{C}{1 + \left|\frac{x}{\varepsilon_n} - y_n\right|^{3+2s}} \leq \frac{C\varepsilon_n^{3+2s}}{\varepsilon_n^{3+2s} + \left|x - \varepsilon_n y_n\right|^{3+2s}} = \frac{C\varepsilon_n^{3+2s}}{\varepsilon_n^{3+2s} + \left|x - \varepsilon_n y_n\right|^{3+2s}}, \quad \text{for all } x \in \mathbb{R}^3.$$

Thus, the proof of Theorem 1.2 is completed. \hfill \Box

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