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The concentration behavior of ground state solutions for a critical fractional Schrödinger–Poisson system

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Abstract
In this paper, we study the following critical fractional Schrödinger–Poisson system

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = P(x)f(u) + Q(x)|u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3, \\ \varepsilon^{2t}(-\Delta)^t \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $s \in \left(\frac{3}{4}, 1\right)$, $t \in (0,1)$ and $2s + 2t > 3$, $2_s^* := \frac{6}{3-2s}$ is the fractional critical exponent for 3-dimension, $V(x) \in C(\mathbb{R}^3)$ has a positive global minimum, and $P(x), Q(x) \in C(\mathbb{R}^3)$ are positive and have global maximums. We obtain the existence of a positive ground state solution by using variational methods, and we determine a concrete set related to the potentials V, P and Q as the concentration position of these ground state solutions as $\varepsilon \to 0^+$. Moreover, we consider some properties of these ground state solutions, such as convergence and decay estimate.

KEYWORDS
concentration, critical growth, fractional Schrödinger–Poisson system, ground state solution

MSC (2010)
35J50, 35Q40, 58E05

1 | INTRODUCTION AND MAIN RESULTS

In this paper, we study the existence and concentration of solutions for the following critical fractional Schrödinger–Poisson system

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = P(x)f(u) + Q(x)|u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3, \\ \varepsilon^{2t}(-\Delta)^t \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $s \in \left(\frac{3}{4}, 1\right)$, $t \in (0,1)$, $2s + 2t > 3$ and $(-\Delta)^\alpha$ is the fractional Laplacian operator, which can be defined by the Fourier transform $(-\Delta)^\alpha u = \mathcal{F}^{-1}\left(|\xi|^{2\alpha}\mathcal{F}u\right)$. In (1.1), the first equation is a nonlinear fractional Schrödinger equation in which the potential ϕ satisfies the second equation which is a fractional Poisson equation. For this reason, (1.1) is refered to as a fractional nonlinear Schrödinger–Poisson system (also called Schrödinger–Maxwell system). When $s = \frac{1}{2}$ and $t = 1$, such a system becomes more interesting in Physics. It comes from the semi-relativistic theory in the repulsive (plasma physics) Coulomb case (see e.g. [29]). If one put the second equation into the first equation, such a system reduces to the semi-relativistic Hartree equation which arises from the quantum theory of boson stars ([21]).

If $\phi(x) = 0$ in the first equation, (1.1) becomes the fractional Schrödinger equation like

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Equation (1.2) is related to standing wave solutions of the fractional time-dependent Schrödinger equation of the form

$$i\varepsilon\frac{\partial\psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s\psi + V(x)\psi - f(x,|\psi|), \quad x \in \mathbb{R}^N,$$

which is a fundamental equation in fractional quantum mechanics (see [20]). It is well known that, different to the classical Laplacian operator, the usual analysis tools for elliptic PDEs can not be directly applied to (1.2) since $(-\Delta)^s$ is a nonlocal operator. To overcome this difficulty, Caffarelli and Silvestre [5] developed a powerful extension method which transfer the nonlocal Equation (1.2) into a local one settled on the half-space $\mathbb{R}^{N+1}_+$.

In the local case that $s = t = 1$, (1.1) reduces to the following system

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

From the point view of Quantum Mechanics, the system (1.3) describes mutual interactions of many particles (see [36]) and also arises in Abelian Gauge Theories. These theories consist of field equations that provide a model to describe the interaction of a nonlinear Schrödinger field with the electromagnetic field (see [7,8]). In the past decades, the system likes or similars to (1.3) has been studied extensively by means of variational tools. See [2,10,16,19,27,40,45] and the references therein for the existence of solutions. The concentration behavior of solutions was studied in some papers. Ruiz constructed a family of solutions which concentrate around a sphere in [26]. In [28], Ruiz and Varia obtained the existence of multi-bump type solutions and showed that the bumps concentrate around a local minimum of the potential for $f(x,u) = |u|^{q-2}u$ and $3 < q < 5$ by applying Lyapunov–Schmidt reduction methods. The critical case was considered in [17], He and Zou proved that system (1.3) possesses a positive ground state solution which concentrates around the global minimum of V. The following semiclassical Schrödinger–Poisson system has also attracted a lot of attention

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \phi u = f(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.4}$$

D'Aprile and Wei [9] constructed a family of positive radially symmetric bound states and showed the concentration around a sphere in $\mathbb{R}^3$ as $\varepsilon \to 0$ for (1.4) with $f(u) = |u|^{q-2}u$, $1 < q < \frac{11}{7}$.

Recently, there is an increasing interest in the existence of solutions to the fractional Schrödinger–Poisson system. A fractional Schrödinger–Poisson system with $V = 0$ and a general nonlinearity in the subcritical and critical case was considered in [44], where a positive solution was obtained by using a perturbation approach, and the asymptotic behavior of solutions for a vanishing parameter was also given. In [34] and [35], Teng adapted the monotonicity trick (see e.g. [18]) to obtain the existence of ground state solutions to the critical and subcritical cases, respectively. In [24], the authors considered the following system

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = g(u), & \text{in } \mathbb{R}^3, \\ \varepsilon^{\theta}(-\Delta)^{\frac{\alpha}{2}}\phi = \gamma_\alpha u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where γ_α is a constant, and they established the multiplicity of solutions for small ε via the Ljusternik–Schnirelmann category theory, where g is subcritical at infinity. However, the concentration behavior of solutions was almost not considered before in literatures. To the best of our knowledge, the only result was due to Liu and Zhang [22], where the authors considered the following system

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = f(u) + |u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3, \\ \varepsilon^{2t}(-\Delta)^t\phi = u^2, & \text{in } \mathbb{R}^3. \end{cases}$$

Under the assumptions

(C) $f:\mathbb{R}\to\mathbb{R}$ is a function of C^1-class,

(M) $\frac{f(t)}{t^3}$ is strictly increasing for $t>0$,

and other suitable conditions, Liu and Zhang obtained the multiplicity of positive solutions which concentrate on the minima of $V(x)$ by the minimax theorems and Ljusternik–Schnirelmann category theory. When f is not a function of C^1-class, the multiplicity of solutions was established in [43]. The concentration behavior of ground state solutions for a subcritical case with two competing potentials was studied in [42].

In this paper, we are concerned with the existence and concentration behavior of ground state solutions for (1.1). We note that (1.1) involves three different potentials which make our problem more complicated than that one in [22]. This brings a competition between the potentials V, P and Q: each one would like to attract ground states to their minimum or maximum points, respectively. It makes difficulties in determining the concentration position of solutions. This kind of problem can be traced back to [38,39] and [6] for the semilinear Schrödinger equation. In [11], the authors found new concentration phenomena for Dirac equations with competing potentials and subcritical or critical nonlinearities, respectively. See also [12,37] and [41] for other related results.

To state our main results, we need the following assumptions

(f_1) $f\in C(\mathbb{R},\mathbb{R})$, $f(t)=o\big(t^3\big)$ as $t\to 0$ and $f(t)=0$ for all $t\le 0$;

(f_2) There exists $4<p<2_s^*$ such that

$$|f(t)|\le c_1\big(1+|t|^{p-1}\big)$$

for all $t\in\mathbb{R}$ and some $c_1>0$;

(f_3) $tf(t)-4F(t)\ge stf(st)-4F(st)$, $\forall t\ge 0, \forall s\in[0,1]$, where $F(t)=\int_0^t f(\tau)\,d\tau$;

(f_4) There exists $4\le\sigma<2_s^*$ such that

$$F(t)\ge c_2t^\sigma$$

for all $t>0$ and some $c_2>0$.

Remark 1.1. If the nonlinearity is differentiable, then it is easy to see that (f_3) is equivalent to the condition (M) by the derivative rules. In the present paper, we only need $f\in C(\mathbb{R},\mathbb{R})$. At the time, the condition (f_3) is weaker than (M). In fact, for any $s\in[0,1], t>0$, let $k(s)=s^4tf(t)-4F(st)$, then $k'(s)=4s^3tf(t)-4tf(st)=4s^3tf(t)-4t\frac{f(st)}{(st)^3}(st)^3$. If (M) holds, then

$$k'(s)\ge 4s^3tf(t)-4t\frac{f(t)}{t^3}(st)^3=0,\quad \text{for all}\quad t\in\mathbb{R}.$$

Therefore, $k(s)$ is increasing on [0,1]. Consequently, $k(1)\ge k(s)$, for all $s\in[0,1]$. Thus, for any $s\in[0,1]$, by (M) we have

$$tf(t)-4F(t)\ge s^4tf(t)-4F(st)=s^4t^4\frac{f(t)}{t^3}-4F(st)\ge stf(st)-4F(st).$$

Here is an example of nonlinearity which satisfies (f_3) but does not satisfy the condition (M). Define

$$F(t)=\begin{cases} t^4\displaystyle\int_0^t\frac{\sin^5\left(\frac{\pi}{2}\tau\right)}{\tau^5}\,d\tau, & t\in[0,1],\\[2ex] t^4\left(\displaystyle\int_0^1\frac{\sin^5\left(\frac{\pi}{2}\tau\right)}{\tau^5}\,d\tau+\int_1^t\frac{1}{\tau^5}\,d\tau\right), & t\ge 1.\end{cases}$$

By a direct computation, one has

$$f(t)=\begin{cases}4t^3\displaystyle\int_0^t\frac{\sin^5\left(\frac{\pi}{2}\tau\right)}{\tau^5}\,d\tau+\frac{\sin^5\left(\frac{\pi}{2}t\right)}{t}, & t\in[0,1],\\ 4t^3\left(\displaystyle\int_0^1\frac{\sin^5\left(\frac{\pi}{2}\tau\right)}{\tau^5}\,d\tau+\int_1^t\frac{1}{\tau^5}\,d\tau\right)+\frac{1}{t}, & t\geq 1.\end{cases}$$

Thereby, for $t\in[0,+\infty)$ and $s\in[0,1]$, as $0\leq t\leq 1$, one has

$$stf(st)-4F(st)=\sin^5\left(\frac{\pi}{2}st\right)\leq\sin^5\left(\frac{\pi}{2}t\right)=tf(t)-4F(t).$$

As $t\geq 1$, if $0\leq st\leq 1$, then

$$stf(st)-4F(st)=\sin^5\left(\frac{\pi}{2}st\right)\leq 1=tf(t)-4F(t),$$

and if $st\geq 1$, then

$$stf(st)-4F(st)=1=tf(t)-4F(t).$$

All in all, $stf(st)-4F(st)\leq tf(t)-4F(t)$ for any $s\in[0,1], t\in[0,+\infty)$.

By the definition of f, we have

$$\frac{f(t)}{t^3}=\begin{cases}4\displaystyle\int_0^t\frac{\sin^5\left(\frac{\pi}{2}\tau\right)}{\tau^5}d\tau+\frac{\sin^5\left(\frac{\pi}{2}t\right)}{t^4}, & t\in[0,1],\\ 4\left(\displaystyle\int_0^1\frac{\sin^5\left(\frac{\pi}{2}\tau\right)}{\tau^5}\,d\tau+\int_1^t\frac{1}{\tau^5}\,d\tau\right)+\frac{1}{t^4}, & t\geq 1,\end{cases}$$

$$=\begin{cases}4\displaystyle\int_0^t\frac{\sin^5\left(\frac{\pi}{2}\tau\right)}{\tau^5}\,d\tau+\frac{\sin^5\left(\frac{\pi}{2}t\right)}{t^4}, & t\in[0,1],\\ 4\displaystyle\int_0^1\frac{\sin^5\left(\frac{\pi}{2}\tau\right)}{\tau^5}\,d\tau+1, & t\geq 1.\end{cases}$$

Notice that $\int_0^1\frac{\sin^5\left(\frac{\pi}{2}\tau\right)}{\tau^5}\,d\tau<+\infty$. Thus, if $t\geq 1$, one has

$$\frac{f(t)}{t^3}\equiv 4\int_0^1\frac{\sin^5\left(\frac{\pi}{2}\tau\right)}{\tau^5}\,d\tau+1.$$

So $\frac{f(t)}{t^3}$ is not strictly increasing for $t>0$, that is, f does not satisfy condition (M).

We need some notations to help us to determine the concentration set of solutions. Set

$$V_{\min}:=\min_{x\in\mathbb{R}^3}V(x),\quad V_{\max}:=\sup_{x\in\mathbb{R}^3}V(x),\quad \mathcal{V}:=\left\{x\in\mathbb{R}^3:V(x)=V_{\min}\right\},\quad V_\infty:=\liminf_{|x|\to\infty}V(x),$$

$$P_{\min}:=\inf_{x\in\mathbb{R}^3}P(x),\quad P_{\max}:=\max_{x\in\mathbb{R}^3}P(x),\quad \mathcal{P}:=\left\{x\in\mathbb{R}^3:P(x)=P_{\max}\right\},\quad P_\infty:=\limsup_{|x|\to\infty}P(x),$$

$$Q_{\min} := \inf_{x\in\mathbb{R}^3} Q(x), \quad Q_{\max} := \max_{x\in\mathbb{R}^3} Q(x), \quad \mathcal{Q} := \left\{x\in\mathbb{R}^3 : Q(x) = Q_{\max}\right\}, \quad Q_\infty := \limsup_{|x|\to\infty} Q(x),$$

$$V_{\mathcal{Q}} := \min_{x\in\mathcal{Q}} V(x), \quad P_{\mathcal{Q}} := \max_{x\in\mathcal{Q}} P(x).$$

Moreover, we assume that V, P and Q satisfy the following conditions:

(A_0) V, P, Q are three continuous and bounded functions with $V_{\min} > 0$, $P_{\min} > 0$ and $Q_{\min} > 0$;
either

(A_1) $P_{\mathcal{Q}} > P_\infty$ and there exists $x_P \in \mathcal{C}_{\mathcal{P}}$ such that $V(x_P) \le V(x)$ for $|x| \ge R$ with $R > 0$ sufficiently large, where $\mathcal{C}_P := \left\{x\in\mathcal{Q} : P(x) = P_{\mathcal{Q}}\right\}$
or

(A_2) $V_{\mathcal{Q}} < V_\infty$ and there exists $x_V \in \mathcal{C}_V$ such that $P(x_V) \ge P(x)$ for $|x| \ge R$ with $R > 0$ sufficiently large, where $\mathcal{C}_V := \left\{x\in\mathcal{Q} : V(x) = V_{\mathcal{Q}}\right\}$.

If (A_1) holds, we set

$$\mathcal{H}_P = \left\{x\in\mathcal{C}_P : V(x) \le V(x_P)\right\} \cup \left\{x\in\mathcal{Q}\setminus\mathcal{C}_P : V(x) < V(x_P)\right\} \cup \left\{x\notin\mathcal{Q} : P(x) > P_{\mathcal{Q}} \text{ or } V(x) < V(x_P)\right\}.$$

If (A_2) holds, we set

$$\mathcal{H}_V = \left\{x\in\mathcal{C}_V : P(x) \ge P(x_V)\right\} \cup \left\{x\in\mathcal{Q}\setminus\mathcal{C}_V : P(x) > P(x_V)\right\} \cup \left\{x\notin\mathcal{Q} : V(x) < V_{\mathcal{Q}} \text{ or } P(x) > P(x_V)\right\}.$$

Clearly, $\mathcal{H}_P$ and $\mathcal{H}_V$ are bounded sets. Moreover, if $\mathcal{V}\cap\mathcal{P}\cap\mathcal{Q} \ne \emptyset$, then $\mathcal{H}_P = \mathcal{H}_V = \mathcal{V}\cap\mathcal{P}\cap\mathcal{Q}$.

Now we state our main results as follows.

Theorem 1.2. *Assume that* (f_1)–(f_4), $s\in\left(\frac{3}{4},1\right)$, $t\in(0,1)$, $2s+2t>3$, (A_0) *and* (A_1) *hold, then for all small* $\varepsilon>0$:

(i) The system (1.1) *has a positive ground state solution* $\left(\omega_\varepsilon, \phi_{\omega_\varepsilon}\right)$;

(ii) ω_ε *possesses a global maximum point* x_ε *such that, up to a subsequence,* $x_\varepsilon \to x_0$ *as* $\varepsilon\to 0$, $\lim_{\varepsilon\to 0} dist(x_\varepsilon, \mathcal{H}_P) = 0$, *and* $\left(v_\varepsilon(x), \psi_\varepsilon(x)\right) := \left(\omega_\varepsilon(\varepsilon x + x_\varepsilon), \phi_\varepsilon(\varepsilon x + x_\varepsilon)\right)$ *converges in* $H^s(\mathbb{R}^3)$ *to a positive ground state solution of*

$$\begin{cases} (-\Delta)^s u + V(x_0)u + \phi u = P(x_0)f(u) + Q(x_0)|u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi = u^2, & \text{in } \mathbb{R}^3.\end{cases}$$

In particular if $\mathcal{V}\cap\mathcal{P}\cap\mathcal{Q}\ne\emptyset$, *then* $\lim_{\varepsilon\to 0} dist(x_\varepsilon, \mathcal{V}\cap\mathcal{P}\cap\mathcal{Q}) = 0$, *and up to a subsequence,* $\left(v_\varepsilon, \psi_\varepsilon\right)$ *converges in* $H^s(\mathbb{R}^3)$ *to a positive ground state solution of*

$$\begin{cases} (-\Delta)^s u + V_{\min}u + \phi u = P_{\max}f(u) + Q_{\max}|u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi = u^2, & \text{in } \mathbb{R}^3.\end{cases}$$

(iii) There exists a constant $C>0$ *such that*

$$\omega_\varepsilon(x) \le \frac{C\varepsilon^{3+2s}}{\varepsilon^{3+2s} + |x-x_\varepsilon|^{3+2s}}, \quad \text{for all} \quad x\in\mathbb{R}^3.$$

Theorem 1.3. *Assume that* (f_1)–(f_4), $s\in\left(\frac{3}{4},1\right)$, $t\in(0,1)$, $2s+2t>3$, (A_0) *and* (A_2) *hold, and we replace* $(\mathcal{H}_P)$ *by* $(\mathcal{H}_V)$, *then all the conclusions of Theorem 1.2 remain true.*

Remark 1.4. Comparing to [22], there are some different points in our paper. First, we do not need f satisfies the smooth condition (S), and this prevents us using the Nehari manifold in a standard way. Second, we do not assume f satisfies the monotonicity condition (M) which plays an important role in [22].

In the sequel, we only give the detailed proof for Theorem 1.2 because the argument for Theorem 1.3 is similar to that for Theorem 1.2.

This paper is organized as follows. In Section 2, we provide some preliminary lemmas which will be used later. In Section 3, we consider the autonomous problem of the sytem (1.1) and prove the existence of positive ground state solutions. In Section 4, we prove the existence of positive ground state solutions of the sytem (1.1) for small $\varepsilon > 0$. In Section 5, we study the concentration phenomenon and convergence of ground state solutions. In Section 6, we obtain the decay estimate of solution, which is polynomial instead of exponential form. Finally, we give the proof of Theorem 1.2.

Notation. In this paper we make use of the following notations.

- For any $R > 0$ and for any $x \in \mathbb{R}^3$, $B_R(x)$ denotes the ball of radius R centered at x;
- $L^p(\mathbb{R}^3)$, $1 \le p \le +\infty$, denotes the Lebesgue space with the following norm

$$\|u\|_p = \begin{cases} \left(\int_{\mathbb{R}^3} |u|^p dx \right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^3} |u(x)|, & \text{if } p = \infty. \end{cases}$$

- C or C_i $(i = 1, 2, \dots)$ denote some positive constants could change from line to line.

2 | PRELIMINARIES

First, we collect some preliminary results for the fractional Laplacian from [3]. We define the homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^3)$ as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{s,2}} := \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dx \, dy \right)^{\frac{1}{2}} = [u]_{H^s}.$$

We denote by $H^s(\mathbb{R}^3)$ the standard fractional Sobolev space, defined as the set of $u \in \mathcal{D}^{s,2}(\mathbb{R}^3)$ satisfying $u \in L^2(\mathbb{R}^3)$ with the norm

$$\|u\|_{H^s}^2 = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dx \, dy + \int_{\mathbb{R}^3} u^2 dx = [u]_{H^s}^2 + \|u\|_2^2.$$

Also, in light of [3] and [25, Proposition 3.4 and Proposition 3.6], we have

$$\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^2 = \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi = \frac{1}{2} C(s) \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dx \, dy,$$

where $\hat{u}$ stands for the Fourier transform of u and

$$C(s) = \left(\int_{\mathbb{R}^3} \frac{1 - cos\xi_1}{|\xi|^{3+2s}} \, d\xi \right)^{-1}, \quad \xi = (\xi_1, \xi_2, \xi_3).$$

As a consequence, the norms on $H^s(\mathbb{R}^3)$ defined below

$$u \mapsto \left(\int_{\mathbb{R}^3} u^2 dx + \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dx \, dy \right)^{\frac{1}{2}},$$

$$u \mapsto \left(\int_{\mathbb{R}^3} u^2 dx + \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}},$$

$$u \mapsto \left(\int_{\mathbb{R}^3} u^2 dx + \left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^2 \right)^{\frac{1}{2}}$$

are all equivalent. Furthermore, it is well known that $H^s(\mathbb{R}^3)$ is continuously embedded into $L^r(\mathbb{R}^3)$ for any $2 \le r \le 2_s^*$ and compactly embedding into $L^r_{loc}(\mathbb{R}^3)$ for any $1 \le r < 2_s^*$ and there exists a best constant $\mathcal{S}_s > 0$ such that

$$\mathcal{S}_s = \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\right)^{\frac{2}{2_s^*}}}.$$

Moreover, $(-\Delta)^s u$ can be equivalently represented as (see [25, Lemma 3.2])

$$(-\Delta)^s u(x) = -\frac{C(s)}{2}\int_{\mathbb{R}^3} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{3+2s}}\, dy, \quad \text{for all} \quad x\in\mathbb{R}^3. \tag{2.1}$$

We denote $\|\cdot\|_{H^s}$ by $\|\cdot\|$ in the sequel for convenience.

Recall that by the Lax–Milgram theorem, we know that for every $u \in H^s(\mathbb{R}^3)$, there exists a unique $\phi_u^t \in \mathcal{D}^{t,2}(\mathbb{R}^3)$ such that $(-\Delta)^t\phi_u^t = u^2$ and ϕ_u^t can be expressed by

$$\phi_u^t(x) = C_t\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}}\, dy, \quad \text{for all} \quad x\in\mathbb{R}^3,$$

which is called t-Riesz potential, where

$$C_t = \frac{1}{\pi^{\frac{3}{2}}}\frac{\Gamma(\frac{3}{2}-t)}{2^{2t}\Gamma(t)}.$$

Making the change of variables $x \mapsto \varepsilon x$, we can rewrite the system (1.1) as the following equivalent system

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x)u + \phi u = P(\varepsilon x)f(u) + Q(\varepsilon x)|u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{2.2}$$

If u is a solution of the system (2.2), then $\omega(x) := u(\frac{x}{\varepsilon})$ is a solution of the system (1.1). Thus, to study the system (1.1), it suffices to study the system (2.2). In view of the presence of potential $V(x)$, we introduce the subspace

$$H_\varepsilon = \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx < \infty \right\},$$

which is a Hilbert space equipped with the inner product

$$(u,v)_\varepsilon = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v\, dx + \int_{\mathbb{R}^3} V(\varepsilon x)uv\, dx,$$

and the equivalent norm

$$\|u\|_\varepsilon^2 = (u,u)_\varepsilon = \int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx.$$

Moreover, it can be proved that $(u,\phi_u^t) \in H_\varepsilon \times \mathcal{D}^{t,2}(\mathbb{R}^3)$ is a solution of (2.2) if and only if $u \in H_\varepsilon$ is a critical point of the functional $\mathcal{I}_\varepsilon : H_\varepsilon \to \mathbb{R}$ defined as

$$\mathcal{I}_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \int_{\mathbb{R}^3} P(\varepsilon x)F(u)\, dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} Q(\varepsilon x)|u|^{2_s^*} dx, \tag{2.3}$$

where ϕ_u^t is the unique solution of the second equation in (2.2). Note that $2 \le \frac{12}{3+2t} \le 2_s^*$ if $4s + 2t \ge 3$, then by the Hölder inequality and the Sobolev inequality, we have

$$\int_{\mathbb{R}^3} \phi_u^t u^2 dx \le \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2t}} dx\right)^{\frac{3+2t}{6}} \left(\int_{\mathbb{R}^3} |\phi_u^t|^{2_t^*} dx\right)^{\frac{1}{2_t^*}} \le \frac{1}{\mathcal{S}_t^{\frac{1}{2}}}\left(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2t}} dx\right)^{\frac{3+2t}{6}} \|\phi_u^t\|_{\mathcal{D}^{t,2}} \le C\|u\|^2\|\phi_u^t\|_{\mathcal{D}^{t,2}} < \infty.$$

Therefore, the functional $\mathcal{I}_\varepsilon$ is well-defined for every $u \in H_\varepsilon$ and belongs to $C^1(H_\varepsilon, \mathbb{R})$. Moreover, for any $u, v \in H_\varepsilon$, we have

$$\langle \mathcal{I}'_\varepsilon(u), v\rangle = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} v\, dx + \int_{\mathbb{R}^3} V(\varepsilon x) uv\, dx + \int_{\mathbb{R}^3} \phi_u^t uv\, dx - \int_{\mathbb{R}^3} P(\varepsilon x) f(u) v\, dx - \int_{\mathbb{R}^3} Q(\varepsilon x)|u|^{2_s^*-2} uv\, dx. \tag{2.4}$$

The properties of the function ϕ_u^t are given in the following lemma (see [35, Lemma 2.3]).

Lemma 2.1. *If $4s + 2t \geq 3$, then for any $u \in H^s(\mathbb{R}^3)$, we have*

(i) $\phi_u^t \geq 0$;
(ii) $\phi_u^t : H^s(\mathbb{R}^3) \to \mathcal{D}^{t,2}(\mathbb{R}^3)$ *is continuous and maps bounded sets into bounded sets;*
(iii) $\int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq C\|u\|^4_{\frac{12}{3+2t}} \leq C\|u\|^4$;
(iv) If $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$, *then* $\phi_{u_n}^t \rightharpoonup \phi_u^t$ *in* $\mathcal{D}^{t,2}(\mathbb{R}^3)$;
(v) If $u_n \to u$ *in* $H^s(\mathbb{R}^3)$, *then* $\phi_{u_n}^t \to \phi_u^t$ *in* $\mathcal{D}^{t,2}(\mathbb{R}^3)$ *and* $\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2\, dx \to \int_{\mathbb{R}^3} \phi_u^t u^2 dx$.

Define $N : H^s(\mathbb{R}^3) \to \mathbb{R}$ by

$$N(u) = \int_{\mathbb{R}^3} \phi_u^t u^2 dx.$$

The next lemma shows that the functional N and N' possesses BL-splitting property which is similar to the well-known Brezis–Lieb lemma ([4]).

Lemma 2.2. *([35, Lemma 2.4]) Assume that $2s + 2t > 3$. Let $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$ and $u_n \to u$ a.e.in $\mathbb{R}^3$. Then*

(i) $N(u_n - u) = N(u_n) - N(u) + o(1)$;
(ii) $N'(u_n - u) = N'(u_n) - N'(u) + o(1)$, *in* $(H^s(\mathbb{R}^3))^*$.

The following vanishing lemma is a version of the concentration-compactness principle proved by P. L. Lions. We can consult [15, Lemma 2.2] and [30, Lemma 2.4].

Lemma 2.3. *If $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$ and it satisfies*

$$\sup_{x\in\mathbb{R}^3} \int_{B_R(x)} |u_n|^2 dx \to 0 \text{ as } n \to \infty,$$

for some $R > 0$. Then $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for any $2 \leq r < 2_s^*$.

In order to find critical points for $\mathcal{I}_\varepsilon$, we will use the Nehari methods. The Nehari mainfold corresponding to $\mathcal{I}_\varepsilon$ is defined by

$$\mathcal{N}_\varepsilon = \{u \in H_\varepsilon \backslash \{0\} : \langle \mathcal{I}'_\varepsilon(u), u\rangle = 0\}.$$

Thus, for any $u \in \mathcal{N}_\varepsilon$, we have that

$$\int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x) u^2 dx + \int_{\mathbb{R}^3} \phi_u^t u^2 dx = \int_{\mathbb{R}^3} P(\varepsilon x) f(u) u\, dx + \int_{\mathbb{R}^3} Q(\varepsilon x)|u|^{2_s^*} dx.$$

Since f is only continuous but not belongs to C^1-class, $\mathcal{N}_\varepsilon$ need not be of class C^1 in our case, so we cannot use standard arguments on the Nehari manifold in the standard way. To overcome the nondifferentiability of the Nehari manifold, we shall use the reduction method developed by Szulkin and Weth in [33].

First, (f_1) and (f_2) imply that for each $\tau > 0$, there is $C_\tau > 0$ such that

$$|f(u)| \leq \tau|u|^3 + C_\tau |u|^{p-1} \quad \text{and} \quad |F(u)| \leq \frac{\tau}{4}|u|^4 + \frac{C_\tau}{q}|u|^p \tag{2.5}$$

for all $u \in H^s(\mathbb{R}^3)$. By (f_1) and (f_3), we deduce that

$$F(u) \geq 0 \quad \text{and} \quad \frac{1}{4} f(u)u - F(u) \geq 0. \tag{2.6}$$

In the following we shall show some properties for $\mathcal{N}_\varepsilon$.

Lemma 2.4. *For any $u \in H_\varepsilon \backslash \{0\}$, we have*

(i) There exists a unique θ_u such that $\theta_u u \in \mathcal{N}_\varepsilon$. Moreover, $\mathcal{I}_\varepsilon(\theta_u u) = \max_{\theta \geq 0} \mathcal{I}_\varepsilon(\theta u)$.
(ii) There exist $T_2 > T_1 > 0$ independent of $\varepsilon > 0$ such that $T_1 \leq \theta_u \leq T_2$.

Proof. (i) For $\theta > 0$, let

$$\begin{aligned} g(\theta) = \mathcal{I}_\varepsilon(\theta u) = &\frac{\theta^2}{2} \int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \frac{\theta^2}{2} \int_{\mathbb{R}^3} V(\varepsilon x) u^2 dx \\ &+ \frac{\theta^4}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx - \int_{\mathbb{R}^3} P(\varepsilon x) F(\theta u)\, dx - \frac{\theta^{2_s^*}}{2_s^*} \int_{\mathbb{R}^3} Q(\varepsilon x) |u|^{2_s^*} dx. \end{aligned}$$

Then, by (2.5) and Sobolev embedding inequality, we have

$$\begin{aligned} g(\theta) &\geq \frac{1}{2}\theta^2 \|u\|_\varepsilon^2 - C\theta^4 \int_{\mathbb{R}^3} |u|^4 dx - C\theta^p \int_{\mathbb{R}^3} |u|^p dx - \frac{\theta^{2_s^*}}{2_s^*} Q_{\max} \int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &\geq \frac{\theta^2}{2} \|u\|_\varepsilon^2 - C\theta^4 \|u\|_\varepsilon^4 - C\theta^p \|u\|_\varepsilon^p - C\theta^{2_s^*} \|u\|_\varepsilon^{2_s^*} \end{aligned}$$

and

$$\begin{aligned} g'(\theta) &\geq \theta \|u\|_\varepsilon^2 - C\theta^3 \int_{\mathbb{R}^3} |u|^4 dx - C\theta^{p-1} \int_{\mathbb{R}^3} |u|^p dx - Q_{\max} \theta^{2_s^*-1} \int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &\geq \theta \|u\|_\varepsilon^2 - C\theta^3 \|u\|_\varepsilon^4 - C\theta^{p-1} \|u\|_\varepsilon^p - C\theta^{2_s^*-1} \|u\|_\varepsilon^{2_s^*}. \end{aligned}$$

Since $4 < p < 2_s^*$, $g(\theta) > 0$ and $g'(\theta) > 0$ for small $\theta > 0$. Moreover, by Lemma 2.1(iii), we get

$$g(\theta) \leq \frac{\theta^2}{2} \|u\|_\varepsilon^2 + C\theta^4 \|u\|_\varepsilon^4 - \frac{Q_{\min}}{2_s^*} \theta^{2_s^*} \int_{\mathbb{R}^3} |u|^{2_s^*} dx.$$

Hence, $g(\theta) \to -\infty$ as $\theta \to \infty$ and g has a positive maximum and there exist $\theta_u > 0$ such that $g'(\theta_u) = 0$, $g'(\theta) > 0$ for $0 < \theta < \theta_u$.

Next we claim that $g'(\theta) \neq 0$ for all $\theta > \theta_u$. Indeed, if the conclusion is false, then, from the above arguments, there exists a $\theta_u < \theta_u'$ such that $g'\left(\theta_u'\right) = 0$ and $g(\theta_u) \geq g\left(\theta_u'\right)$. However, (f_3) implies that

$$\begin{aligned} g\left(\theta_u'\right) &= g\left(\theta_u'\right) - \frac{\theta_u'}{4} g'\left(\theta_u'\right) \\ &= \frac{\theta_u'^2}{4} \|u\|_\varepsilon^2 + \frac{1}{4} \int_{\mathbb{R}^3} P(\varepsilon x)\left[f\left(\theta_u' u\right)\theta_u' u - 4F\left(\theta_u' u\right)\right] dx + \frac{4s-3}{12} \theta_u'^{2_s^*} \int_{\mathbb{R}^3} Q(\varepsilon x)|u|^{2_s^*} dx \\ &> \frac{\theta_u^2}{4} \|u\|_\varepsilon^2 + \frac{1}{4} \int_{\mathbb{R}^3} P(\varepsilon x)\left[f(\theta_u u)\theta_u u - 4F(\theta_u u)\right] dx + \frac{4s-3}{12} \theta_u^{2_s^*} \int_{\mathbb{R}^3} Q(\varepsilon x)|u|^{2_s^*} dx \\ &= g(\theta_u) - \frac{\theta_u}{4} g'(\theta_u) \\ &= g(\theta_u), \end{aligned}$$

here we use $s > \frac{3}{4}$, this is a contradiction. This claim is proved and then g has a unique maximum at θ_u. Moreover, notice that $g'(\theta) = \theta^{-1} \langle \mathcal{I}_\varepsilon'(\theta u), \theta u \rangle$, then $g'(\theta_u) = 0$ implies $\theta_u u \in \mathcal{N}_\varepsilon$. Thus (i) holds.

(ii) By $\theta_u u \in \mathcal{N}_\varepsilon$ and Lemma 2.1(iii), we have

$$\begin{aligned} C_1\theta_u^2\|u\|^2 + C_2\theta_u^4\|u\|^4 &\ge \theta_u^2\|u\|_\varepsilon^2 + \theta_u^4\int_{\mathbb{R}^3}\phi_u^t u^2\,dx \\ &= \theta_u\int_{\mathbb{R}^3}P(\varepsilon x)f(\theta_u u)u\,dx + \theta_u^{2_s^*}\int_{\mathbb{R}^3}Q(\varepsilon x)|u|^{2_s^*}dx \\ &\ge C_3\theta_u^{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx. \end{aligned}$$

Thus, there exists a $T_2 > 0$ independent of ε such that $\theta_u \le T_2$.

On the other hand, using $\theta_u u \in \mathcal{N}_\varepsilon$ again and Lemma 2.1(i), we have

$$C_4\theta_u^2\|u\|^2 \le \theta_u^2\|u\|_\varepsilon^2 \le C_5\theta_u^\sigma\int_{\mathbb{R}^3}|u|^\sigma dx + C_6\theta_u^{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx \le C\theta_u^\sigma\|u\|^\sigma + C\theta_u^{2_s^*}\|u\|^{2_s^*},$$

which yields that there exists a $T_1 > 0$ independent of ε such that $\theta_u \ge T_1$. □

Lemma 2.5. *For any fixed $\varepsilon > 0$, we have the following facts:*

(i) There exist $\rho > 0$ such that $c_\varepsilon = \inf_{\mathcal{N}_\varepsilon}\mathcal{I}_\varepsilon \ge \inf_{S_\rho}\mathcal{I}_\varepsilon > 0$, where $S_\rho = \{u \in H_\varepsilon : \|u\|_\varepsilon = \rho\}$.

(ii) There exists $r^* > 0$ such that

$$\|u\|_\varepsilon \ge r^*, \quad \textit{for all} \quad u \in \mathcal{N}_\varepsilon.$$

Proof.

(i) For any $u \in H_\varepsilon\backslash\{0\}$, then by Lemma 2.1$(i)$ and (2.5), we have

$$\begin{aligned} \mathcal{I}_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}V(\varepsilon x)u^2dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2dx - \int_{\mathbb{R}^3}F(u)dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx \\ &\ge \frac{1}{2}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}V(\varepsilon x)u^2dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx - C\left(\int_{\mathbb{R}^3}u^4dx + \int_{\mathbb{R}^3}|u|^p dx\right) \\ &\ge \frac{1}{2}\|u\|_\varepsilon^2 - C\left(\|u\|_\varepsilon^4 + \|u\|_\varepsilon^p + \|u\|_\varepsilon^{2_s^*}\right). \end{aligned}$$

Hence, $\inf_{S_\rho}\mathcal{I}_\varepsilon > 0$ for sufficiently small ρ. Moreover, for any $u \in \mathcal{N}_\varepsilon$, Lemma 2.4 implies that $\mathcal{I}_\varepsilon(u) = \max_{\theta\ge 0}\mathcal{I}_\varepsilon(\theta u)$. Taking a $t_0 > 0$ with $t_0 u \in S_\rho$. Then

$$\mathcal{I}_\varepsilon(u) \ge \mathcal{I}_\varepsilon(t_0 u) \ge \inf_{v\in S_\rho}\mathcal{I}_\varepsilon(v).$$

This completes the proof of (i).

(ii) For any $u \in \mathcal{N}_\varepsilon$, similar to (i), we have

$$0 = \left\langle \mathcal{I}_\varepsilon'(u), u\right\rangle \ge \|u\|_\varepsilon^2 - C\left(\|u\|_\varepsilon^4 + \|u\|_\varepsilon^p + \|u\|_\varepsilon^{2_s^*}\right),$$

from which we obtain that

$$\|u\|_\varepsilon \ge r^* > 0$$

for some $r^* > 0$ in view of $u \in \mathcal{N}_\varepsilon \subset H_\varepsilon\backslash\{0\}$.

□

Lemma 2.6. *If $\mathcal{W}$ is a compact subset of $H_\varepsilon \setminus \{0\}$, then there exists $R > 0$ such that $\mathcal{I}_\varepsilon(u) \le 0$ on $\left(\mathbb{R}^+\mathcal{W}\right)\backslash B_R(0)$ for each $u \in \mathcal{W}$.*

Proof. Assume that this is not true. Then there exist sequences $\{u_n\} \subset \mathcal{W}$ and $\{t_n\} \subset \mathbb{R}^+$ such that $\mathcal{I}_\varepsilon(t_n u_n) \geq 0$ and $t_n \to +\infty$ as $n \to \infty$. By the compactness of $\mathcal{W}$, we can assume that $u_n \to u \in \mathcal{W}$ in H_ε and $\|u_n\|_\varepsilon \leq C$ for all n. Set $\Omega := \{x \in \mathbb{R}^3 : u(x) \neq 0\}$. Then $meas(\Omega) > 0$. Hence, for $x \in \Omega$, $|t_n u_n(x)| \to +\infty$. Consequently, by Fatou's lemma, one has

$$\int_{\mathbb{R}^3} t_n^{2_s^*-4} Q(\varepsilon x)|u_n|^{2_s^*} dx \geq Q_{\min} \int_\Omega t_n^{2_s^*-4}|u_n|^{2_s^*} dx \to +\infty.$$

Therefore,

$$\begin{aligned} 0 \leq \frac{\mathcal{I}_\varepsilon(t_n u_n)}{t_n^4} &= \frac{1}{2t_n^2}\|u_n\|_\varepsilon^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \frac{t_n^{2_s^*-4}}{2_s^*}\int_{\mathbb{R}^3} Q(\varepsilon x)|u_n|^{2_s^*} dx - \frac{1}{t_n^4}\int_{\mathbb{R}^3} P(\varepsilon x) F(t_n u_n)\, dx \\ &\leq C - \frac{Q_{\min} t_n^{2_s^*-4}}{2_s^*}\int_\Omega |u_n|^{2_s^*} dx \to -\infty, \end{aligned}$$

a contradiction. This completes the proof. □

Lemma 2.7. *$\mathcal{I}_\varepsilon$ is coercive on $\mathcal{N}_\varepsilon$, i.e., $\mathcal{I}_\varepsilon(u) \to \infty$ as $\|u\|_\varepsilon \to \infty, u \in \mathcal{N}_\varepsilon$.*

Proof. Since $u \in \mathcal{N}_\varepsilon$, we have

$$\begin{aligned} \mathcal{I}_\varepsilon(u) &= \mathcal{I}_\varepsilon(u) - \frac{1}{4}\langle \mathcal{I}_\varepsilon'(u), u\rangle \\ &= \frac{1}{4}\int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} V(\varepsilon x) u_n^2\, dx + \left(\frac{1}{4} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^3} Q(\varepsilon x)|u_n|^{2_s^*} dx \\ &+ \frac{1}{4}\int_{\mathbb{R}^3} P(\varepsilon x)\left[f(u_n)u_n - 4F(u_n)\right] dx \\ &\geq \frac{1}{4}\|u\|_\varepsilon^2. \end{aligned}$$

Thus, $\mathcal{I}_\varepsilon$ is coercive on $\mathcal{N}_\varepsilon$. □

Define the mapping $\tilde{m}_\varepsilon : H_\varepsilon \backslash \{0\} \to \mathcal{N}_\varepsilon$ and $m_\varepsilon : S_\varepsilon \to \mathcal{N}_\varepsilon$ by setting

$$\tilde{m}_\varepsilon(u) = \theta_u u \quad \text{and} \quad m_\varepsilon = \tilde{m}_\varepsilon|_{S_\varepsilon},$$

where θ_u is as in Lemma 2.4, $S_\varepsilon = \{u \in H_\varepsilon : \|u\|_\varepsilon = 1\}$.

We also consider the functionals $\tilde{\Upsilon}_\varepsilon : H_\varepsilon \backslash \{0\} \to \mathbb{R}$ and $\Upsilon_\varepsilon : S_\varepsilon \to \mathbb{R}$ defined by

$$\tilde{\Upsilon}_\varepsilon(u) = \mathcal{I}_\varepsilon(\tilde{m}_\varepsilon(u)) \quad \text{and} \quad \Upsilon_\varepsilon = \tilde{\Upsilon}_\varepsilon|_{S_\varepsilon}. \tag{2.7}$$

Since H_ε is a Hilbert space, Lemma 2.4, Lemma 2.5(*ii*) and Lemma 2.6 imply that the hypotheses (A_1), (A_2) and A_3 in [33] (see, Chapter 3) are satisfied. Hence, we have the following Lemmas 2.8–2.9.

Lemma 2.8. *(See[33]) The mapping $\tilde{m}_\varepsilon : H_\varepsilon \backslash \{0\} \to \mathcal{N}_\varepsilon$ is continuous and m_ε is a homeomorphism between S_ε and $\mathcal{N}_\varepsilon$, and the inverse of m_ε is given by $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon}$.*

Lemma 2.9. *(See[33]) For each $\varepsilon > 0$, we have*

(i) $\Upsilon_\varepsilon \in C^1(S_\varepsilon, \mathbb{R})$ *and for each $w \in S_\varepsilon$, one has*

$$\langle \Upsilon_\varepsilon'(w), z\rangle = \|m_\varepsilon(w)\|_\varepsilon \langle \mathcal{I}_\varepsilon'(m_\varepsilon(w)), z\rangle$$

for all $z \in T_w(S_\varepsilon) = \{v \in H_\varepsilon : \langle w, v\rangle = 0\}$.

(ii) If $\{w_n\}$ is a (PS) sequence for Υ_ε, then $\{m_\varepsilon(w_n)\}$ is a (PS) sequence for $\mathcal{I}_\varepsilon$. If $\{u_n\} \subset \mathcal{N}_\varepsilon$ is a bounded (PS) sequence for $\mathcal{I}_\varepsilon$, then $\{m_\varepsilon^{-1}(u_n)\}$ is a (PS) sequence for Υ_ε.

(iii) *w is a critical point of Υ_ε if and only if $m_\varepsilon(w)$ is a nontrivial point of $\mathcal{I}_\varepsilon$. Moreover, the corresponding values of Υ_ε and $\mathcal{I}_\varepsilon$ coincide and $\inf_{S_\varepsilon}\Upsilon_\varepsilon = \inf_{\mathcal{N}_\varepsilon}\mathcal{I}_\varepsilon$.*

Moreover, we also have

Lemma 2.10.

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon}\mathcal{I}_\varepsilon(u) = \inf_{u\in H_\varepsilon\setminus\{0\}}\max_{\theta\geq 0}\mathcal{I}_\varepsilon(\theta u) = \inf_{u\in S_\varepsilon}\max_{\theta\geq 0}\mathcal{I}_\varepsilon(\theta u) > 0.$$

3 | THE AUTONOMOUS PROBLEM

In the section, we shall prove some properties of the least energy solutions of the autonomous problem. Precisely, for any $a, b, c > 0$, we consider the following constant coefficient problem

$$\begin{cases} (-\Delta)^s u + au + \phi u = bf(u) + c|u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3, \\ (-\Delta)^t\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{3.1}$$

and the corresponding energy functional

$$\mathcal{I}_{abc}(u) = \frac{1}{2}\|u\|_a^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2\,dx - b\int_{\mathbb{R}^3}F(u)\,dx - \frac{c}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\,dx,$$

defined for $u \in H^s(\mathbb{R}^3)$, where $\|u\|_a = \left(\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx + a\int_{\mathbb{R}^3}u^2dx\right)^{\frac{1}{2}}$. The Nehari mainfold corresponding to $\mathcal{I}_{abc}$ is defined by

$$\mathcal{N}_{abc} = \left\{u \in H^s(\mathbb{R}^3)\setminus\{0\} : \left\langle \mathcal{I}'_{abc}(u), u\right\rangle = 0\right\}.$$

We define the least energy associated with (3.1) by

$$\gamma_{abc} = \inf_{u\in\mathcal{N}_{abc}}\mathcal{I}_{abc}(u).$$

The number γ_{abc} and the manifold $\mathcal{N}_{abc}$ have properties similar to those of c_ε and $\mathcal{N}_\varepsilon$ stated in Lemmas 2.4–2.7. Hence, for each $u \in H^s(\mathbb{R}^3)\setminus\{0\}$, there exists a unique $\theta_u > 0$ such that $\theta_u u \in \mathcal{N}_{abc}$. Recall that $S_a = \left\{u \in H^s(\mathbb{R}^3) : \|u\|_a = 1\right\}$ and define the mapping $\tilde{m}_{abc} : H^s(\mathbb{R}^3)\setminus\{0\} \to \mathcal{N}_{abc}$ by $\tilde{m}_{abc}(u) = \theta_u u$, and $m_{abc} = \tilde{m}_{abc}|_{S_a}$. Moreover, the inverse of m_{abc} is given by $m_{abc}^{-1}(u) = \frac{u}{\|u\|_a}$. Let the functional $\tilde{\Upsilon}_{abc} : H^s(\mathbb{R}^3)\setminus\{0\} \to \mathbb{R}$ be

$$\tilde{\Upsilon}_{abc}(u) = \mathcal{I}_{abc}\left(\tilde{m}_{abc}(u)\right) \quad\text{and}\quad \Upsilon_{abc} = \tilde{\Upsilon}_{abc}|_{S_a}.$$

Moreover, we also have

$$\gamma_{abc} = \inf_{u\in\mathcal{N}_{abc}}\mathcal{I}_{abc}(u) = \inf_{u\in H^s(\mathbb{R}^3)\setminus\{0\}}\max_{\theta\geq 0}\mathcal{I}_{abc}(\theta u) = \inf_{u\in S_a}\max_{\theta\geq 0}\mathcal{I}_{abc}(\theta u) > 0.$$

Lemma 3.1. *For any $a, b, c > 0$, the following inequality holds:*

$$0 < \gamma_{abc} < \frac{s}{3c^{\frac{3-2s}{2s}}}\mathcal{S}_s^{\frac{3}{2s}}.$$

Proof. The proof is similar to the proof of Lemma 3.3 in [35]. For the sake of completeness, we give the details here.

We define

$$u_\varepsilon(x) = \psi(x)U_\varepsilon(x), \quad x \in \mathbb{R}^3,$$

where $U_\varepsilon(x)=\varepsilon^{-\frac{3-2s}{2}}u^*\left(\frac{x}{\varepsilon}\right), u^*(x)=\frac{\tilde{u}\left(x/S_s^{\frac{1}{2s}}\right)}{\|\tilde{u}\|_{2_s^*}}$, $\kappa\in\mathbb{R}\backslash\{0\}$, $\mu_0>0$ and $x_0\in\mathbb{R}^3$ are fixed constants, $\tilde{u}(x)=\kappa\left(\mu_0^2+|x-x_0|^2\right)^{-\frac{3-2s}{2}}$ (see [31, Section 4]), and $\psi\in C^\infty\left(\mathbb{R}^3\right)$ such that $0\le\psi\le 1$ in $\mathbb{R}^3$, $\psi\equiv 1$ in B_r and $\psi\equiv 0$ in $\mathbb{R}^3\setminus B_{2r}$. From [31, Proposition 21 and Proposition 22], we know that

$$\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u_\varepsilon(x)\right|^2dx\le S_s^{\frac{3}{2s}}+O\left(\varepsilon^{3-2s}\right), \tag{3.2}$$

$$\int_{\mathbb{R}^3}|u_\varepsilon(x)|^{2_s^*}dx=S_s^{\frac{3}{2s}}+O\left(\varepsilon^3\right), \tag{3.3}$$

and

$$\int_{\mathbb{R}^3}|u_\varepsilon(x)|^r dx=\begin{cases}O\left(\varepsilon^{\frac{3(2-r)+2sr}{2}}\right), & r>\frac{3}{3-2s},\\ O\left(\varepsilon^{\frac{3(2-r)+2sr}{2}}|\log\varepsilon|\right), & r=\frac{3}{3-2s},\\ O\left(\varepsilon^{\frac{3-2s}{2}r}\right), & r<\frac{3}{3-2s}.\end{cases} \tag{3.4}$$

From (f_4), we have

$$\begin{aligned}\mathcal{I}_{abc}(u)&=\frac12\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx+\frac{a}{2}\int_{\mathbb{R}^3}u^2dx+\frac14\int_{\mathbb{R}^3}\phi_u^t u^2dx-b\int_{\mathbb{R}^3}F(u)\,dx-\frac{c}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx\\&\le\frac12\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx+\frac{a}{2}\int_{\mathbb{R}^3}u^2dx+\frac14\int_{\mathbb{R}^3}\phi_u^t u^2dx-c_2b\int_{\mathbb{R}^3}|u|^\sigma dx-\frac{c}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx:=\Psi_{abc}(u).\end{aligned}$$

By a direct calculation, we have

$$\Psi_{abc}(\theta u)=\frac{\theta^2}{2}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx+\frac{\theta^2a}{2}\int_{\mathbb{R}^3}u^2dx+\frac{\theta^4}{4}\int_{\mathbb{R}^3}\phi_u^t u^2dx-c_2b\theta^\sigma\int_{\mathbb{R}^3}|u|^\sigma dx-\frac{\theta^{2_s^*}c}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx.$$

Define $g(\theta)=\frac{\theta^2}{2}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u_\varepsilon\right|^2dx-\frac{\theta^{2_s^*}c}{2_s^*}\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}dx$ for $\theta\ge 0$. We note that $g(\theta)$ attains its maximum at

$$\theta_0=\left(\frac{\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u_\varepsilon\right|^2dx}{c\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}dx}\right)^{\frac{3-2s}{4s}}.$$

Moreover, by (3.2)–(3.3), using the elementary inequality $(\alpha+\beta)^q\le\alpha^q+q(\alpha+\beta)^{q-1}\beta$ which holds for $q\ge 1$ and $\alpha,\beta\ge 0$, we deduce that

$$\begin{aligned}\max_{\theta\ge 0}g(\theta)=g(\theta_0)&=\frac12\left(\frac{\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2dx}{c\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}dx}\right)^{\frac{3-2s}{2s}}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u_\varepsilon\right|^2dx-\frac{1}{2_s^*}\left(\frac{\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2dx}{c\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}dx}\right)^{\frac{3}{2s}}c\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}dx\\&=\frac{s}{3}\frac{\|(-\Delta)^{\frac{s}{2}}u_\varepsilon\|_2^{\frac{3}{s}}}{c^{\frac{3-2s}{2s}}\|u_\varepsilon\|_{2_s^*}^{\frac{3}{s}}}\end{aligned}$$

$$\leq \frac{s}{3c^{\frac{3-2s}{2s}}} \frac{\left(S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s})\right)^{\frac{3}{2s}}}{\left(S_s^{\frac{3}{2s}} + O(\varepsilon^3)\right)^{\frac{3-2s}{2s}}}$$

$$\leq \frac{s}{3c^{\frac{3-2s}{2s}}} \frac{\left(S_s^{\frac{3}{2s}}\right)^{\frac{3}{2s}} + O(\varepsilon^{3-2s})}{\left(S_s^{\frac{3}{2s}} + O(\varepsilon^3)\right)^{\frac{3-2s}{2s}}}$$

$$\leq \frac{s}{3c^{\frac{3-2s}{2s}}} S_s^{\frac{3}{2s}} + O\left(\varepsilon^{3-2s}\right). \tag{3.5}$$

Since $\mathcal{I}_{abc}\left(\theta u_\varepsilon\right) \to -\infty$ as $\theta \to \infty$, by standard argument, there exists $\theta_\varepsilon > 0$ such that

$$0 < \gamma_{abc} \leq \max_{\theta\geq 0} \mathcal{I}_{abc}\left(\theta u_\varepsilon\right) = \mathcal{I}_{abc}\left(\theta_\varepsilon u_\varepsilon\right) \leq \Psi_{abc}\left(\theta_\varepsilon u_\varepsilon\right), \tag{3.6}$$

which implies that $\theta_\varepsilon \geq A_1 > 0$ for some constant A_1. On the other hand, from (3.2)–(3.4), for any $\varepsilon > 0$, we have that

$$0 < \gamma_{abc} \leq \Psi_{abc}\left(\theta_\varepsilon u_\varepsilon\right) \leq C_1\theta_\varepsilon^2 + C_2\theta_\varepsilon^4 - C_3\theta_\varepsilon^{2_s^*},$$

which implies that there exists $A_2 > 0$ such that $\theta_\varepsilon \leq A_2$ and thus $0 < A_1 \leq \theta_\varepsilon \leq A_2$ for any $\varepsilon > 0$.

Now, by (3.2)–(3.6), we deduce that

$$\begin{aligned}
\Psi_{abc}\left(\theta_\varepsilon u_\varepsilon\right) &\leq \frac{s}{3c^{\frac{3-2s}{2s}}} S_s^{\frac{3}{2s}} + O\left(\varepsilon^{3-2s}\right) + \frac{\theta_\varepsilon^2 a}{2}\int_{\mathbb{R}^3} u_\varepsilon^2\,dx + \frac{\theta_\varepsilon^4}{4}\int_{\mathbb{R}^3} \phi_{u_\varepsilon}^t u_\varepsilon^2\,dx - c_2 b\theta_\varepsilon^\sigma \int_{\mathbb{R}^3} |u_\varepsilon|^\sigma dx \\
&\leq \frac{s}{3c^{\frac{3-2s}{2s}}} S_s^{\frac{3}{2s}} + O\left(\varepsilon^{3-2s}\right) + \frac{A_2^2 a}{2}\int_{\mathbb{R}^3} u_\varepsilon^2\,dx + \frac{A_2^4}{4}\int_{\mathbb{R}^3} \phi_{u_\varepsilon}^t u_\varepsilon^2\,dx - c_2 b A_1^\sigma \int_{\mathbb{R}^3} |u_\varepsilon|^\sigma dx \\
&\leq \frac{s}{3c^{\frac{3-2s}{2s}}} S_s^{\frac{3}{2s}} + O\left(\varepsilon^{3-2s}\right) + \frac{A_2^2 a}{2}\int_{\mathbb{R}^3} u_\varepsilon^2\,dx + CA_2^4\left(\int_{\mathbb{R}^3} |u_\varepsilon|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{3}} - A_1^\sigma \int_{\mathbb{R}^3} |u_\varepsilon|^\sigma dx.
\end{aligned}$$

Since $s > \frac{3}{4}$, then $\frac{3}{3-2s} > 2$ and

$$\int_{\mathbb{R}^3} u_\varepsilon^2\,dx = O\left(\varepsilon^{3-2s}\right).$$

Therefore,

$$\Psi_{abc}\left(\theta_\varepsilon u_\varepsilon\right) \leq \frac{s}{3c^{\frac{3-2s}{2s}}} S_s^{\frac{3}{2s}} + O\left(\varepsilon^{3-2s}\right) + C\left(\int_{\mathbb{R}^3} |u_\varepsilon|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{3}} - C\int_{\mathbb{R}^3} |u_\varepsilon|^\sigma dx.$$

Moreover, we deduce that

$$\lim_{\varepsilon\to 0} \frac{\left(\int_{\mathbb{R}^3} |u_\varepsilon(x)|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{3}}}{\varepsilon^{3-2s}} = \begin{cases} \lim\limits_{\varepsilon\to 0} \dfrac{O\left(\varepsilon^{4s+2t-3}\right)}{\varepsilon^{3-2s}} = 0, & \frac{12}{3+2t} > \frac{3}{3-2s}, \\ \lim\limits_{\varepsilon\to 0} \dfrac{O\left(\varepsilon^{4s+2t-3}|\log\varepsilon|^{\frac{3+2t}{3}}\right)}{\varepsilon^{3-2s}} = 0, & \frac{12}{3+2t} = \frac{3}{3-2s}, \\ \lim\limits_{\varepsilon\to 0} \dfrac{O\left(\varepsilon^{6-4s}\right)}{\varepsilon^{3-2s}} = 0, & \frac{12}{3+2t} < \frac{3}{3-2s}, \end{cases}$$

and we also have $\frac{3}{3-2s} < \frac{4s}{3-2s} < 4 \le \sigma < 2_s^*$, then we deduce that

$$\lim_{\varepsilon\to 0}\frac{\int_{\mathbb{R}^3}|u_\varepsilon(x)|^\sigma dx}{\varepsilon^{3-2s}} = \lim_{\varepsilon\to 0}\frac{O\left(\varepsilon^{3-\frac{3-2s}{2}\sigma}\right)}{\varepsilon^{3-2s}} = +\infty.$$

Therefore, the above arguments imply that

$$0 < \gamma_{abc} \le \mathcal{I}_{abc}\left(\theta_\varepsilon u_\varepsilon\right) \le \Psi_{abc}\left(\theta_\varepsilon u_\varepsilon\right) < \frac{s}{3c^{\frac{3-2s}{2s}}}S_s^{\frac{3}{2s}}.$$

Thus we complete the proof. □

Lemma 3.2. *For any $a, b, c > 0$, sytem* (3.1) *has a positive ground state solution in $H^s(\mathbb{R}^3)$.*

Proof. If $u \in \mathcal{N}_{abc}$ satisfies $\mathcal{I}_{abc}(u) = \gamma_{abc}$, then

$$\Upsilon_{abc}\left(m_{abc}^{-1}(u)\right) = \mathcal{I}_{abc}\left(m_{abc}\left(m_{abc}^{-1}(u)\right)\right) = \mathcal{I}_{abc}(u) = \gamma_{abc} = \inf_{S_a}\Upsilon_{abc}(u).$$

That is, $m_{abc}^{-1}(u)$ is a minimizer of Υ_{abc}, and hence a critical point of Υ_{abc}. Therefore, similar to Lemma 2.9, we see that u is a critical point of $\mathcal{I}_{abc}$. It remains to show that there exists a minimizer u of $\mathcal{I}_{abc}|_{\mathcal{N}_{abc}}$. By Ekeland's variational principle in [14], there exists a sequence $\{w_n\} \subset S_a$ with $\Upsilon_{abc}(w_n) \to \gamma_{abc}$, $\Upsilon'_{abc}(w_n) \to 0$ as $n \to \infty$. In fact, set

$$g_a(u) = \|u\|_a^2 - 1, \quad \text{for all} \quad u \in H^s(\mathbb{R}^3).$$

Notice that $S_a = \left\{u \in H^s\left(\mathbb{R}^3\right) : g_a(u) = 0\right\}$ and for each $u \in S_a$, one has

$$\left\langle g_a'(u), u\right\rangle = 2\|u\|_a^2 = 2 > 0.$$

By Proposition 9 in [33], we know that $\tilde{\Upsilon}_{abc} \in C^1\left(H^s(\mathbb{R}^3)\setminus\{0\}, \mathbb{R}\right)$ and

$$\left\langle \tilde{\Upsilon}'_{abc}(u), v\right\rangle = \frac{\|\tilde{m}(u)\|_a}{\|u\|_a}\left\langle \mathcal{I}'_{abc}(\tilde{m}_{abc}(u)), v\right\rangle, \quad \text{for all} \quad 0 \ne u, v \in H^s(\mathbb{R}^3).$$

Hence, by Corollary 3.4 in [14] there exists a sequence $\{w_n\} \subset S_a$ such that $\Upsilon_{abc}(w_n) \to \gamma_{abc}$ and there exists $\alpha_n \in R$ such that $\left\|\Upsilon'_{abc}(w_n) - \alpha_n g_a'(w_n)\right\|_a \to 0$. It implies

$$\alpha_n = \frac{\left\langle \Upsilon'_{abc}(w_n), g_a'(w_n)\right\rangle}{\|g_a'(w_n)\|_a^2} + o(1).$$

Hence, $\Upsilon'_{abc}(w_n) - \frac{\left\langle \Upsilon'_{abc}(w_n), g_a'(w_n)\right\rangle}{\|g_a'(w_n)\|_a^2} g_a'(w_n) = o(1)$, i.e., $\Upsilon'_{abc}(w_n) = o(1)$. Let $u_n = m_{abc}(w_n)$, by the definition of m_{abc}, we know $u_n \in \mathcal{N}_{abc}$ for all $n \in \mathbb{N}$. Similar to Lemma 2.9, one has $\mathcal{I}_{abc}(u_n) \to \gamma_{abc}$, $\mathcal{I}'_{abc}(u_n) \to 0$ as $n \to \infty$. Similar to Lemma 2.7, we know that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$.

Next, we claim that there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ and $R, \delta > 0$ such that

$$\int_{B_R(y_n)} |u_n|^2 dx \ge \delta, \quad n \in \mathbb{N}. \tag{3.7}$$

Otherwise, by Lemma 2.3, we have

$$u_n \to 0 \text{ in } L^r\left(\mathbb{R}^3\right) \text{ for } 2 < r < 2_s^*.$$

Thus, by (2.5), we have

$$\int_{\mathbb{R}^3} F(u_n)\, dx \to 0, \quad \int_{\mathbb{R}^3} f(u_n)u_n\, dx \to 0 \text{ as } n \to \infty. \tag{3.8}$$

Moreover, by Lemma 2.1 (iii), we can obtain

$$\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 \, dx \to 0 \text{ as } n \to \infty. \tag{3.9}$$

Notice that

$$\mathcal{I}_{abc}(u_n) - \frac{1}{2_s^*}\langle \mathcal{I}_{abc}'(u_n), u_n \rangle = \frac{s}{3}\|u_n\|^2 + \frac{4s-3}{12}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 \, dx - b\int_{\mathbb{R}^3} F(u_n)\, dx + \frac{b}{2_s^*}\int_{\mathbb{R}^3} f(u_n) u_n \, dx.$$

Therefore, (3.8)–(3.9) imply that

$$\int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 dx \le \frac{3}{s}\gamma_{abc} + o(1).$$

Similarly, we have

$$\int_{\mathbb{R}^3} |u_n|^{2_s^*} dx = \frac{3}{sc}\gamma_{abc} + o(1).$$

Moreover,

$$\int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 dx - c\int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \le o(1),$$

which implies

$$\gamma_{abc} \ge \frac{s}{3c^{\frac{3-2s}{2s}}} S_s^{\frac{3}{2s}},$$

which is a contradiction with Lemma 3.1. Let $v_n(x) = u_n(x + y_n)$, then $\{v_n\}$ is bounded in $H^s(\mathbb{R}^3)$ by the boundedness of $\{u_n\}$ and, up to a subsequence, we assume that $v_n \rightharpoonup v$ in $H^s(\mathbb{R}^3)$. By (3.7), we see that $v \neq 0$ and it is easy check that $\mathcal{I}_{abc}(v) = \gamma_{abc}$. Moreover, by Lemma 2.2$(ii)$ and Lemma 2.3, we can obtain $\mathcal{I}_{abc}'(v) = 0$

Next we only need to prove that v is positive. Put $v^\pm = \max\{\pm v, 0\}$, the positive (negative) part of v. We note that all the calculations above can be repeated word by word, replacing $\mathcal{I}_{abc}^+(u)$ with the functional

$$\mathcal{I}_{abc}^+(v) = \frac{1}{2}\int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} v\right|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} a v^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_v^s v^2 dx - b\int_{\mathbb{R}^3} F(v^+)\, dx - \frac{c}{2_s^*}\int_{\mathbb{R}^3} |v^+|^{2_s^*} dx.$$

In this way we get a ground state solution v of the equation

$$(-\Delta)^s v + av + \phi_v^t v = bf(v^+) + c|v^+|^{2_s^*-2} v^+, \text{ in } \mathbb{R}^3. \tag{3.10}$$

Using v^- as a test function in (3.10) we obtain

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} v \cdot (-\Delta)^{\frac{s}{2}} v^- dx + \int_{\mathbb{R}^3} a|v^-|^2 dx + \int_{\mathbb{R}^3} \phi_v^t (v^-)^2 dx = 0. \tag{3.11}$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} v \cdot (-\Delta)^{\frac{s}{2}} v^- dx &= \frac{1}{2}C(s)\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(v(x)-v(y))(v^-(x)-v^-(y))}{|x-y|^{3+2s}}\, dx\, dy \\
&\ge \frac{1}{2}C(s)\left[\iint_{\{v>0\}\times\{v<0\}} \frac{(v(x)-v(y))(-v^-(y))}{|x-y|^{3+2s}}\, dx\, dy \right.\\
&\quad \left. + \iint_{\{v<0\}\times\{v<0\}} \frac{(v^-(x)-v^-(y))^2}{|x-y|^{3+2s}}\, dx\, dy + \iint_{\{v<0\}\times\{v>0\}} \frac{(v(x)-v(y))v^-(x)}{|x-y|^{3+2s}}\, dx\, dy\right] \\
&\ge 0.
\end{aligned}$$

Thus, it follows from (3.11) and Lemma 2.1(i), we have $v^- = 0$ and $v \geq 0$. Moreover, if $v(y_0) = 0$ for some $y_0 \in \mathbb{R}^3$, then $(-\Delta)^s v(y_0) = 0$ and by (2.1), we have

$$(-\Delta)^s v(y_0) = -\frac{C(s)}{2}\int_{\mathbb{R}^3} \frac{v(y_0+y)+v(y_0-y)-2v(y_0)}{|y|^{3+2s}}\,dy,$$

therefore,

$$\int_{\mathbb{R}^3} \frac{v(y_0+y)+v(y_0-y)}{|y|^{3+2s}}\,dy = 0,$$

yielding $v \equiv 0$, a contradiction. □

The following lemma describes a comparison between the mountain pass values for different parameters $a, b, c > 0$, which will play an important role in proving the existence results in Section 4.

Lemma 3.3. *Let $a_j > 0$ and $b_j > 0, j = 1, 2$, with $a_1 \leq a_2, b_1 \geq b_2$ and $c_1 \geq c_2$. Then $\gamma_{a_1b_1c_1} \leq \gamma_{a_2b_2c_2}$. In particular, if one of inequalities is strict, then $\gamma_{a_1b_1c_1} < \gamma_{a_2b_2c_2}$.*

Proof. Let $u \in \mathcal{N}_{a_2b_2c_2}$ be such that

$$\gamma_{a_2b_2c_2} = \mathcal{I}_{a_2b_2c_2}(u) = \max_{\theta\geq 0} \mathcal{I}_{a_2b_2c_2}(\theta u).$$

Let $u_0 = \theta_1 u$ be such that $\mathcal{I}_{a_1b_1c_1}\left(u_0\right) = \max_{\theta\geq 0} \mathcal{I}_{a_1b_1c_1}(\theta u)$. One has

$$\begin{aligned}\gamma_{a_2b_2c_2} &= \mathcal{I}_{a_2b_2c_2}(u) \geq \mathcal{I}_{a_2b_2c_2}\left(u_0\right)\\ &= \mathcal{I}_{a_1b_1c_1}\left(u_0\right) + \frac{1}{2}\left(a_2-a_1\right)\int_{\mathbb{R}^3}|u_0|^2dx + \left(b_1-b_2\right)\int_{\mathbb{R}^3}F\left(u_0\right)dx + \frac{1}{2_s^*}\left(c_1-c_2\right)\int_{\mathbb{R}^3}|u_0|^{2_s^*}dx\\ &\geq \gamma_{a_1b_1c_1}.\end{aligned}$$

The second part can be obtained similarly. Thus, we complete the proof. □

4 | EXISTENCE OF GROUND STATE SOLUTIONS

In the section, we will prove the existence of ground state solutions to the sytem (2.2). Observing that for any $x_P \in \mathcal{C}_P$, we set $\tilde{V}(x) = V(x+x_P)$, $\tilde{P}(x) = P(x+x_P)$ and $\tilde{Q}(x) = Q(x+x_P)$. Clearly, if $\tilde{u}(x)$ is a solution of

$$\begin{cases}(-\Delta)^s\tilde{u} + \tilde{V}(\varepsilon x)\tilde{u} + \phi\tilde{u} = \tilde{P}(\varepsilon x)f(\tilde{u}) + \tilde{Q}(\varepsilon x)|\tilde{u}|^{2_s^*-2}\tilde{u}, & \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi = \tilde{u}^2, & \text{in } \mathbb{R}^3,\end{cases}$$

then $u(x) = \tilde{u}(x - x_P)$ solves (2.2). Thus, without loss of generality, we may assume that

$$x_P = 0 \in \mathcal{C}_P,$$

so

$$Q(0) = Q_{\max}, \quad P(0) = P_Q \quad \text{and} \quad \nu := V(0) \leq V(x) \text{ for all } |x| \geq R. \tag{4.1}$$

Lemma 4.1. $\limsup_{\varepsilon\to 0} c_\varepsilon \leq \gamma_{\nu P_Q Q_{\max}}$.

Proof. Denote $V_\varepsilon^a(x) = \max\{a, V(\varepsilon x)\}$, $P_\varepsilon^b(x) = \min\{b, P(\varepsilon x)\}$ and $Q_\varepsilon^c(x) = \min\{c, Q(\varepsilon x)\}$, where a, b, c are three positive constants. Define the auxiliary functional as follows:

$$\mathcal{I}_\varepsilon^{abc}(u) := \frac{1}{2}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}V_\varepsilon^a(x)u^2dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2dx - \int_{\mathbb{R}^3}P_\varepsilon^b(x)F(u)\,dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3}Q_\varepsilon^c(x)|u|^{2_s^*}dx,$$

for any $u \in H^s(\mathbb{R}^3)$, which implies that $\mathcal{I}_{abc}(u) \le \mathcal{I}_\varepsilon^{abc}(u)$, and thus $\gamma_{abc} \le c_\varepsilon^{abc}$, where c_ε^{abc} is the least energy of $\mathcal{I}_\varepsilon^{abc}$. By the definition of $V_{\min}$, $P_{\max}$ and $Q_{\max}$, we get $V_\varepsilon^{V_{\min}}(x) = V(\varepsilon x)$, $P_\varepsilon^{P_{\max}}(x) = P(\varepsilon x)$ and $Q_\varepsilon^{Q_{\max}}(x) = Q(\varepsilon x)$. Therefore, we have

$$\mathcal{I}_\varepsilon^{V_{\min}P_{\max}Q_{\max}}(u) = \mathcal{I}_\varepsilon(u), \tag{4.2}$$

and $V_\varepsilon^{V_{\min}}(x) \to V(0) = \nu$, $P_\varepsilon^{P_{\max}}(x) \to P(0) = P_Q$, $Q_\varepsilon^{Q_{\max}}(x) \to Q(0) = Q_{\max}$ on bounded sets of x as $\varepsilon \to 0$.

Now, we claim $\limsup_{\varepsilon\to 0} c_\varepsilon^{V_{\min}P_{\max}Q_{\max}} \le \gamma_{\nu P_Q Q_{\max}}$.

Indeed, let w be a ground state solution of $\mathcal{I}_{\nu P_Q Q_{\max}}$ by Lemma 3.2, that is, $\mathcal{I}_{\nu P_Q Q_{\max}}(w) = \gamma_{\nu P_Q Q_{\max}}$, then there exists $\theta_\varepsilon > 0$ such that $\theta_\varepsilon w \in \mathcal{N}_\varepsilon^{V_{\min}P_{\max}Q_{\max}}$, where $\mathcal{N}_\varepsilon^{V_{\min}P_{\max}Q_{\max}}$ is the Nehari manifold of the functional $\mathcal{I}_\varepsilon^{V_{\min}P_{\max}Q_{\max}}$. Thus

$$c_\varepsilon^{V_{\min}P_{\max}Q_{\max}} \le \mathcal{I}_\varepsilon^{V_{\min}P_{\max}Q_{\max}}(\theta_\varepsilon w) = \max_{\theta\ge 0} \mathcal{I}_\varepsilon^{V_{\min}P_{\max}Q_{\max}}(\theta w).$$

One has

$$\begin{aligned}\mathcal{I}_\varepsilon^{V_{\min}P_{\max}Q_{\max}}(\theta_\varepsilon w) &= \mathcal{I}_{\nu P_Q Q_{\max}}(\theta_\varepsilon w) + \frac{1}{2}\int_{\mathbb{R}^3}\left(V_\varepsilon^{V_{\min}}(x) - \nu\right)|\theta_\varepsilon w|^2 dx \\ &\quad + \int_{\mathbb{R}^3}\left(P_Q - P_\varepsilon^{P_{\max}}(x)\right)F(\theta_\varepsilon w)\,dx + \frac{1}{2_s^*}\int_{\mathbb{R}^3}\left(Q_{\max} - Q_\varepsilon^{Q_{\max}}(x)\right)|\theta_\varepsilon w|^{2_s^*} dx.\end{aligned} \tag{4.3}$$

By Lemma 2.4(ii), we can assume that $\theta_\varepsilon \to \theta_0$ as $\varepsilon \to 0$. Since $w \in L^2(\mathbb{R}^3)$, for any $\tau > 0$, there exists a $R > 0$ such that

$$\int_{\mathbb{R}^3\setminus B_R(0)} |w|^2 dx < \tau.$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^3}\left(V_\varepsilon^{V_{\min}}(x) - \nu\right)|\theta_\varepsilon w|^2 dx &= \int_{\mathbb{R}^3}\left(V_\varepsilon^{V_{\min}}(x) - \nu\right)|\theta_0 w|^2 dx + o(1) \\ &= \int_{\mathbb{R}^3\setminus B_R(0)}\left(V_\varepsilon^{V_{\min}}(x) - \nu\right)|\theta_0 w|^2 dx + \int_{B_R(0)}\left(V_\varepsilon^{V_{\min}}(x) - \nu\right)|\theta_0 w|^2 dx + o(1) \\ &\le C\theta_0^2\tau + o(1),\end{aligned}$$

here we use the fact that $V_\varepsilon^{V_{\min}}(x) \to \nu$ in $x \in B_R(0)$. Thus, we obtain

$$\int_{\mathbb{R}^3}\left(V_\varepsilon^{V_{\min}}(x) - \nu\right)|\theta_\varepsilon w|^2 dx = o(1).$$

Similarly, we have

$$\int_{\mathbb{R}^3}\left(P_Q - P_\varepsilon^{P_{\max}}(x)\right)F(\theta_\varepsilon w)\,dx = o(1), \quad \int_{\mathbb{R}^3}\left(Q_{\max} - Q_\varepsilon^{Q_{\max}}(x)\right)|\theta_\varepsilon w|^{2_s^*} dx = o(1).$$

Thus, by (4.3), we have

$$\mathcal{I}_\varepsilon^{V_{\min}P_{\max}Q_{\max}}(\theta_\varepsilon w) = \mathcal{I}_{\nu P_Q Q_{\max}}(\theta_\varepsilon w) + o(1) \to \mathcal{I}_{\nu P_Q Q_{\max}}(\theta_0 w) \quad \text{as} \quad \varepsilon \to 0. \tag{4.4}$$

Consequently

$$c_\varepsilon^{V_{\min}P_{\max}Q_{\max}} \le \mathcal{I}_\varepsilon^{V_{\min}P_{\max}Q_{\max}}(\theta_\varepsilon w) \to \mathcal{I}_{\nu P_Q Q_{\max}}(\theta_0 w) \le \max_{\theta\ge 0}\mathcal{I}_{\nu P_Q Q_{\max}}(\theta w) = \mathcal{I}_{\nu P_Q Q_{\max}}(w) = \gamma_{\nu P_Q Q_{\max}}.$$

From (4.2), we obtain $c_\varepsilon^{V_{\min}P_{\max}Q_{\max}} = c_\varepsilon$. This completes the proof. □

Next we only truncate the functional $V(x)$ and $P(x)$ with $a = \nu$ and $b \in (P_\infty, P_Q)$ and consider the truncated energy functional

$$\tilde{\mathcal{I}}_\varepsilon^{\nu b}(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon^\nu(x)u^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2 dx - \int_{\mathbb{R}^3} P_\varepsilon^b(x)F(u)dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} Q(\varepsilon x)|u|^{2_s^*} dx.$$

The corresponding Nehari manifold and least energy are $\tilde{\mathcal{N}}_\varepsilon^{\nu b}$ and $\tilde{c}_\varepsilon^{\nu b}$, respectively.

We have an important lower bound for the least energy $\tilde{c}_\varepsilon^{\nu b}$.

Lemma 4.2. $\tilde{c}_\varepsilon^{\nu b} \geq \gamma_{\nu b Q_{\max}}$.

Proof. Since $V_\varepsilon^\nu(x) \geq \nu$, $P_\varepsilon^b(x) \leq b$, $Q(\varepsilon x) \leq Q_{\max}$, from the characterization of the value $\gamma_{\nu b Q_{\max}}$, we know that

$$\inf_{u \in H_\varepsilon \setminus \{0\}} \max_{\theta \geq 0} \tilde{\mathcal{I}}_\varepsilon^{\nu b}(\theta u) \geq \inf_{u \in H_\varepsilon \setminus \{0\}} \max_{\theta \geq 0} \mathcal{I}_{\nu b Q_{\max}}(\theta u),$$

which gives

$$\tilde{c}_\varepsilon^{\nu b} \geq \gamma_{\nu b Q_{\max}}.$$

This completes the proof. □

Lemma 4.3. *c_ε is attained at some positive u_ε for small $\varepsilon > 0$.*

Proof. Similar to the arguments of Lemma 3.2, there exists a sequence $\{w_n\} \subset S_\varepsilon$ with $\Upsilon_\varepsilon(w_n) \to c_\varepsilon$, $\Upsilon_\varepsilon'(w_n) \to 0$ as $n \to \infty$. Let $u_n = m_\varepsilon(w_n)$, by the definition of m_ε, we know $u_n \in \mathcal{N}_\varepsilon$ for all $n \in \mathbb{N}$. By Lemma 2.9, one has $\mathcal{I}_\varepsilon(u_n) \to c_\varepsilon$, $\mathcal{I}_\varepsilon'(u_n) \to 0$ as $n \to \infty$. Moreover, we know that $\{u_n\}$ is bounded in H_ε by Lemma 2.7. Assume that $u_n \rightharpoonup u_\varepsilon$ in H_ε, then by Lemma 2.2(ii) and Lemma 2.3, we have $\mathcal{I}_\varepsilon'(u_\varepsilon) = 0$. If $u_\varepsilon \neq 0$, it is easy to check that $\mathcal{I}_\varepsilon(u_\varepsilon) = c_\varepsilon$. Next we show that $u_\varepsilon \neq 0$ for small $\varepsilon > 0$. Assume by contradiction that there exists a sequence $\varepsilon_j \to 0$ such that $u_{\varepsilon_j} = 0$, then $u_n \rightharpoonup 0$ in H_ε, and thus $u_n \to 0$ in $L_{loc}^r(\mathbb{R}^3)$ for $r \in [1, 2_s^*)$ and $u_n(x) \to 0$ a.e. in $x \in \mathbb{R}^3$.

By (A_1), choose $b \in (P_\infty, P_Q)$ and consider the functional $\tilde{\mathcal{I}}_{\varepsilon_j}^{\nu b}$, where ν is defined in (4.1). Let $\theta_n > 0$ be such that $\theta_n u_n \in \tilde{\mathcal{N}}_{\varepsilon_j}^{\nu b}$, from Lemma 2.4($ii$), $\{\theta_n\}$ is bounded. Assume $\theta_n \to \theta_0$ as $n \to \infty$. By (A_1) again, the set $\{x \in \mathbb{R}^3 : V_\varepsilon(x) < \nu\}$ is bounded. Thus,

$$\int_{\mathbb{R}^3} \left(V_\varepsilon^\nu(x) - V(\varepsilon_j x)\right)|\theta_n u_n|^2 dx = \int_{\{V_\varepsilon(x) < \nu\}} \left(\nu - V(\varepsilon_j x)\right)|\theta_n u_n|^2 dx = o(1). \tag{4.5}$$

Similarly, since $b > P_\infty$ implies $\{x \in \mathbb{R}^3 : P_\varepsilon(x) \geq b\}$ is bounded and f is subcritical growth, we have

$$\int_{\mathbb{R}^3} \left(P(\varepsilon_j x) - P_{\varepsilon_j}^b(x)\right) F(\theta_n u_n)\, dx = o(1). \tag{4.6}$$

Therefore, by (4.5)–(4.6) and $\mathcal{I}_{\varepsilon_j}(\theta_n u_n) \leq \mathcal{I}_{\varepsilon_j}(u_n)$, we have

$$\begin{aligned}
\tilde{c}_{\varepsilon_j}^{\nu b} &\leq \tilde{\mathcal{I}}_{\varepsilon_j}^{\nu b}(\theta_n u_n) \\
&= \mathcal{I}_{\varepsilon_j}(\theta_n u_n) + \frac{1}{2}\int_{\mathbb{R}^3} \left(V_\varepsilon^\nu(x) - V(\varepsilon_j x)\right)|\theta_n u_n|^2 dx + \int_{\mathbb{R}^3} \left(P(\varepsilon_j x) - P_{\varepsilon_j}^b(x)\right) F(\theta_n u_n)\, dx \\
&= \mathcal{I}_{\varepsilon_j}(\theta_n u_n) + o(1) \leq \mathcal{I}_{\varepsilon_j}(u_n) + o(1),
\end{aligned}$$

which implies that $\tilde{c}_{\varepsilon_j}^{\nu b} \leq c_{\varepsilon_j}$ as $n \to \infty$. Notice that $\tilde{c}_{\varepsilon_j}^{\nu b} \geq \gamma_{\nu b Q_{\max}}$ by Lemma 4.2. Thus, we have

$$\gamma_{\nu b Q_{\max}} \leq c_{\varepsilon_j}.$$

In virtue of Lemma 4.1, letting $\varepsilon_j \to 0$ yields

$$\gamma_{\nu b Q_{\max}} \leq \gamma_{\nu P_Q Q_{\max}}.$$

Applying Lemma 3.3 and the fact that $b < P_Q$ yield a contradiction. Therefore, c_ε is attained at some $u_\varepsilon \neq 0$ for small $\varepsilon > 0$. Moreover, similar to Lemma 3.2, u_ε is a positive solution of the system (2.2) and the proof is completed. □

5 | CONCENTRATION AND CONVERGENCE OF GROUND STATE SOLUTIONS

In this section, we are devoted to the concentration behavior of the ground state solutions u_ε as $\varepsilon \to 0$. We will prove the following results.

Theorem 5.1. *Let u_ε be a solution of the system* (2.2) *given by Lemma 4.3, then u_ε possesses a global maximum point y_ε such that, up to a subsequence, $\varepsilon y_\varepsilon \to x_0$ as $\varepsilon \to 0$, $\lim_{\varepsilon\to 0} dist(\varepsilon y_\varepsilon, \mathcal{H}_P) = 0$ and $v_\varepsilon(x) := u_\varepsilon(x+y_\varepsilon)$ converges in $H^s(\mathbb{R}^3)$ to a positive ground state solution of*

$$\begin{cases} (-\Delta)^s u + V(x_0)u + \phi u = P(x_0)f(u) + Q(x_0)|u|^{2_s^*-2}u, & in\ \mathbb{R}^3, \\ (-\Delta)^t\phi = u^2, & in\ \mathbb{R}^3. \end{cases}$$

In particular, if $\mathcal{V}\cap\mathcal{P}\cap\mathcal{Q} \neq \emptyset$, then $\lim_{\varepsilon\to 0} dist(\varepsilon y_\varepsilon, \mathcal{V}\cap\mathcal{P}\cap\mathcal{Q}) = 0$, and up to a subsequence, v_ε converges in $H^s(\mathbb{R}^3)$ to a positive ground state solution of

$$\begin{cases} (-\Delta)^s u + V_{\min}u + \phi u = P_{\max}f(u) + Q_{\max}|u|^{2_s^*-2}u, & in\ \mathbb{R}^3, \\ (-\Delta)^t\phi = u^2, & in\ \mathbb{R}^3. \end{cases}$$

Lemma 5.2. *There exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*)$, there exist $\{y_\varepsilon\} \subset \mathbb{R}^3$ and $\tilde{R}$, $\delta > 0$ such that*

$$\int_{B_{\tilde{R}}(y_\varepsilon)} u_\varepsilon^2\, dx \geq \delta.$$

Proof. Assume by contradiction that there exists a sequence $\varepsilon_j \to 0$ as $j \to \infty$, such that for any $R > 0$,

$$\lim_{j\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} u_{\varepsilon_j}^2\, dx = 0.$$

Thus, by Lemma 2.3, we have

$$u_{\varepsilon_j} \to 0 \text{ in } L^r(\mathbb{R}^3) \text{ for } 2 < r < 2_s^*,$$

Thus, since the potential function P is bounded and (2.5), we have

$$\int_{\mathbb{R}^3} P(\varepsilon_j x)F(u_{\varepsilon_j})\, dx \to 0, \int_{\mathbb{R}^3} P(\varepsilon_j x)f(u_{\varepsilon_j})u_{\varepsilon_j}\, dx \to 0 \quad \text{as} \quad j \to \infty. \tag{5.1}$$

Moreover, since $4s + 2t > 3$, we have that $2 < \frac{12}{3+2t} < 2_s^*$, and by Lemma 2.1 (iii), we have

$$\int_{\mathbb{R}^3} \phi_{u_{\varepsilon_j}}^t u_{\varepsilon_j}^2\, dx \to 0 \quad \text{as} \quad j \to \infty. \tag{5.2}$$

Notice that

$$\begin{aligned} &\mathcal{I}_{\varepsilon_j}(u_{\varepsilon_j}) - \frac{1}{2_s^*}\left\langle \mathcal{I}'_{\varepsilon_j}(u_{\varepsilon_j}), u_{\varepsilon_j}\right\rangle \\ &= \frac{s}{3}\|u_{\varepsilon_j}\|_{\varepsilon_j}^2 + \frac{4s-3}{12}\int_{\mathbb{R}^3} \phi_{u_{\varepsilon_j}}^t u_{\varepsilon_j}^2\, dx - \int_{\mathbb{R}^3} P(\varepsilon_j x)F(u_{\varepsilon_j})\, dx + \frac{1}{2_s^*}\int_{\mathbb{R}^3} P(\varepsilon_j x)f(u_{\varepsilon_j})u_{\varepsilon_j}\, dx. \end{aligned}$$

Thus, by (5.1)–(5.2), we have

$$\int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}}u_{\varepsilon_j}\right|^2 dx \leq \frac{3}{s}c_{\varepsilon_j} + o(1).$$

Similarly, we have

$$\int_{\mathbb{R}^3} Q(\varepsilon_j x)|u_{\varepsilon_j}|^{2_s^*}\, dx = \frac{3}{s}c_{\varepsilon_j} + o(1).$$

Moreover,

$$\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u_{\varepsilon_j}\right|^2dx-\int_{\mathbb{R}^3}Q\left(\varepsilon_jx\right)\left|u_{\varepsilon_j}\right|^{2_s^*}dx\le o(1).$$

Thus, by the best constant of the Sobolev imbedding, we get

$$\begin{aligned}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u_{\varepsilon_j}\right|^2dx&\le\int_{\mathbb{R}^3}Q\left(\varepsilon_jx\right)\left|u_{\varepsilon_j}\right|^{2_s^*}dx+o(1)\\&=\left(\int_{\mathbb{R}^3}Q\left(\varepsilon_jx\right)\left|u_{\varepsilon_j}\right|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}\left(\int_{\mathbb{R}^3}Q\left(\varepsilon_jx\right)\left|u_{\varepsilon_j}\right|^{2_s^*}dx\right)^{\frac{2s}{3}}+o(1)\\&=\left(\int_{\mathbb{R}^3}Q_{\max}\left|u_{\varepsilon_j}\right|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}\left(\int_{\mathbb{R}^3}Q\left(\varepsilon_jx\right)\left|u_{\varepsilon_j}\right|^{2_s^*}dx\right)^{\frac{2s}{3}}+o(1)\\&\le\frac{Q_{\max}^{\frac{2}{2_s^*}}}{\mathcal{S}_s}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u_{\varepsilon_j}\right|^2dx\left(\int_{\mathbb{R}^3}Q\left(\varepsilon_jx\right)\left|u_{\varepsilon_j}\right|^{2_s^*}dx\right)^{\frac{2s}{3}}+o(1),\end{aligned}$$

which implies

$$\liminf_{j\to\infty}c_{\varepsilon_j}\ge\frac{s}{3Q_{\max}^{\frac{3-2s}{2s}}}\mathcal{S}_s^{\frac{3}{2s}}$$

a contradiction with Lemma 3.1 and Lemma 4.1. □

Set $v_\varepsilon(x):=u_\varepsilon(x+y_\varepsilon)$, then v_ε satisfies

$$(-\Delta)^sv_\varepsilon+V(\varepsilon(x+y_\varepsilon))v_\varepsilon+\phi_{v_\varepsilon}^tv_\varepsilon=P(\varepsilon(x+y_\varepsilon))f(v_\varepsilon)+Q(\varepsilon(x+y_\varepsilon))|v_\varepsilon|^{2_s^*-2}v_\varepsilon,\tag{5.3}$$

with energy

$$\begin{aligned}\mathcal{J}_\varepsilon(v_\varepsilon)=&\frac12\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}v_\varepsilon\right|^2dx+\frac12\int_{\mathbb{R}^3}V(\varepsilon(x+y_\varepsilon))v_\varepsilon^2\,dx+\frac14\int_{\mathbb{R}^3}\phi_{v_\varepsilon}^tv_\varepsilon^2\,dx\\&-\int_{\mathbb{R}^3}P(\varepsilon(x+y_\varepsilon))F(v_\varepsilon)\,dx-\frac{1}{2_s^*}\int_{\mathbb{R}^3}Q(\varepsilon(x+y_\varepsilon))|v_\varepsilon|^{2_s^*}dx\\=&\mathcal{J}_\varepsilon(v_\varepsilon)-\frac14\langle\mathcal{J}_\varepsilon'(v_\varepsilon),v_\varepsilon\rangle\\=&\frac14\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}v_\varepsilon\right|^2dx+\frac14\int_{\mathbb{R}^3}V(\varepsilon(x+y_\varepsilon))v_\varepsilon^2\,dx\\&+\frac14\int_{\mathbb{R}^3}P(\varepsilon(x+y_\varepsilon))[f(v_\varepsilon)v_\varepsilon-4F(v_\varepsilon)]\,dx+\frac{4s-3}{12}\int_{\mathbb{R}^3}Q(\varepsilon(x+y_\varepsilon))|v_\varepsilon|^{2_s^*}dx\\=&\mathcal{I}_\varepsilon(u_\varepsilon)-\frac14\langle\mathcal{I}_\varepsilon'(u_\varepsilon),u_\varepsilon\rangle=\mathcal{I}_\varepsilon(u_\varepsilon)=c_\varepsilon.\end{aligned}$$

We may assume $v_\varepsilon\rightharpoonup u$ in H_ε, and $v_\varepsilon\to u$ in $L_{loc}^r\left(\mathbb{R}^3\right)$ for $r\in\left[1,2_s^*\right)$ with $u\neq0$.

By condition $\left(A_0\right)$, without loss of generality, we may assume that $V(\varepsilon y_\varepsilon)\to V_0$, $P(\varepsilon y_\varepsilon)\to P_0$ and $Q(\varepsilon y_\varepsilon)\to Q_0$ as $\varepsilon\to0$.

Lemma 5.3. *u is a positive ground state solution of*

$$(-\Delta)^su+V_0u+\phi_u^tu=P_0f(u)+Q_0|u|^{2_s^*-2}u,\ \ in\ \mathbb{R}^3.\tag{5.4}$$

Proof. By (5.3), for any $\varphi\in C_0^\infty\left(\mathbb{R}^3\right)$, there holds that

$$0=\lim_{\varepsilon\to0}\int_{\mathbb{R}^3}\left((-\Delta)^sv_\varepsilon+V(\varepsilon(x+y_\varepsilon))v_\varepsilon+\phi_{v_\varepsilon}^tv_\varepsilon-P(\varepsilon(x+y_\varepsilon))f(v_\varepsilon)-Q(\varepsilon(x+y_\varepsilon))|v_\varepsilon|^{2_s^*-2}v_\varepsilon\right)\varphi\,dx.\tag{5.5}$$

Since V, P, Q are all continuous and bounded, we have

$$\int_{\mathbb{R}^3} V(\varepsilon(x+y_\varepsilon))v_\varepsilon \varphi\,dx \to V_0\int_{\mathbb{R}^3} u\varphi\,dx, \quad \int_{\mathbb{R}^3} P(\varepsilon(x+y_\varepsilon))f(v_\varepsilon)\varphi\,dx \to P_0\int_{\mathbb{R}^3} f(u)\varphi\,dx$$

and

$$\int_{\mathbb{R}^3} Q(\varepsilon(x+y_\varepsilon))|v_\varepsilon|^{2_s^*-2}v_\varepsilon\varphi\,dx \to Q_0\int_{\mathbb{R}^3}|u|^{2_s^*-2}u\varphi\,dx,$$

which combined with (5.5) implies that

$$(-\Delta)^s u + V_0 u + \phi_u^t u = P_0 f(u) + Q_0|u|^{2_s^*-2}u, \text{ in } \mathbb{R}^3,$$

that is, u solves (5.4) with energy

$$\begin{aligned}
\mathcal{I}_{V_0P_0Q_0}(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \frac{1}{2}V_0\int_{\mathbb{R}^3} u^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2 dx - P_0\int_{\mathbb{R}^3}F(u)\,dx - \frac{1}{2_s^*}Q_0\int_{\mathbb{R}^3}|u|^{2_s^*}dx\\
&= \mathcal{I}_{V_0P_0Q_0}(u) - \frac{1}{4}\left\langle \mathcal{I}'_{V_0P_0Q_0}(u), u\right\rangle\\
&= \frac{1}{4}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \frac{1}{4}V_0\int_{\mathbb{R}^3}u^2 dx + \frac{1}{4}P_0\int_{\mathbb{R}^3}[f(u)u - 4F(u)]\,dx + \frac{4s-3}{12}Q_0\int_{\mathbb{R}^3}|u|^{2_s^*}dx\\
&\geq \gamma_{V_0P_0Q_0}.
\end{aligned}$$

By Fatou's lemma and the proof of Lemma 4.1, we have

$$\begin{aligned}
\gamma_{V_0P_0Q_0} &\leq \frac{1}{4}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \frac{1}{4}V_0\int_{\mathbb{R}^3}u^2 dx + \frac{1}{4}P_0\int_{\mathbb{R}^3}[f(u)u - 4F(u)]\,dx + \frac{4s-3}{12}Q_0\int_{\mathbb{R}^3}|u|^{2_s^*}dx\\
&\leq \liminf_{\varepsilon\to 0}\Bigg[\frac{1}{4}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}v_\varepsilon\right|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}V(\varepsilon(x+y_\varepsilon))v_\varepsilon^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3}P(\varepsilon(x+y_\varepsilon))[f(v_\varepsilon)v_\varepsilon - 4F(v_\varepsilon)]\,dx\\
&\quad + \frac{4s-3}{12}\int_{\mathbb{R}^3}Q(\varepsilon(x+y_\varepsilon))|v_\varepsilon|^{2_s^*}dx\Bigg]\\
&= \liminf_{\varepsilon\to 0}\mathcal{J}_\varepsilon(v_\varepsilon)\\
&\leq \limsup_{\varepsilon\to 0}\mathcal{I}_\varepsilon(u_\varepsilon)\\
&\leq \gamma_{V_0P_0Q_0}.
\end{aligned}$$

Consequently,

$$\lim_{\varepsilon\to 0}\mathcal{J}_\varepsilon(v_\varepsilon) = \lim_{\varepsilon\to 0}c_\varepsilon = \mathcal{I}_{V_0P_0Q_0}(u) = \gamma_{V_0P_0Q_0}. \tag{5.6}$$

Thus, u is a ground state solution of Equation (5.4). As in the proof of Lemma 3.2, u is positive. □

Lemma 5.4. *$\{\varepsilon y_\varepsilon\}$ is bounded.*

Proof. Suppose to the contrary that, after passing to a subsequence, $|\varepsilon y_\varepsilon| \to \infty$. Since $P(0) = P_Q$ and $\nu = V(0) \leq V(x)$ for all $|x| \geq R$, we deduce that $P_0 < P_Q$ and $\nu \leq V_0$. So it follows from Lemma 3.3 that $\gamma_{V_0P_0Q_0} > \gamma_{\nu P_Q Q_{\max}}$. However, by (5.6) and Lemma 4.1, $c_\varepsilon \to \gamma_{V_0P_0Q_0} \leq \gamma_{\nu P_Q Q_{\max}}$, which is a contradiction. Therefore, $\{\varepsilon y_\varepsilon\}$ is bounded. □

After extracting a subsequence, we may assume $\varepsilon y_\varepsilon \to x_0$ as $\varepsilon \to 0$, then $V_0 = V\left(x_0\right)$, $P_0 = P\left(x_0\right)$ and $Q_0 = Q\left(x_0\right)$.

Lemma 5.5. $\lim\limits_{\varepsilon\to 0} dist\left(\varepsilon y_\varepsilon, \mathcal{H}_P\right) = 0$.

Proof. It suffices to show that $x_0 \in \mathcal{H}_P$. We argue by contradiction, if $x_0 \notin \mathcal{H}_P$, then it is easy to check that $\gamma_{V(x_0)P(x_0)Q(x_0)} > \gamma_{\nu P_Q Q_{\max}}$ by (A_1) and Lemma 3.3. Therefore, by Lemma 4.1, we have

$$\lim_{\varepsilon\to 0} c_\varepsilon = \gamma_{V(x_0)P(x_0)Q(x_0)} > \gamma_{\nu P_Q Q_{\max}} \geq \lim_{\varepsilon\to 0} c_\varepsilon,$$

which is absurd. □

Lemma 5.6. *$v_\varepsilon \to u$ in $H^s(\mathbb{R}^3)$.*

Proof. Recall that u is a ground state solution of (5.4), we have

$$\begin{aligned}
\frac{1}{4}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx &\leq \liminf_{\varepsilon\to 0}\frac{1}{4}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}v_\varepsilon\right|^2dx\\
&\leq \limsup_{\varepsilon\to 0}\frac{1}{4}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}v_\varepsilon\right|^2dx\\
&\leq \limsup_{\varepsilon\to 0}\frac{1}{4}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}v_\varepsilon\right|^2dx + \liminf_{\varepsilon\to 0}\frac{1}{4}\int_{\mathbb{R}^3}V(\varepsilon(x+y_\varepsilon))v_\varepsilon^2\,dx - \frac{1}{4}V_0\int_{\mathbb{R}^3}u^2dx\\
&\quad + \liminf_{\varepsilon\to 0}\frac{1}{4}\int_{\mathbb{R}^3}P(\varepsilon(x+y_\varepsilon))[f(v_\varepsilon)v_\varepsilon - 4F(v_\varepsilon)]\,dx - \frac{1}{4}P_0\int_{\mathbb{R}^3}[f(u)u-4F(u)]\,dx\\
&\quad + \liminf_{\varepsilon\to 0}\frac{4s-3}{12}\int_{\mathbb{R}^3}Q(\varepsilon(x+y_\varepsilon))|v_\varepsilon|^{2_s^*}dx - \frac{4s-3}{12}Q_0\int_{\mathbb{R}^3}|u|^{2_s^*}dx\\
&\leq \limsup_{\varepsilon\to 0}\left[\frac{1}{4}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}v_\varepsilon\right|^2dx + \frac{1}{4}\int_{\mathbb{R}^3}V(\varepsilon(x+y_\varepsilon))v_\varepsilon^2\,dx\right.\\
&\quad \left.+\frac{1}{4}\int_{\mathbb{R}^3}P(\varepsilon(x+y_\varepsilon))[f(v_\varepsilon)v_\varepsilon - 4F(v_\varepsilon)]\,dx + \frac{4s-3}{12}\int_{\mathbb{R}^3}Q(\varepsilon(x+y_\varepsilon))|v_\varepsilon|^{2_s^*}dx\right]\\
&\quad -\frac{1}{4}V_0\int_{\mathbb{R}^3}u^2dx - \frac{1}{4}P_0\int_{\mathbb{R}^3}[f(u)u-4F(u)]\,dx - \frac{4s-3}{12}Q_0\int_{\mathbb{R}^3}|u|^{2_s^*}dx\\
&= \frac{1}{4}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx.
\end{aligned}$$

Consequently,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}v_\varepsilon\right|^2dx = \int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx.$$

Similarly, we have

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}V(\varepsilon(x+y_\varepsilon))v_\varepsilon^2\,dx = V_0\int_{\mathbb{R}^3}u^2dx.$$

Notice that

$$\lim_{\varepsilon\to 0}\left(\int_{\mathbb{R}^3}V\big(\varepsilon(x+y_\varepsilon)\big)v_\varepsilon^2\,dx - V_0\int_{\mathbb{R}^3}v_\varepsilon^2dx\right) = 0.$$

Thus

$$\lim_{\varepsilon\to 0}\left\{\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}v_\varepsilon\right|^2dx + V_0\int_{\mathbb{R}^3}v_\varepsilon^2dx\right\} = \int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx + V_0\int_{\mathbb{R}^3}u^2dx.$$

Together with $v_\varepsilon \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, we have $v_\varepsilon \to u$ in $H^s(\mathbb{R}^3)$. □

To establish the L^∞-estimate of ground state solutions, we first recall the following result which can be found in [13, (5.1.1) and (5.1.2)]. (See [42] for the proof.)

Lemma 5.7. *Suppose that $g : \mathbb{R} \to \mathbb{R}$ is convex and Lipschitz continuous with the Lipschitz constant L, $g(0) = 0$. Then for each $u \in H^s(\mathbb{R}^3)$, $g(u) \in H^s(\mathbb{R}^3)$ and*

$$(-\Delta)^s g(u) \leq g'(u)(-\Delta)^s u \tag{5.7}$$

in the weak sense.

Remark 5.8. In fact, from the above arguments, one can see that (5.7) holds for a.e. $x \in \mathbb{R}^3$. Moreover, Lemma 5.7 is true for general dimension N.

The following lemma plays a fundamental role in the study of behavior of the maximum points of the solutions, whose proof is related to the Moser iterative method [23].

Lemma 5.9. *Let $\varepsilon_n \to 0$ and v_{ε_n} be a solution of the following equation*

$$(-\Delta)^s v_{\varepsilon_n} + V\left(\varepsilon_n\left(x + y_{\varepsilon_n}\right)\right)v_{\varepsilon_n} + \phi^t_{v_{\varepsilon_n}} v_{\varepsilon_n} = P\left(\varepsilon_n\left(x + y_{\varepsilon_n}\right)\right) f\left(v_{\varepsilon_n}\right) + Q\left(\varepsilon_n\left(x + y_{\varepsilon_n}\right)\right)|v_{\varepsilon_n}|^{2^*_s - 2} v_{\varepsilon_n}, \ \text{in } \mathbb{R}^3, \tag{5.8}$$

where y_{ε_n} is given in Lemma 5.2. Then $v_{\varepsilon_n} \in L^\infty(\mathbb{R}^3)$ and there exists $C > 0$ such that

$$\|v_{\varepsilon_n}\|_\infty \leq C, \ \text{uniformly in } n \in \mathbb{N}.$$

Moreover, $v_{\varepsilon_n} \to u$ in $L^q(\mathbb{R}^3)$, for all $q \in [2, +\infty)$.

Proof. For simplicity of notations, we denote v_{ε_n} and y_{ε_n} by v_n and y_n, respectively. Define

$$h(x, v_n) := P(\varepsilon_n(x + y_n)) f(v_n) + Q(\varepsilon_n(x + y_n))|v_n|^{2^*_s - 2} v_n - V(\varepsilon_n(x + y_n)) v_n - \phi^t_{v_n} v_n.$$

From Lemma 5.6, $\{v_n\}$ is bounded in $H^s(\mathbb{R}^3)$, and hence in $L^q(\mathbb{R}^3)$ for any $q \in [2, 2^*_s]$. So there exists some $C > 0$ such that

$$\|v_n\|_q \leq C,$$

uniformly in n. Since v_n is a solution of (5.8), then

$$\begin{aligned}
\phi^t_{v_n}(x) &= \int_{\mathbb{R}^3} \frac{v_n^2(y)}{|x - y|^{3-2t}}\, dy = \int_{\{|x-y|\leq 1\}} \frac{v_n^2(y)}{|x - y|^{3-2t}}\, dy + \int_{\{|x-y|>1\}} \frac{v_n^2(y)}{|x - y|^{3-2t}}\, dy \\
&\leq \int_{\{|x-y|\leq 1\}} \frac{v_n^2(y)}{|x - y|^{3-2t}}\, dy + \int_{\{|x-y|>1\}} v_n^2(y)\, dy \\
&\leq \left(\int_{\{|x-y|\leq 1\}} \frac{1}{|x - y|^{(3-2t)r'}}\, dy\right)^{\frac{1}{r'}} \left(\int_{\{|x-y|\leq 1\}} v_n^{2r}(y)\, dy\right)^{\frac{1}{r}} + C \\
&\leq C,
\end{aligned}$$

where $r'(3 - 2t) < 3$, $2r \in \left[2, 2^*_s\right]$, $\frac{1}{r} + \frac{1}{r'} = 1$ since $2s + 2t > 3$. Therefore, we have

$$|h(x, v_n)| \leq C\left(|v_n| + |v_n|^{p-1}\right) \leq C\left(1 + |v_n|^{2^*_s - 1}\right). \tag{5.9}$$

Let $T > 0$, we follow [13] and define

$$H(\theta) = \begin{cases} 0, & \text{if } \theta \leq 0, \\ \theta^\beta, & \text{if } 0 < \theta < T, \\ \beta T^{\beta - 1}(\theta - T) + T^\beta, & \text{if } \theta \geq T, \end{cases}$$

with $\beta > 1$ to be determined later. Since H is convex and Lipschitz with constant $L_0 = \beta T^{\beta-1}$ and $H(0) = 0$, by Lemma 5.7, we have $H(v_n) \in \mathcal{D}^{s,2}\left(\mathbb{R}^3\right)$ and

$$(-\Delta)^s H(v_n) \leq H'(v_n)(-\Delta)^s v_n \tag{5.10}$$

in the weak sense. Thus, from $H(v_n) \in \mathcal{D}^{s,2}\left(\mathbb{R}^3\right)$, the self-adjointness of the operator $(-\Delta)^{s/2}$ and (5.9)–(5.10), we have

$$\begin{aligned}
\|H(v_n)\|_{2_s^*}^2 &\leq C \int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} H(v_n)\right|^2 dx = C \int_{\mathbb{R}^3} H(v_n)(-\Delta)^s H(v_n)\, dx \\
&\leq C \int_{\mathbb{R}^3} H(v_n)H'(v_n)(-\Delta)^s v_n\, dx = C \int_{\mathbb{R}^3} H(v_n)H'(v_n)h(x, v_n)\, dx \\
&\leq C \int_{\mathbb{R}^3} H(v_n)H'(v_n)\, dx + C \int_{\mathbb{R}^3} H(v_n)H'(v_n)v_n^{2_s^*-1} dx.
\end{aligned}$$

Using the fact that $H(v_n)H'(v_n) \leq \beta^2 v_n^{2\beta-1}$ and $v_n H'(v_n) \leq \beta H(v_n)$, we have

$$\left(\int_{\mathbb{R}^3} \left(H(v_n)\right)^{2_s^*} dx\right)^{\frac{2}{2_s^*}} \leq C\beta^2 \left(\int_{\mathbb{R}^3} v_n^{2\beta-1} dx + \int_{\mathbb{R}^3} \left(H(v_n)\right)^2 v_n^{2_s^*-2} dx\right), \tag{5.11}$$

where C is a positive constant that does not depend on β. Notice that the last integral is well defined for T in the definition of H. Indeed

$$\begin{aligned}
\int_{\mathbb{R}^3} \left(H(v_n)\right)^2 v_n^{2_s^*-2} dx &= \int_{\{v_n \leq T\}} \left(H(v_n)\right)^2 v_n^{2_s^*-2} dx + \int_{\{v_n > T\}} \left(H(v_n)\right)^2 v_n^{2_s^*-2} dx \\
&\leq T^{2\beta-2} \int_{\mathbb{R}^3} v_n^{2_s^*} dx + C \int_{\mathbb{R}^3} v_n^{2_s^*} dx < \infty.
\end{aligned}$$

We choose now β in (5.11) such that $2\beta - 1 = 2_s^*$, and we name it β_1, that is

$$\beta_1 := \frac{2_s^* + 1}{2}. \tag{5.12}$$

Let $\hat{R} > 0$ to be fixed later. Attending to the last integral in (5.11) and applying Holder's inequality with exponents $\gamma := \frac{2_s^*}{2}$ and $\gamma' := \frac{2_s^*}{2_s^*-2}$,

$$\begin{aligned}
\int_{\mathbb{R}^3} \left(H(v_n)\right)^2 v_n^{2_s^*-2} dx &= \int_{\{v_n \leq \hat{R}\}} \left(H(v_n)\right)^2 v_n^{2_s^*-2} dx + \int_{\{v_n > \hat{R}\}} \left(H(v_n)\right)^2 v_n^{2_s^*-2} dx \\
&\leq \int_{\{v_n \leq \hat{R}\}} \frac{\left(H(v_n)\right)^2}{v_n} \hat{R}^{2_s^*-1} dx + \left(\int_{\mathbb{R}^3} \left(H(v_n)\right)^{2_s^*} dx\right)^{\frac{2}{2_s^*}} \left(\int_{\{v_n > \hat{R}\}} v_n^{2_s^*} dx\right)^{\frac{2_s^*-2}{2_s^*}}.
\end{aligned} \tag{5.13}$$

By the monotone convergence theorem, we can choose $\hat{R}$ large enough so that

$$\left(\int_{\{v_n > \hat{R}\}} v_n^{2_s^*} dx\right)^{\frac{2_s^*-2}{2_s^*}} \leq \frac{1}{2C\beta_1^2},$$

where C is the constant appearing in (5.11). Therefore, we can absorb the last term in (5.13) by the left hand side of (5.11) to get

$$\left(\int_{\mathbb{R}^3} \left(H(v_n)\right)^{2_s^*} dx\right)^{\frac{2}{2_s^*}} \leq 2C\beta_1^2 \left(\int_{\mathbb{R}^3} v_n^{2_s^*} dx + \hat{R}^{2_s^*-1} \int_{\mathbb{R}^3} \frac{\left(H(v_n)\right)^2}{v_n} dx\right).$$

Now we use the fact that $H(v_n) \le v_n^{\beta_1}$ and (5.12) once again in the right hand side and we take $T \to \infty$ we obtain

$$\left(\int_{\mathbb{R}^3} v_n^{2_s^*\beta_1} dx\right)^{\frac{2}{2_s^*}} \le 2C\beta_1^2\left(\int_{\mathbb{R}^3} v_n^{2_s^*} dx + \hat{R}^{2_s^*-1}\int_{\mathbb{R}^3} v_n^{2_s^*} dx\right),$$

and therefore

$$v_n \in L^{2_s^*\beta_1}\left(\mathbb{R}^3\right), \quad \text{for all} \quad n, \tag{5.14}$$

and

$$\|v_n\|_{2_s^*\beta_1} \le C, \tag{5.15}$$

uniformly in n.

Let us suppose now $\beta > \beta_1$. Thus, using that $H(v_n) \le v_n^{\beta}$ in the right hand side of (5.11) and letting $T \to \infty$ we get

$$\left(\int_{\mathbb{R}^3} v_n^{2_s^*\beta} dx\right)^{\frac{2}{2_s^*}} \le C\beta^2\left(\int_{\mathbb{R}^3} v_n^{2\beta-1} dx + \hat{R}^{2_s^*-1}\int_{\mathbb{R}^3} v_n^{2\beta+2_s^*-2} dx\right). \tag{5.16}$$

Set $r_0 := \frac{2_s^*(2_s^*-1)}{2(\beta-1)}$ and $r_1 := 2\beta - 1 - r_0$. Notice that, since $\beta > \beta_1$, then $0 < r_0 < 2_s^*, r_1 > 0$. Hence, applying Young's inequality with exponents $\gamma := 2_s^*/r_0$ and $\gamma' := 2_s^*/2_s^* - r_0$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} v_n^{2\beta-1} dx &\le \frac{r_0}{2_s^*}\int_{\mathbb{R}^3} v_n^{2_s^*} dx + \frac{2_s^*}{2_s^* - r_0}\int_{\mathbb{R}^3} v_n^{\frac{2_s^* r_1}{2_s^*-r_0}} dx \\
&\le \int_{\mathbb{R}^3} v_n^{2_s^*} dx + \int_{\mathbb{R}^3} v_n^{2\beta+2_s^*-2} dx \\
&\le C\left(1 + \int_{\mathbb{R}^3} v_n^{2\beta+2_s^*-2} dx\right),
\end{aligned}$$

with $C > 0$ independent of β. Plugging into (5.16),

$$\left(\int_{\mathbb{R}^3} v_n^{2_s^*\beta} dx\right)^{\frac{2}{2_s^*}} \le C\beta^2\left(1 + \int_{\mathbb{R}^3} v_n^{2\beta+2_s^*-2} dx\right),$$

with C changing from line to line, but remaining independent of β. Therefore

$$\left(1 + \int_{\mathbb{R}^3} v_n^{2_s^*\beta} dx\right)^{\frac{1}{2_s^*(\beta-1)}} \le \left(C\beta^2\right)^{\frac{1}{2(\beta-1)}}\left(1 + \int_{\mathbb{R}^3} v_n^{2\beta+2_s^*-2} dx\right)^{\frac{1}{2(\beta-1)}}. \tag{5.17}$$

Repeating this argument we will define a sequence $\beta_m, m \ge 1$ such that

$$2\beta_{m+1} + 2_s^* - 2 = 2_s^*\beta_m.$$

Thus,

$$\beta_{m+1} - 1 = \left(\frac{2_s^*}{2}\right)^m (\beta_1 - 1).$$

Replacing it in (5.17) one has

$$\left(1 + \int_{\mathbb{R}^3} v_n^{2_s^*\beta_{m+1}} dx\right)^{\frac{1}{2_s^*(\beta_{m+1}-1)}} \le \left(C\beta_{m+1}^2\right)^{\frac{1}{2(\beta_{m+1}-1)}}\left(1 + \int_{\mathbb{R}^3} v_n^{2_s^*\beta_m} dx\right)^{\frac{1}{2_s^*(\beta_m-1)}}.$$

Defining $C_{m+1} := C\beta_{m+1}^2$ and

$$A_m := \left(1 + \int_{\mathbb{R}^3} v_n^{2_s^*\beta_m} dx\right)^{\frac{1}{2_s^*(\beta_m - 1)}}.$$

So

$$A_{m+1} \le (C_{m+1})^{\frac{1}{2(\beta_{m+1}-1)}} A_m, \quad m = 1, 2, \dots .$$

Now from an iterative procedure we conclude that there exists a constant $C_0 > 0$ independent of m, such that

$$A_m \le \prod_{k=1}^{m} C_k^{\frac{1}{2(\beta_k - 1)}} A_1 \le C_0 A_1, \quad \text{for all} \quad m.$$

Thus, from (5.14),

$$\|v_n\|_\infty \le C_0 A_1 < \infty, \tag{5.18}$$

and hence $v_n \in L^\infty(\mathbb{R}^3)$. By (5.15),

$$\|v_n\|_\infty \le C, \tag{5.19}$$

uniformly in $n \in \mathbb{N}$, Finally, by interpolation on the L^q-spaces and $v_n \to u$ in $L^2(\mathbb{R}^3)$, we have $v_n \to u$ in $L^q(\mathbb{R}^3)$, for all $q \in [2, +\infty)$. This finishes the proof of Lemma 5.9. □

Lemma 5.10. *$v_n(x) \to 0$ as $|x| \to \infty$ uniformly in n.*

Proof. Since v_n satisfies the equation

$$(-\Delta)^s v_n + v_n = \Upsilon_n, \quad x \in \mathbb{R}^3,$$

where

$$\Upsilon_n(x) = v_n(x) - V(\varepsilon_n(x + y_n))v_n(x) - \phi_{v_n}^t v_n(x) + P(\varepsilon_n(x + y_n)) f(v_n(x)) + Q(\varepsilon_n(x + y_n)) v_n^{2_s^*-1}(x), \ x \in \mathbb{R}^3.$$

Putting $\Upsilon(x) = u(x) - V(x_0)u(x) - \phi_u^t u(x) + P(x_0) f(u(x)) + Q(x_0) u^{2_s^*-1}(x)$, by Lemma 5.9, we see that

$$\Upsilon_n \to \Upsilon \text{ in } L^q(\mathbb{R}^3), \quad \text{for all} \quad q \in [2, +\infty),$$

and there exists a $C_2 > 0$ such that

$$\|\Upsilon_n\|_\infty \le C_2, \quad \text{for all} \quad n \in \mathbb{N}.$$

From [15], we have that

$$v_n(x) = \mathcal{G} * \Upsilon_n = \int_{\mathbb{R}^3} \mathcal{G}(x - y)\Upsilon_n(y)\, dy,$$

where $\mathcal{G}$ is the Bessel kernel

$$\mathcal{G}(x) = \mathcal{F}^{-1}\left(\frac{1}{1 + |\xi|^{2s}}\right).$$

It is known from [15, Theorem 3.3] that, $\mathcal{G}$ is positive, radially symmetric and smooth in $\mathbb{R}^3 \backslash \{0\}$; there is $C > 0$ such that $\mathcal{G}(x) \le \frac{C}{|x|^{3+2s}}$, and $\mathcal{G} \in L^q(\mathbb{R}^3)$, for all $q \in \left[1, \frac{3}{3-2s}\right)$. Now argue as in the proof of [1, Lemma 2.6], we conclude that

$$v_n(x) \to 0 \quad \text{as} \quad |x| \to \infty, \tag{5.20}$$

uniformly in $n \in \mathbb{N}$. □

Proof of Theorem 5.1. First we claim that there exists a $\rho_0 > 0$ such that $\|v_n\|_\infty \geq \rho_0, \quad \text{for all} \quad n \in \mathbb{N}$. In fact, suppose that $\|v_n\|_\infty \to 0$ as $n \to \infty$. Then, by (2.5), we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} v_n\right|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon_n(x+y_n)) v_n^2 dx \\
&\leq \int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}} v_n\right|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon_n(x+y_n)) v_n^2\, dx + \int_{\mathbb{R}^3} \phi_{v_n}^t v_n^2\, dx \\
&= \int_{\mathbb{R}^3} P(\varepsilon_n(x+y_n)) f(v_n) v_n\, dx + \int_{\mathbb{R}^3} Q(\varepsilon_n(x+y_n)) |v_n|^{2_s^*} dx \\
&\leq P_{\max} \int_{\mathbb{R}^3} C\left(v_n^4 + v_n^p\right) dx + Q_{\max} \int_{\mathbb{R}^3} v_n^{2_s^*} dx \\
&\leq C P_{\max} \|v_n\|_\infty^2 \int_{\mathbb{R}^3} v_n^2\, dx + C P_{\max} \|v_n\|_\infty^{p-2} \int_{\mathbb{R}^3} v_n^2\, dx + Q_{\max} \|v_n\|_\infty^{2_s^*-2} \int_{\mathbb{R}^3} v_n^2\, dx \\
&= \left(C P_{\max} \|v_n\|_\infty^2 + C P_{\max} \|v_n\|_\infty^{p-2} + Q_{\max} \|v_n\|_\infty^{2_s^*-2}\right) \int_{\mathbb{R}^3} v_n^2\, dx \\
&\leq \frac{V_{\min}}{2} \int_{\mathbb{R}^3} v_n^2\, dx
\end{aligned}$$

for n large enough. This implies that $\|v_n\| = 0$ for n large enough, which is impossible because $v_n \to u$ in $H^s\left(\mathbb{R}^3\right)$ and $u \neq 0$. Then, the claim is true.

From [32, Proposition 2.9], we see that $v_n \in C^{1,\alpha}\left(\mathbb{R}^3\right)$ for any $\alpha < 2s - 1$. Thus, we know that v_n has a global maximum point p_n by (5.20) and the claim above, we also see that $p_n \in B_{R_0}(0)$ for some $R_0 > 0$. Hence, the global maximum point of u_{ε_n} given by $p_n + y_n$. Define $\psi_n(x) := u_{\varepsilon_n}(x + y_n + p_n)$, where $u_{\varepsilon_n}(x) = v_n(x + y_n)$. Since $\{p_n\} \subset B_{R_0}(0)$ is bounded, then we know that $\{\varepsilon_n(p_n + y_n)\}$ is bounded and $\varepsilon_n(p_n + y_n) \to x_0 \in \mathcal{H}_P$. It follows from the boundedness of $\left\{u_{\varepsilon_n}\right\}$ that $\{\psi_n\}$ is bounded in $H^s\left(\mathbb{R}^3\right)$, and we assume that $\psi_n \rightharpoonup \psi$ in $H^s\left(\mathbb{R}^3\right)$, $\psi_n \to \psi$ in $L^q_{loc}\left(\mathbb{R}^3\right)$ for $q \in \left[1, 2_s^*\right)$. On the other hand, by Lemma 4.1, we have

$$\int_{B_{\bar{R}+R_0}(0)} \psi_n^2(x)\, dx \geq \int_{\{|x+p_n|<\bar{R}\}} \psi_n^2(x)\, dx = \int_{B_{\bar{R}}(y_n)} u_{\varepsilon_n}^2(x)\, dx \geq \sigma,$$

so we obtain $\psi \neq 0$. Moreover, similar to the argument above, we know that ψ is a ground state solution of (5.4) and $\psi_n \to \psi$ in $H^s\left(\mathbb{R}^3\right)$. Therefore, ψ_n possesses a same properties as v_n, and we can assume that y_n is a global maximum point of u_{ε_n}. Then, by Lemma 5.2–5.6 above, one can obtain Theorem 5.1. □

6 | DECAY ESTIMATES

In this section, we estimate the decay properties of v_n.

Lemma 6.1. *There exist $C > 0$ such that*

$$v_n(x) \leq \frac{C}{1 + |x|^{3+2s}}, \quad \textit{for all} \quad x \in \mathbb{R}^3.$$

Proof. According to [15, Lemma 4.2], there exists a continuous function $\bar{\omega}$ such that

$$0 < \bar{\omega}(x) \leq \frac{C}{1 + |x|^{3+2s}}, \tag{6.1}$$

and

$$(-\Delta)^s \bar{\omega} + \frac{V_{\min}}{2} \bar{\omega} = 0, \text{ in } \mathbb{R}^3 \backslash B_{\bar{R}}(0) \tag{6.2}$$

for some suitable $\bar{R} > 0$. Thanks to (5.20), we have that $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in n. Therefore, for some large $R_1 > 0$, we obtain

$$\begin{aligned}(-\Delta)^s v_n + \frac{V_{\min}}{2} v_n &= (-\Delta)^s v_n + V(\varepsilon_n(x+y_n))v_n - \left(V(\varepsilon_n(x+y_n)) - \frac{V_{\min}}{2}\right) v_n \\ &= -\phi^t_{v_n} v_n + P(\varepsilon_n(x+y_n)) f(v_n) + Q(\varepsilon_n(x+y_n))|v_n|^{2^*_s-2} v_n - \left(V(\varepsilon_n(x+y_n)) - \frac{V_{\min}}{2}\right) v_n \\ &\le \left(C P_{\max}\left(v_n^3 + v_n^{p-1}\right) + Q_{\max}|v_n|^{2^*_s-2} v_n - \frac{V_{\min}}{2}\right) v_n \\ &= \left(C P_{\max}\left(v_n^2 + v_n^{p-2}\right) + Q_{\max} v_n^{2^*_s-2} - \frac{V_{\min}}{2}\right) v_n \\ &\le 0, \end{aligned} \tag{6.3}$$

for $x \in \mathbb{R}^3 \backslash B_{R_1}(0)$. Now we take $R_2 := \max\left\{\bar{R}, R_1\right\}$ and set

$$z_n := (m+1)\bar{\omega} - b v_n, \tag{6.4}$$

where $m := \sup_{n\in\mathbb{N}} \|v_n\|_\infty < \infty$ and $b := \min_{\bar{B}_{R_2}(0)} \bar{\omega} > 0$. We next show that $z_n \ge 0$ in $\mathbb{R}^3$. For this we suppose by contradiction that, there is a sequence $\left\{x_n^j\right\}$ such that

$$\inf_{x\in\mathbb{R}^3} z_n(x) = \lim_{j\to\infty} z_n\left(x_n^j\right) < 0. \tag{6.5}$$

Notice that

$$\lim_{|x|\to\infty} \bar{\omega}(x) = 0.$$

Jointly with (5.20), we obtain

$$\lim_{|x|\to\infty} z_n(x) = 0,$$

uniformly in $n \in \mathbb{N}$. Consequently, the sequence $\left\{x_n^j\right\}$ is bounded and therefore, up to a subsequence, we may assume that $x_n^j \to x_n^*$ as $j \to \infty$ for some $x_n^* \in \mathbb{R}^3$. Hence (6.5) becomes

$$z_n\left(x_n^*\right) = \inf_{x\in\mathbb{R}^3} z_n(x) < 0. \tag{6.6}$$

From (6.6) and (2.1), we have

$$(-\Delta)^s z_n\left(x_n^*\right) = -\frac{C(s)}{2}\int_{\mathbb{R}^3} \frac{z_n(x_n^*+y) + z_n(x_n^*-y) - 2z_n\left(x_n^*\right)}{|y|^{3+2s}}\, dy \le 0. \tag{6.7}$$

By (6.4), we get

$$z_n(x) \ge mb + \bar{\omega} - mb > 0, \quad \text{in } B(0, R_2).$$

Therefore, combining this with (6.6), we see that

$$x_n^* \in \mathbb{R}^3 \backslash B_{R_2}(0). \tag{6.8}$$

From (6.2)–(6.3), we conclude that

$$(-\Delta)^s z_n + \frac{V_{\min}}{2} z_n \ge 0, \quad in\, \mathbb{R}^3 \backslash B_{R_2}(0). \tag{6.9}$$

Thinks to (6.8), we can evaluate (6.9) at the point x_n^*, and recall (6.6), (6.7), we conclude that

$$0 \leq (-\Delta)^s z_n\left(x_n^*\right) + \frac{V_{\min}}{2} z_n\left(x_n^*\right) < 0,$$

this is a contradiction, so $z_n(x) \geq 0$ in $\mathbb{R}^3$. That is to say, $v_n \leq (m+1)b^{-1}\bar{\omega}$, which together with (6.1), implies that

$$v_n(x) \leq \frac{C}{1+|x|^{3+2s}}, \quad \text{for all} \quad x \in \mathbb{R}^3.$$

Then the proof is completed. □

Proof of Theorem 6.2. Define $\omega_n(x) := u_n\left(\frac{x}{\varepsilon_n}\right)$, then ω_n is a positive ground state solution of the sytem (1.1) and $x_{\varepsilon_n} := \varepsilon_n y_n$ is a maximum point of ω_n, and by Theorem 5.1, we know that the Theorem 1.2(i), (ii) hold. Moreover, we have

$$\begin{aligned} \omega_n(x) = u_n\left(\frac{x}{\varepsilon_n}\right) &= v_n\left(\frac{x}{\varepsilon_n} - y_n\right) \\ &\leq \frac{C}{1+\left|\frac{x}{\varepsilon_n} - y_n\right|^{3+2s}} \\ &= \frac{C\varepsilon_n^{3+2s}}{\varepsilon_n^{3+2s} + |x - \varepsilon_n y_n|^{3+2s}} \\ &= \frac{C\varepsilon_n^{3+2s}}{\varepsilon_n^{3+2s} + |x - x_{\varepsilon_n}|^{3+2s}}, \quad \text{for all} \quad x \in \mathbb{R}^3. \end{aligned}$$

Thus, the proof of Theorem 1.2 is completed. □

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