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Concentration behavior of ground state solutions for a fractional Schrödinger–Poisson system involving critical exponent

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We are concerned with the existence and concentration behavior of ground state solutions of the fractional Schrödinger–Poisson system with critical nonlinearity

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = \lambda|u|^{p-2}u + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ \varepsilon^{2t}(-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $\lambda > 0$, $\frac{4s+2t}{s+t} < p < 2_s^* = \frac{6}{3-2s}$, $(-\Delta)^\alpha$ denotes the fractional Laplacian of order $\alpha = s, t \in (0,1)$ and satisfies $2t + 2s > 3$. The potential V is continuous and positive, and has a local minimum. We obtain a positive ground state solution for $\varepsilon > 0$ small, and we show that these ground state solutions concentrate around a local minimum of V as $\varepsilon \to 0$.

Keywords: Concentration; fractional Schrödinger–Poisson equation; critical point; critical exponent.

Mathematics Subject Classification 2010: 35Q40, 35J50, 58E05

1. Introduction and the Main Results

In this paper, we study the existence and concentration of ground state solutions for the following fractional Schrödinger–Poisson system:

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = \lambda|u|^{p-2}u + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ \varepsilon^{2t}(-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $\lambda > 0$, $\frac{4s+2t}{s+t} < p < 2_s^*$, $s, t \in (0,1)$ and $2s+2t > 3$, $2_s^* := \frac{6}{3-2s}$ is the fractional critical exponent. For $\varepsilon > 0$ sufficiently small, these

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standing waves are referred to as semiclassical states. In what follows, we assume that the potential function V satisfies the following conditions:

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} V(x) > 0$.
(V_2) There is a bounded domain Λ such that

$$V_0 := \inf_{\Lambda} V(x) < \min_{\partial\Lambda} V(x).$$

Without loss of generality, we may assume that $\mathcal{M} = \{x \in \Lambda : V(x) = V_0\} \neq \emptyset$ and $0 \in \mathcal{M}$.

Our motivation to study (1.1) mainly comes from the fact that solutions $(u(x), \phi(x))$ of (1.1) corresponding to standing wave solutions $(e^{-iEt/\hbar}u(x), \phi(x))$ of the time-dependent system

$$\begin{cases} i\hbar\dfrac{\partial\Psi}{\partial t} = \hbar^{2s}(-\Delta)^s\Psi + \widetilde{V}(x)\Psi + \mu\phi\Psi - \widetilde{f}(x, |\Psi|)\Psi & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ \hbar^{2t}(-\Delta)^t\phi = |\Psi|^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where i is the imaginary unit, $\hbar$ is the Planck constant, $\widetilde{V}(x) = V(x) + E$ and $\widetilde{f}(x, |u|)u = f(x, u)$.

The first equation in (1.2) was introduced by Laskin [37], which is the so-called fractional Schrödinger equation, describes quantum (nonrelativistic) particles interacting with the electromagnetic field generated by the motion. An interesting Schrödinger equation class is when the potential $\phi(x)$ is determined by the charge of wave function itself, that is, when the second equation in (1.2) (Poisson equation) holds. For this reason, (1.2) is referred to as a fractional nonlinear Schrödinger–Poisson system (also called fractional Schrödinger–Maxwell system). When $s = \frac{1}{2}$ and $t = 1$, such a system becomes more interesting in Physics. It comes from the semi-relativistic theory in the repulsive (plasma physics) Coulomb case (see e.g. [1]). If one put the second equation into the first equation, such a system reduces to the semi-relativistic Hartree equation which arise in the quantum theory of boson stars [38].

When $s = t = 1$, (1.1) is the following classical Schrödinger–Poisson system:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \mu\phi u = f(x, u) & \text{in } \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

which was proposed by Benci and Fortunato [9] in 1998 on a bounded domain, and is related to the Hartree equation [39]. In the past several years, the existence and multiplicity of solutions to the systems similar to (1.3) with $\varepsilon = 1$ has been studied extensively by means of variational tools, we refer the interested readers to see [3, 5, 6, 16, 29, 33, 57] and the references therein. In particular, when $f(x, u) = u^{p-1}(2 < p < 6)$, $V \equiv 1$ and $\mu > 0$ is a positive parameter, Ruiz [44] obtained some general results about existence and nonexistence of positive solutions. In the case $p < 4$, the problem (1.3) becomes more delicate since the corresponding energy function does not possesses the mountain pass geometry in general. To overcome this difficulty, Ruiz considered a new constrained minimization problem on a new manifold which is obtained by combining the usual Nehari manifold and the Pohožaevs identity. After that, Wang and Zhou [51] proved that (1.3) has a positive solution for μ small and has no any nontrivial solution for μ large when the nonlinearity $f(x,s)$ is asymptotically linear with respect to s at infinity. The existence of solutions of (1.3) involving nonconstant positive potentials was considered independently in [7, 58]. Ambrosetti and Ruiz [4] constructed multiple solutions to (1.3) with a potential vanishing at infinity. A system under the effect of a general nonlinear term was considered in [5, 6]. The existence of sign-changing solutions for (1.3) was established in [17, 28, 31, 35, 47, 52] for different conditions on $V(x)$ and $f(x,u)$. Recently, in [40], Liu, Wang and Zhang obtained the existence of infinitely many sign-changing solutions to (1.3) with a general nonlinearity $f(u) \sim |u|^{p-1}u(3 < p < 5)$ and a coercive potential by using the method of invariant sets of descending flow.

There are also some works concerning with the semiclassical state of (1.3). In [18], D'Aprile and Wei constructed a family of radially symmetric solutions concentrating around a sphere. Ianni and Vaira [30] proved the existence of single-bump solutions which concentrates around the critical points of $V(x)$. When the potential V satisfies the global condition

(V_3) $0 < \inf_{x\in\mathbb{R}^N} V(x) < \liminf_{|x|\to\infty} V(x) = V_\infty$,

which was introduced by Rabinowitz [43], He [23] studied the multiplicity of positive solutions and proved that these positive solutions concentrate around the global minimum of the potential V. Wang *et al.* [50] studied the existence and the concentration behavior of ground state solutions for a subcritical problem with competing potentials. The critical case was considered in [26], He and Zou proved that system (1.3) possesses a positive ground state solution which concentrates around the global minimum of V. In [25], under the local condition (V_2), the author studied the existence and concentration of positive ground state solutions for the following system involving critical exponent:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \phi u = \lambda|u|^{p-2}u + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

with $p \in (3,4]$. Using a version of quantitative deformation lemma due to Figueiredo, Ikoma and Santos Junior [22], they construct a special bounded Palais–Smale sequence and recover the compactness by using a penalization method which was introduced in [14].

To the best of our knowledge, there are only few papers that considered the existence and multiplicity of solutions to the fractional Schrödinger–Poisson system (1.1). The system (1.1) is different with the local one (1.4) since the fractional Laplacian operator is a nonlocal one. Therefore, the standard techniques that were developed for the local Laplacian do not work immediately. In [56], the authors studied the existence of radial solutions by using the constrained minimization methods for system (1.1) with $\varepsilon = 1$, $V(x) = 0$ and Berestycki–Lions-type conditions [10]. In [48, 49], Teng consider the fractional Schrödinger–Poisson system (1.1) with subcritical and critical nonlinearity, respectively. By the monotone trick, concentration-compactness principle and a global compactness lemma he establishes the existence of ground state solutions.

It seems that the only works concerning the concentration behavior of solutions are due to Liu and Zhang [41], and Yu, Zhao and Zhao [54]. Assuming the global condition (V_3) and $f(u) \sim |u|^{p-2}u(4 < p < 2_s^*)$, the authors obtained the multiplicity and concentration of positive solutions for the following system:

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = f(u) + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ \varepsilon^{2t}(-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$

via the standard Nehari manifold method. The concentration behavior of ground state solutions for a subcritical case with two competing positive potentials was obtained in [54].

Different to [41, 54], in this paper, we devote to establishing the existence and concentration of positive ground state solutions for the fractional Schrödinger–Poisson system (1.1) under the local condition (V_2). The main motivations for considering the critical problem comes from the famous paper [12] due to Brézis and Nirenberg in 1983. Since the nonlinearity $\lambda|u|^{p-2}u + |u|^{2_s^*-2}u$ with $p \in (\frac{4s+2t}{s+t}, 2_s^*)$ does not satisfy Ambrosetti–Rabinowitz condition and the fact that the function $\frac{\lambda|u|^{p-1}+|u|^{2_s^*-1}}{u^3}$ is not increasing for $u > 0$, these prevent us from obtaining a bounded Palais–Smale sequence and using the Nehari manifold in a standard way. So the arguments in [41] cannot be applied in our case.

To overcome these difficulties, inspired by [13, 22, 25], we use a version of quantitative deformation lemma to construct a special bounded and convergent Palais–Smale sequence. And we need to use a penalization method introduced in [14], which helps us to overcome the obstacle caused by the non-compactness due to the unboundedness of the domain and the lack of Ambrosetti–Rabinowitz condition. Proceeding by the standard arguments, the existence of ground state solution u_ε follows. Finally, we use some estimate to verity that the critical point u_ε is indeed a solution of the original problem (1.1).

Now we state our main results as follows.

Theorem 1.1. *Assume that V satisfies (V_1) and (V_2), if $p \in (\frac{4s+2t}{s+t}, \frac{4s}{3-2s}]$, then there exist $\varepsilon^* > 0$ and $\lambda^* > 0$ such that for each $\lambda \in [\lambda^*, \infty)$ and $\varepsilon \in (0, \varepsilon^*)$, the system* (1.1) *possesses a positive ground state solution $(u_\varepsilon, \phi_\varepsilon) \in H^s(\mathbb{R}^3) \times \mathcal{D}^{t,2}(\mathbb{R}^3)$. And if $p \in (\frac{4s}{3-2s}, 2_s^*)$, then there exists $\varepsilon^* > 0$ such that for any $\lambda > 0$ and $\varepsilon \in (0, \varepsilon^*)$, the system* (1.1) *possesses a positive ground state solution. Moreover, if $x_\varepsilon \in \Lambda$ is a maximum point of u_ε, then*

$$\lim_{\varepsilon \to 0} V(x_\varepsilon) = V_0,$$

and there exists a constant $C > 0$ (*independent of* ε) *such that*

$$u_\varepsilon(x) \le \frac{C\varepsilon^{3+2s}}{\varepsilon^{3+2s} + |x - x_\varepsilon|^{3+2s}}, \quad \forall\, x \in \mathbb{R}^3.$$

This paper is organized as follows. In Sec. 2, besides describing the functional setting to study problem (1.1), we prove some preliminary Lemmas which will be used later. In Sec. 3, we study the limit problem associated with (1.1) and we prove the existence of positive ground state solutions. In Sec. 4, we study the existence and concentration phenomenon of these ground state solutions for system (1.1). Finally, we give the decay estimate of solution, which is polynomial instead of exponential form.

2. Variational Settings and Preliminary Results

Throughout this paper, we denote $\|\cdot\|_p$ the usual norm of the space $L^p(\mathbb{R}^3)$, $1 \le p < \infty$, $B_r(x)$ denotes the open ball with center at x and radius r, C or C_i $(i = 1, 2, \ldots)$ denote some positive constants may change from line to line. $a_n \rightharpoonup a$ and $a_n \to a$ mean the weak and strong convergence, respectively, as $n \to \infty$.

2.1. *The functional space setting*

First, fractional Sobolev spaces are the convenient setting for our problem, so we will give some sketches of the fractional Sobolev spaces and the complete introduction can be found in [19]. We recall that, for any $\alpha \in (0, 1)$, the fractional Sobolev space $H^\alpha(\mathbb{R}^3) = W^{\alpha,2}(\mathbb{R}^3)$ is defined as follows:

$$H^\alpha(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\xi|^{2\alpha}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2)d\xi < \infty \right\},$$

whose norm is defined as

$$\|u\|^2_{H^\alpha(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (|\xi|^{2\alpha}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2)d\xi,$$

where $\mathcal{F}$ denotes the Fourier transform. We also define the homogeneous fractional Sobolev space $\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ as the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} := \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2\alpha}} dxdy \right)^{\frac{1}{2}} = [u]_{H^\alpha(\mathbb{R}^3)}.$$

The embedding $\mathcal{D}^{\alpha,2}(\mathbb{R}^3) \hookrightarrow L^{2^*_\alpha}(\mathbb{R}^3)$ is continuous and for any $\alpha \in (0, 1)$, there exists a best constant $S_\alpha > 0$ such that

$$S_\alpha := \inf_{u\in\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} \frac{\|u\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)}}{\|u\|^2_{2^*_\alpha(\mathbb{R}^3)}}.$$

The fractional Laplacian, $(-\Delta)^\alpha u$ of a smooth function $u : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$\mathcal{F}((-\Delta)^\alpha u)(\xi) = |\xi|^{2\alpha}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^3,$$

that is

$$\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-i\xi\cdot x}\phi(x)dx,$$

for functions ϕ in the Schwartz class. Also $(-\Delta)^\alpha u$ can be equivalently represented as (see [19])

$$(-\Delta)^\alpha u(x) = -\frac{1}{2}C(\alpha)\int_{\mathbb{R}^3} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{3+2\alpha}}dy, \quad \forall x\in\mathbb{R}^3,$$

where

$$C(\alpha) = \left(\int_{\mathbb{R}^3} \frac{(1-\cos\xi_1)}{|\xi|^{3+2\alpha}}d\xi\right)^{-1}, \quad \xi = (\xi_1,\xi_2,\xi_3).$$

Also, by the Plancherel formula in Fourier analysis, we have

$$[u]^2_{H^\alpha(\mathbb{R}^3)} = \frac{2}{C(\alpha)}\|(-\Delta)^{\frac{\alpha}{2}}u\|_2^2.$$

As a consequence, the norms on $H^\alpha(\mathbb{R}^3)$ defined above

$$u \mapsto \left(\int_{\mathbb{R}^3}|u|^2dx + \iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}}dxdy\right)^{\frac{1}{2}};$$

$$u \mapsto \left(\int_{\mathbb{R}^3}(|\xi|^{2\alpha}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2)d\xi\right)^{\frac{1}{2}};$$

$$u \mapsto \left(\int_{\mathbb{R}^3}|u|^2dx + \|(-\Delta)^{\frac{\alpha}{2}}u\|_2^2\right)^{\frac{1}{2}}$$

are equivalent.

Making the change of variable $x \mapsto \varepsilon x$, we can rewrite the system (1.1) as the following equivalent system:

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x)u + \phi u = \lambda|u|^{p-2}u + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^t\phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.1}$$

If u is a solution of the system (2.1), then $\omega(x) := u(\frac{x}{\varepsilon})$ is a solution of the system (1.1). Thus, to study the system (1.1), it suffices to study the system (2.1). In view of the presence of potential $V(x)$, we introduce the subspace

$$H_\varepsilon = \left\{u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x)u^2dx < +\infty\right\},$$

which is a Hilbert space equipped with the inner product

$$(u,v)_{H_\varepsilon} = \int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}vdx + \int_{\mathbb{R}^3}V(\varepsilon x)uvdx,$$

and the norm

$$\|u\|^2_{H_\varepsilon} = (u,u) = \int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx + \int_{\mathbb{R}^3}V(\varepsilon x)u^2dx.$$

We denote $\|\cdot\|_{H_\varepsilon}$ by $\|\cdot\|$ in the sequel for convenience.

For the reader's convenience, we review the main embedding result for this class of fractional Sobolev spaces.

Lemma 2.1 ([19]). *Let $0 < \alpha < 1$, then there exists a constant $C = C(\alpha) > 0$, such that*

$$\|u\|^2_{L^{2^*_\alpha}(\mathbb{R}^3)} \leq C[u]^2_{H^\alpha(\mathbb{R}^3)}$$

for every $u \in H^\alpha(\mathbb{R}^3)$, where $2^*_\alpha = \frac{6}{3-2\alpha}$ is the fractional critical exponent. Moreover, the embedding $H^\alpha(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ is continuous for any $r \in [2, 2^*_\alpha]$ and is locally compact whenever $r \in [2, 2^*_\alpha)$.

Lemma 2.2 ([49]). *If $\{u_n\}$ is bounded in $H^\alpha(\mathbb{R}^3)$ and for some $R > 0$ we have*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^{2^*_\alpha} dx = 0,$$

then $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for any $2 < r \leq 2^*_\alpha$.

2.2. *The reduction method*

It is clear that the system (2.1) is the Euler–Lagrange equations of the functional $J : H_\varepsilon \times \mathcal{D}^{t,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$J(u,\phi) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\phi|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi u^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{1}{2^*_s}\int_{\mathbb{R}^3} |u|^{2^*_s} dx. \tag{2.2}$$

Evidently, the action functional $J \in C^1(H_\varepsilon \times \mathcal{D}^{t,2}(\mathbb{R}^3), \mathbb{R})$ and its critical points are the solutions of (2.1). It is easy to know that J exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinitely dimensional subspaces. This indefiniteness can be removed using the reduction method described in [9]. First of all, for a fixed $u \in H_\varepsilon$, there exists a unique $\phi^t_u \in \mathcal{D}^{t,2}(\mathbb{R}^3)$ which is the solution of

$$(-\Delta)^t\phi = u^2 \quad \text{in } \mathbb{R}^3.$$

We can write an integral expression for ϕ^t_u in the form

$$\phi^t_u(x) = C_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} dy, \quad \forall\, x \in \mathbb{R}^3,$$

which is called t-Riesz potential (see [36]), where

$$C_t = \frac{1}{\pi^{\frac{3}{2}}} \frac{\Gamma(3-2t)}{2^{2t}\Gamma(s)}.$$

Then the system (2.1) can be reduced to the first equation with ϕ represented by the solution of the fractional Poisson equation. This is the basic strategy of solving (2.1). To be more precise about the solution ϕ of the fractional Poisson equation, we have the following lemma.

Lemma 2.3 ([49]). *For any* $u \in H^s(\mathbb{R}^3)$ *and* $4s + 2t \geq 3$, *we have*:

(i) $\phi_u^t \geq 0$;
(ii) $\phi_u^t : H^s(\mathbb{R}^3) \to \mathcal{D}^{t,2}(\mathbb{R}^3)$ *is continuous and maps bounded sets into bounded sets*;
(iii) $\int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq S_t^2 \|u\|^4_{\frac{12}{3+2t}} \leq C\|u\|^4$;
(iv) *If* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$, *then* $\phi_{u_n}^t \rightharpoonup \phi_u^t$ *in* $\mathcal{D}^{t,2}(\mathbb{R}^3)$;
(v) *If* $u_n \to u$ *in* $H^s(\mathbb{R}^3)$, *then* $\phi_{u_n}^t \to \phi_u^t$ *in* $\mathcal{D}^{t,2}(\mathbb{R}^3)$ *and* $\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \to \int_{\mathbb{R}^3} \phi_u^t u^2 dx$.

Define $N : H^s(\mathbb{R}^3) \to \mathbb{R}$ by

$$N(u) = \int_{\mathbb{R}^3} \phi_u^t u^2 dx,$$

it is clear that $N(u(\cdot + y)) = N(u)$ for any $y \in \mathbb{R}^3$, $u \in H^s(\mathbb{R}^3)$ and N is weakly lower semi-continuous in $H^s(\mathbb{R}^3)$. Moreover, similarly to the well-known Brezis–Lieb lemma [11], we have the next lemma.

Lemma 2.4 ([49]). *Let* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$ *and* $u_n \to u$ *a.e.in* $\mathbb{R}^3$ *with* $2s+2t > 3$. *Then*:

(i) $N(u_n - u) = N(u_n) - N(u) + o(1)$;
(ii) $N'(u_n - u) = N'(u_n) - N'(u) + o(1)$, *in* $(H^s(\mathbb{R}^3))^{-1}$.

Putting $\phi = \phi_u^t$ into the first equation of (2.1), we obtain a semilinear elliptic equation

$$(-\Delta)^s u + V(\varepsilon x)u + \phi_u^t u = \lambda |u|^{p-2}u + |u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^3,$$

with a nonlocal term. The corresponding functional $I : H_\varepsilon \to \mathbb{R}$ is defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx.$$

Note that if $4s + 2t \geq 3$, there holds $2 \leq \frac{12}{3+2t} \leq 2_s^*$ and thus $H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$, then by the Hölder inequality and the Sobolev inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \phi_u^t u^2 dx &\leq \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2t}} dx\right)^{\frac{3+2t}{6}} \left(\int_{\mathbb{R}^3} |\phi_u^t|^{2_t^*} dx\right)^{\frac{1}{2_t^*}} \\ &\leq \mathcal{S}_t^{-\frac{1}{2}} \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2t}} dx\right)^{\frac{3+2t}{6}} \|\phi_u^t\|_{\mathcal{D}^{t,2}} \\ &\leq C\|u\|^2 \|\phi_u^t\|_{\mathcal{D}^{t,2}} < \infty.\end{aligned}$$

Therefore, the functional I is well-defined for every $u \in H^s(\mathbb{R}^3)$ and belongs to $C^1(H^s(\mathbb{R}^3), \mathbb{R})$. Moreover, for any $u, v \in H^s(\mathbb{R}^3)$, we have

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} V(\varepsilon x) uv dx + \int_{\mathbb{R}^3} \phi_u^t uv dx \\ - \lambda \int_{\mathbb{R}^3} |u|^{p-2} uv dx - \int_{\mathbb{R}^3} |u|^{2_s^*-2} uv dx.$$

It is standard to verify that a critical point u of the functional I corresponds to a weak solution (u, ϕ) of (1.1) with $\phi = \phi_u^t$. Hence in the following, we consider critical points of I using variational method.

2.3. *A local compactness result*

The next lemma is a variant of [8, Lemma 2.7], for reader's convenience, we give a detailed proof.

Lemma 2.5. *Let $N \geq 2s$ and $\{u_n\} \subset H^s_{\mathrm{loc}}(\mathbb{R}^N)$ be a bounded sequence of functions such that $u_n \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$. Suppose that there exist a bounded open set $Q \subset \mathbb{R}^N$ and a positive constant $\gamma > 0$ such that*

$$\int_Q |(-\Delta)^s u_n|^2 dx \geq \gamma > 0, \quad \int_Q |u_n|^{2_s^*} dx \geq \gamma > 0. \tag{2.3}$$

Moreover, suppose that

$$(-\Delta)^s u_n - |u_n|^{2_s^*-2} u_n = \chi_n, \tag{2.4}$$

where $\chi_n \in H^{-s}(\mathbb{R}^N)$ and

$$|\langle \chi_n, \varphi\rangle| \leq \varepsilon_n \|\varphi\|_{H^s(\mathbb{R}^N)}, \quad \forall \varphi \in C_0^\infty(U), \tag{2.5}$$

where U is an open neighborhood of Q and ε_n is a sequence of positive numbers converging to 0. Then there exist a sequence of points $\{y_n\} \subset \mathbb{R}^N$ and a sequence of positive numbers $\{\sigma_n\}$ such that

$$v_n := \sigma_n^{\frac{(N-2s)}{2}} u_n(\sigma_n x + y_n)$$

converges weakly in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ to a nontrivial solution v of

$$(-\Delta)^s u = |u|^{2_s^*-2} u, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N). \tag{2.6}$$

Moreover,

$$y_n \to \bar{y} \in \bar{Q} \quad \textit{and} \quad \sigma_n \to 0.$$

Because of the presence of nonlocal operator $(-\Delta)^s$, the proof is different from the one in [8, 24]. Indeed, the definition of nonlocal operator causes some techniques developed for local case cannot be adapted immediately to nonlocal case. To overcome these difficulties, we will use an approach due to Caffarelli and Silvestre [15], that is, we will apply the s-harmonic extension technique to transform a nonlocal problem to a local one.

For this, we will denote $\mathbb{R}_+^{N+1} := \mathbb{R}^N \times (0, +\infty)$. Also, for a point $A \in \mathbb{R}_+^{N+1}$, we will use the notation $A = (x, y)$, with $x \in \mathbb{R}^N$ and $y > 0$. Moreover, for $A \in \mathbb{R}_+^{N+1}$ and $r > 0$, we will denote by $B_r^{N+1}(A)$ the ball in $\mathbb{R}_+^{N+1}$ centered at A with radius r.

Definition 2.1. For any $u \in H^s(\mathbb{R}^N)$, we define that $w = E_s(u)$ is its s-harmonic extension to the upper half-space R_+^{N+1}, if w is a solution of the problem

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ w = u & \text{on } \mathbb{R}^N \times \{y = 0\}. \end{cases}$$

Moreover, we define the spaces $X^s(R_+^{N+1})$ and $\dot{H}^s(\mathbb{R}^N)$ as the completion of $C_0^\infty(\overline{\mathbb{R}_+^{N+1}})$ and $C_0^\infty(\mathbb{R}^N)$ under the norms

$$\|w\|_{X^s}^2 := \int_{R_+^{N+1}} \kappa_s y^{1-2s} |\nabla w|^2 dxdy,$$

$$\|w\|_{\dot{H}^s}^2 := \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx,$$

where $\kappa_s > 0$ is a normalization constant.

Now we may reformulate the nonlocal problem (2.4) in a local way, that is

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w_n) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ -\kappa_s \dfrac{\partial w_n}{\partial \nu}(x, y) = |w_n|^{2_s^*-2} w_n + \chi_n & \text{on } \mathbb{R}^N \times \{y = 0\}, \end{cases} \tag{2.7}$$

where

$$\frac{\partial w_n}{\partial \nu} = \lim_{y \to 0^+} \frac{\partial w_n}{\partial y}(x, y) = -\frac{1}{\kappa_s}(-\Delta)^s u_n(x).$$

If w_n is a solution of (2.7), then the trace $u_n(x) = Tr(w_n) = w_n(x, 0)$ is a solution of (2.4). The converse is also true. Therefore, both formulations are equivalent. In the sequel, we will use them both whenever we may take some advantage.

In order to establish the local compactness results, we need an extension of a concentration-compactness result by Lions, that was proved in [20]. For this, we recall the following definition.

Definition 2.2. We say a sequence $\{w_n\}$ is tight in $X^s(\mathbb{R}_+^{N+1})$ if for every $\delta > 0$ there exists $\rho > 0$ such that

$$\int_{\mathbb{R}_+^{N+1} \setminus B_\rho^+} y^{1-2s} |\nabla w_n|^2 dxdy \le \delta \quad \text{for any } n \in \mathbb{N}.$$

Lemma 2.6 ([20] Concentration-compactness Principle). *Let $\{w_n\}$ be a bounded tight sequence in $X^s(\mathbb{R}_+^{N+1})$, such that $\{w_n\}$ converges weakly to w in*

$X^s(\mathbb{R}^{N+1}_+)$. Let μ, ν be two nonnegative measures on $\mathbb{R}^{N+1}_+$ and $\mathbb{R}^N$ respectively and such that

$$\lim_{n\to\infty} y^{1-2s}|\nabla w_n|^2 = \mu, \quad \lim_{n\to\infty} |w_n|^{2_s^*} = \nu,$$

in the sense of measures. Then there exist an at most countable set J and three families $\{x_j\} \subset \mathbb{R}^N$, $\{\mu_j\}$, $\{\nu_j\}$, $\mu_j, \nu_j \geq 0$ such that

(i) $\nu = |w(x,0)|^{2_s^*} + \sum_{j\in J} \nu_j \delta_{x_j}$;

(ii) $\mu \geq y^{1-2s}|\nabla w_n|^2 + \sum_{j\in J} \mu_j \delta_{(x_j,0)}$;

(iii) $\mu_j \geq S_s \nu_j^{2/2_s^*}$,

for all $j \in J$, where δ_{x_j} is the Dirac mass at $x_j \in \mathbb{R}^N$.

Proof of Lemma 2.5. It is easy to see that $\{w_n\}$ is bounded and tight in $X^s(\mathbb{R}^{N+1}_+)$, then by Lemma 2.6, we obtain an at most countable index set J, sequences $\{x_j\} \subset \mathbb{R}^N$ and $\nu_j \subset (0,\infty)$ such that

$$|w_n|^{2_s^*} \rightharpoonup \sum_{j\in J} \nu_j \delta_{x_j},$$

then there is at least one $j_0 \in J$ such that $x_{j_0} \in \bar{Q}$ with $\nu_{j_0} > 0$. Otherwise, $w_n \to 0$ in $L^{2_s^*}(Q)$, which contradicts (2.3).

We define the concentration function

$$G_n(r) = \sup_{x\in\bar{Q}} \int_{B_r(x)} |w_n|^{2_s^*} dx.$$

Fixing a small $\tau \in (0, S_s^{\frac{N}{2s}})$ and choosing $\sigma_n = \sigma_n(\tau) > 0$, $y_n \in \bar{Q}$ such that

$$\int_{B_{\sigma_n}(y_n)} |w_n|^{2_s^*} dx = G_n(\sigma_n) = \tau. \tag{2.8}$$

Denoting $v_n(x) = \sigma_n^{\frac{N-2s}{2}} w_n(\sigma_n x + y_n)$, we see that

$$\tilde{G}_n(r) := \sup_{x\in\bar{Q}_n} \int_{B_r(x)} |v_n|^{2_s^*} dx = \sup_{x\in\bar{Q}} \int_{B_{\sigma_n r}(x)} |w_n|^{2_s^*} dx = G_n(\sigma_n r), \tag{2.9}$$

where $\bar{Q}_n := \{x \in \mathbb{R}^N : \sigma_n x + y_n \in \bar{Q}\}$. Equations (2.8) and (2.9) imply that

$$\tilde{G}_n(1) = \int_{B_1(0)} |v_n|^{2_s^*} dx = \int_{B_{\sigma_n}(y_n)} |w_n|^{2_s^*} dx = G_n(\sigma_n) = \tau. \tag{2.10}$$

Now, we prove that there is a small $\tau \in (0, S_s^{\frac{N}{2s}})$ such that $\sigma_n(\tau) \to 0$ as $n \to \infty$. Otherwise, for any $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that $\sigma_n(\varepsilon) \geq M_\varepsilon$, then

$$\int_{B_{M_\varepsilon}(x_{j_0})} |w_n|^{2_s^*} dx \leq \sup_{x\in\bar{Q}} \int_{B_{\sigma_n(\varepsilon)}(x)} |w_n|^{2_s^*} dx = G_n(\sigma_n(\varepsilon)) = \varepsilon.$$

In particular,

$$\nu_{j_0} \leq \int_{B_{M_\varepsilon}(x_{j_0})} |w_n|^{2_s^*} dx + o(1) \leq \varepsilon + o(1), \quad \forall\, \varepsilon > 0, \tag{2.11}$$

where $o(1) \to 0$ as $n \to \infty$. Letting $n \to \infty$ and $\varepsilon \to 0$ in (2.11), we see that $\nu_{j_0} \le 0$ which contradicts to $\nu_{j_0} > 0$.

Since $\|v_n\|_{\dot{H}^s(\mathbb{R}^N)} = \|(-\Delta)^{\frac{s}{2}} u\|_2$ and $\{w_n\}$ is bounded in $X^s(\mathbb{R}^{N+1}_+)$, up to a subsequence, there exists a $v \in \dot{H}^s(\mathbb{R}^N)$ such that

$$v_n \rightharpoonup v \quad \text{in } \dot{H}^s(\mathbb{R}^N). \tag{2.12}$$

For any $\varphi \in C_0^\infty(\mathbb{R}^N)$, denote $\tilde{\varphi}_n(x) := \sigma_n^{\frac{N-2s}{2}} \varphi(\frac{x-y_n}{\sigma_n})$. Note that $\sigma_n \to 0$ and $y_n \in \bar{Q}$ imply that $\tilde{\varphi}_n(x) \in C_0^\infty(U)$ for n large. Then, we get from (2.5) that

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla v_n \nabla \varphi dx dy - \int_{\mathbb{R}^N} |v_n|^{2_s^*-2} v_n \varphi dx \\
&\quad = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \tilde{\varphi}_n dx - \int_{\mathbb{R}^N} |u_n|^{2_s^*-2} u_n \tilde{\varphi}_n dx \\
&\quad = o(1) \|\tilde{\varphi}_n\|_{H^s(\mathbb{R}^N)}.
\end{aligned} \tag{2.13}$$

Letting $n \to \infty$ in (2.13), we see that v is a solution of (2.6).

Next, we claim that up to a subsequence,

$$w_n \to w \quad \text{in } L^{2_s^*}(B_1(0)). \tag{2.14}$$

By Lemma 2.6, we obtain an at most countable index set J, sequences $\{x_j\} \subset \mathbb{R}^N$ and $\{\mu_j\}, \{\nu_j\} \subset (0, \infty)$ such that

$$\mu \ge y^{1-2s} |\nabla w_n|^2 + \sum_{j \in J} \mu_j \delta_{(x_j, 0)}, \quad \nu = |w(x,0)|^{2_s^*} + \sum_{j \in J} \nu_j \delta_{x_j} \quad \text{and} \quad S_s \nu_j^{\frac{N-2s}{N}} \le \mu_j. \tag{2.15}$$

To prove (2.14), it suffices to show that $\{x_j\} \cap \overline{B_1(0)} = \emptyset$. Suppose, by contradiction, that there is a $x_{j_0} \in \overline{B_1(0)}$ for some $j_0 \in J$. Define for $\rho > 0$, the function $\varphi_\rho(x,y) := \varphi(\frac{x - x_{j_0}}{\rho}, \frac{y}{\rho})$, where $\varphi \in C_0^\infty(\mathbb{R}^{N+1}_+, [0,1])$ is such that $\varphi \equiv 1$ on $B_1^{N+1}(0)$, $\varphi \equiv 0$ on $\mathbb{R}^{N+1}_+ \backslash B_2^{N+1}(0)$ and $|\nabla \varphi| \le C$. We suppose that ρ is chosen in such way that the support of $\varphi_\rho(x,0)$ is contained in $B_1(0)$. Denote $\tilde{\varphi}_{\rho,n}(x,y) := \varphi_\rho((\frac{x-y_n}{\sigma_n}), \frac{y}{\sigma_n})$ by the facts that $y_n \in \bar{Q}$, $x_{j_0} \in \overline{B_1(0)}$ and $\sigma_n \to 0$ as $n \to \infty$, we see that for n large, $\operatorname{supp} \tilde{\varphi}_{\rho,n} \subset B_{2\sigma_n \rho}(y_n + \sigma_n x_{j_0}) \subset U_+^{N+1} \subset \mathbb{R}^{N+1}_+$, then $\tilde{\varphi}_{\rho,n} w_n \in X_0^s(U_+^{N+1})$. Direct computations show that $\{\tilde{\varphi}_{\rho,n} w_n\}$ is bounded and the bound is independent of ρ. By (2.5) and the fact that $C_0^\infty(U_+^{N+1})$ is dense in $X_0^s(U_+^{N+1})$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla w_n \nabla(\varphi_\rho w_n) dx dy - \int_{\mathbb{R}^N} |w_n(x,0)|^{2_s^*-2} w_n \varphi_\rho w_n dx \\
&\quad = \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla w_n|^2 \varphi_\rho dx dy + \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla w_n \nabla(\varphi_\rho) w_n dx dy \\
&\qquad - \int_{\mathbb{R}^N} |w_n(x,0)|^{2_s^*-2} w_n \varphi_\rho w_n dx
\end{aligned}$$

$$= \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w_n|^2\tilde{\varphi}_{\rho,n}dxdy + \int_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla w_n\nabla(\tilde{\varphi}_{\rho,n})w_n dxdy$$
$$- \int_{\mathbb{R}^N} |w_n(x,0)|^{2_s^*-2}w_n\tilde{\varphi}_{\rho,n}w_n dx$$
$$= \int_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla w_n\nabla(\tilde{\varphi}_{\rho,n}w_n)dxdy - \int_{\mathbb{R}^N} |w_n(x,0)|^{2_s^*-2}w_n\tilde{\varphi}_{\rho,n}w_n dx$$
$$= o(1)\|\tilde{\varphi}_{\rho,n}w_n\|_{X^s} = o(1). \tag{2.16}$$

As $\rho \to 0$,

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^{N+1}_+} y^{1-2s}(\nabla w_n\nabla\varphi_\rho)w_n dxdy\right|$$
$$\le \limsup_{n\to\infty}\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w_n|^2 dxdy\right)^{\frac{1}{2}}\left(\int_{B_{2\rho}^{N+1}(x_{j_0})} y^{1-2s}|\nabla\varphi_\rho|^2 w_n^2 dxdy\right)^{\frac{1}{2}}$$
$$\le C\left(\int_{B_{2\rho}^{N+1}(x_{j_0})} |\nabla\varphi_\rho|^2 w^2 dxdy\right)^{\frac{1}{2}}$$
$$\le C\left(\int_{B_{2\rho}^{N+1}(x_{j_0})} |\nabla\varphi_\rho|^{\frac{N}{s}} dxdy\right)^{\frac{s}{N}}\left(\int_{B_{2\rho}^{N+1}(x_{j_0})} w^{2_s^*} dxdy\right)^{\frac{1}{2_s^*}}$$
$$\le C\left(\int_{B_{2\rho}^{N+1}(x_{j_0})} w^{2_s^*} dxdy\right)^{\frac{1}{2_s^*}} \to 0.$$

Hence, we see from (2.16) that $\mu_{j_0} \le \nu_{j_0}$, then by (2.15) we get $\nu_{j_0} \ge S_s^{\frac{N}{2s}}$. By (2.10), there holds

$$S_s^{\frac{N}{2s}} \le \nu_{j_0} \le \int_{B_1(0)} |v_n|^{2_s^*}dx + o(1) = \tau + o(1), \tag{2.17}$$

where $o(1) \to 0$ as $n \to \infty$. Letting $n \to \infty$ in (2.17), we see that $S_s^{\frac{N}{2s}} \le \tau$, which contradicts $S_s^{\frac{N}{2s}} > \tau$, hence $\{x_j\} \cap \overline{B_1(0)} = \emptyset$, (2.14) holds. Equations (2.10) and (2.14) imply that

$$\int_{B_1(0)} |v|^{2_s^*}dx = \lim_{n\to\infty}\int_{B_1(o)} |v_n|^{2_s^*}dx = \tau > 0,$$

which means that v is nontrivial. □

We collect some regularity results which are useful in our problem.

Lemma 2.7 ([20]). *Let $u \in \dot{H}^s(\mathbb{R}^N)$ be a nonnegative solution to the problem*

$$(-\Delta)^s u = f(x,u) \quad \text{in } \mathbb{R}^N,$$

and assume that $|f(x,t)| \leq C(1+|t|^p)$, *for some* $1 \leq p \leq 2_s^* - 1$ *and* $C > 0$. *Then* $u \in L^\infty(\mathbb{R}^N)$.

3. The Limiting Problem

In this section, we consider the existence of a ground state solution to the following limiting problem, that is the constant potential case

$$\begin{cases} (-\Delta)^s u + au + \phi_u^t u = \lambda |u|^{p-2}u + |u|^{2_s^*-2}u, \\ u > 0, \quad u \in H^s(\mathbb{R}^3), \end{cases} \tag{3.1}$$

where $a > 0$ and the norm on the $H^s(\mathbb{R}^3)$ is taken as

$$\|u\| = \left(\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + au^2) dx \right)^{\frac{1}{2}}.$$

We define the energy functional for the limiting problem (3.1) by

$$\begin{aligned} I_a(u) &= \frac{1}{2} \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + au^2) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\lambda}{p} \int_{\mathbb{R}^3} |u|^p dx - \frac{1}{2_s^*} \int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &= \frac{1}{2}\|u\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\lambda}{p} \int_{\mathbb{R}^3} |u|^p dx - \frac{1}{2_s^*} \int_{\mathbb{R}^3} |u|^{2_s^*} dx, \end{aligned}$$

which is of C^1 class and whose derivative is given by

$$\begin{aligned} \langle I_a'(u), v \rangle = &\int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + auv) dx + \int_{\mathbb{R}^3} \phi_u^t uv dx - \lambda \int_{\mathbb{R}^3} |u|^{p-2} uv dx \\ &- \int_{\mathbb{R}^3} |u|^{2_s^*-2} uv dx, \end{aligned}$$

for all $v \in H^s(\mathbb{R}^3)$. Hence the critical points of I_a in $H^s(\mathbb{R}^3)$ are weak solutions of problem (3.1).

In view of [48], if $u \in H^s(\mathbb{R}^3)$ is a weak solution to problem (3.1), then we have the following Pohožaev identity:

$$\begin{aligned} P_a(u) = &\frac{3-2s}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{3a}{2} \int_{\mathbb{R}^3} u^2 dx + \frac{2t+3}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx \\ &- \frac{3\lambda}{p} \int_{\mathbb{R}^3} |u|^p dx - \frac{3}{2_s^*} \int_{\mathbb{R}^3} |u|^{2_s^*} dx = 0. \end{aligned}$$

Similarly in [44], define $G_a : H^s(\mathbb{R}^3) \to \mathbb{R}$ as

$$\begin{aligned} G_a(u) &= (s+t)\langle I_a'(u), u \rangle - P_a(u) \\ &= \frac{4s+2t-3}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{2s+2t-3}{2} \int_{\mathbb{R}^3} au^2 dx \\ &\quad + \frac{4s+2t-3}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{p(s+t)-3}{p} \lambda \int_{\mathbb{R}^3} |u|^p dx \\ &\quad - \frac{(s+t)2_s^*-3}{2_s^*} \int_{\mathbb{R}^3} |u|^{2_s^*} dx. \end{aligned}$$

Next we study the functional I_a restricted on the manifold M_a defined as

$$M_a := \{u \in H^s(\mathbb{R}^3)\backslash\{0\} : G_a(u) = 0\}.$$

Obviously, if $u \in H^s(\mathbb{R}^3)$ is a nontrivial critical point of I_a, then $u \in M_a$. Hence, if $(u, \phi) \in H^s(\mathbb{R}^3) \times \mathcal{D}^{t,2}(\mathbb{R}^3)$ is a solution of (3.1), then $u \in M_a$.

Lemma 3.1. *For any $u \in H^s(\mathbb{R}^3)\backslash\{0\}$, there is a unique $\theta_0 > 0$ such that $u_{\theta_0} \in M_a$, where $u_{\theta_0}(x) := \theta_0^{s+t}u(\theta_0 x)$. Moreover,*

$$I_a(u_{\theta_0}) = \max_{\theta>0} I_a(u_\theta).$$

Proof. For any $u \in H^s(\mathbb{R}^3)\backslash\{0\}$ and $\theta > 0$, set $u_\theta(x) := \theta^{s+t}u(\theta x)$. Consider the function

$$\tau(\theta) := I_a(u_\theta) = \frac{\theta^{4s+2t-3}}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx + \frac{\theta^{2s+2t-3}}{2}\int_{\mathbb{R}^3} au^2 dx$$
$$+ \frac{\theta^{4s+2t-3}}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\theta^{(s+t)p-3}}{p}\lambda\int_{\mathbb{R}^3} |u|^p dx - \frac{\theta^{(s+t)2_s^*-3}}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx.$$

Clearly, by elementary computations, $\tau(\theta)$ is positive for small θ and tends to $-\infty$ as $\theta \to +\infty$. Moreover, $\tau(\theta)$ has a unique critical point $\theta_0 > 0$ corresponding to its maximum, that is

$$\tau(\theta_0) = \max_{\theta>0} \tau(\theta) \quad \text{and} \quad \tau'(\theta_0) = 0,$$

which means that $u_{\theta_0} \in M_a$. □

Lemma 3.2. *I_a possesses the mountain pass geometry, that is*

(i) *there exist $\alpha, \rho > 0$ such that $I_a(u) \geq \alpha$ for $\|u\| = \rho$;*
(ii) *there exists an $e \in H^s(\mathbb{R}^3)$ satisfying $\|e\| > \rho$ such that $I_a(e) < 0$.*

Proof. (i) There exist $\alpha, \rho > 0$ small and constants $C_1, C_2 > 0$ such that

$$\begin{aligned} I_a(u) &= \frac{1}{2}\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}}u|^2 + au^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^3} |u|^p dx \\ &\quad - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &\geq \frac{1}{2}\|u\|^2 - C_1\lambda\|u\|^p - C_2\|u\|^{2_s^*} \geq \alpha > 0 \end{aligned}$$

for $\|u\| = \rho > 0$.

(ii) Fix $u \in H^s(\mathbb{R}^3)\backslash\{0\}$, set $u_\theta(x) := \theta^{s+t}u(\theta x)$, then we have

$$I_a(u_\theta) = \frac{\theta^{4s+2t-3}}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx + \frac{\theta^{2s+2t-3}}{2}\int_{\mathbb{R}^3} au^2 dx$$

$$+\frac{\theta^{4s+2t-3}}{4}\int_{\mathbb{R}^3}\phi_u^t u^2dx-\frac{\theta^{(s+t)p-3}}{p}\lambda\int_{\mathbb{R}^3}|u|^pdx$$
$$-\frac{\theta^{(s+t)2_s^*-3}}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx<0,$$

for $\theta>0$ large, then there exists $\theta_0>0$ such that $e=u_{\theta_0}$ and $I_a(e)<0$. □

Hence we can define the mountain pass level of I_a:

$$c_a:=\inf_{\gamma\in\Gamma_a}\sup_{h\in[0,1]}I_a(\gamma(h)),$$

where the set of paths is defined as

$$\Gamma_a:=\{\gamma\in C([0,1],H^s(\mathbb{R}^3)):\gamma(0)=0\text{ and }I_a(\gamma(1))<0\}.$$

Next, we will construct a Palais–Smale sequence $\{u_n\}$ for I_a at the level c_a which satisfies $G_a(u_n)\to 0$ as $n\to\infty$.

Lemma 3.3. *There exists a sequence $\{u_n\}\subset H^s(\mathbb{R}^3)$ such that*

$$I_a(u_n)\to c_a,\quad I_a'(u_n)\to 0,\quad G_a(u_n)\to 0\quad\text{as } n\to\infty. \tag{3.2}$$

Proof. In order to construct a Pohožaev–Palais–Smale sequence, following Jeanjean [32], for $\theta\in\mathbb{R}$, $v\in H^s(\mathbb{R}^3)$ and $x\in\mathbb{R}^3$ we define the map $\Phi:\mathbb{R}\times H^s(\mathbb{R}^3)$ by

$$\Phi(\theta,v):=e^{(s+t)\theta}v(e^\theta x).$$

Then the functional $I_a\circ\Phi$ is computed as

$$I_a(\Phi(\theta,v))=\frac{e^{(4s+2t-3)\theta}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}v|^2dx+\frac{e^{(2s+2t-3)\theta}}{2}\int_{\mathbb{R}^3}av^2dx$$
$$+\frac{e^{(4s+2t-3)\theta}}{4}\int_{\mathbb{R}^3}\phi_v^t v^2dx-\frac{e^{((s+t)p-3)\theta}}{p}\lambda\int_{\mathbb{R}^3}|v|^pdx$$
$$-\frac{e^{((s+t)2_s^*-3)\theta}}{2_s^*}\int_{\mathbb{R}^3}|v|^{2_s^*}dx.$$

In view of Lemma 3.2, we can easily check that $I_a(\Phi(\theta,v))>0$ for all (θ,v) with $|\theta|,\|v\|$ small and $I_a(\Phi(0,e))<0$. That is $I_a\circ\Phi$ possesses the mountain pass geometry in $\mathbb{R}\times H^s(\mathbb{R}^3)$. Hence we can define the mountain pass level of $I_a\circ\Phi$:

$$\tilde{c}_a:=\inf_{\tilde{\gamma}\in\tilde{\Gamma}_a}\sup_{h\in[0,1]}I_a(\tilde{\gamma}(h)),$$

where the set of paths is defined as

$$\tilde{\Gamma}_a:=\{\tilde{\gamma}\in C([0,1],H^s(\mathbb{R}^3)):\tilde{\gamma}(0)=0\text{ and }I_a(\tilde{\gamma}(1))<0\}.$$

Note that $\Gamma_a=\{\Phi\circ\tilde{\gamma}:\tilde{\gamma}\in\tilde{\Gamma}_a\}$, we see that mountain pass levels of I_a and $I_a\circ\Phi$ coincide, i.e. $c_a=\tilde{c}_a$. By the General Minimax Principle [53, Theorem 2.8], there exists a sequence $\{(\theta_n,v_n)\}\subset\mathbb{R}\times H^s(\mathbb{R}^3)$ such that as $n\to\infty$,

$$(I_a\circ\Phi)(\theta_n,v_n)\to c_a,\quad (I_a\circ\Phi)'(\theta_n,v_n)\to 0\quad\text{in }(\mathbb{R}\times H^s(\mathbb{R}^3))^{-1},\quad \theta_n\to 0.$$

Because for every $(h, w) \in \mathbb{R} \times H^s(\mathbb{R}^3)$,

$$\langle (I_a \circ \Phi)'(\theta_n, v_n), (h, w)\rangle = \langle I_a'(\Phi(\theta_n, v_n)), \Phi(\theta_n, w)\rangle + G_a(\Phi(\theta_n, v_n))h.$$

We take $h = 1, w = 0$, then as $n \to \infty$, we get

$$G_a(\Phi(\theta_n, v_n)) \to 0. \tag{3.3}$$

For any $v \in H^s(\mathbb{R}^3)$, set $w(x) = e^{-(s+t)\theta_n} v(e^{-\theta_n} x)$, $h = 0$ in (3.3), we get

$$\langle I_a'(\Phi(\theta_n, v_n)), \Phi(\theta_n, w)\rangle = \langle I_a'(\Phi(\theta_n, v_n)), v\rangle = o(1)\|v\|.$$

Denote $u_n := \Phi(\theta_n, v_n)$, then we get a bounded sequence $\{u_n\} \subset H^s(\mathbb{R}^3)$ that satisfies (3.2). $\square$

Set

$$c_a^* = \inf_{u \in H^s(\mathbb{R}^3)\setminus\{0\}} \max_{\theta > 0} I_a(u_\theta), \quad c_a^{**} = \inf_{u \in M_a} I_a(u).$$

Similarly to the proof of [48, Lemma 3.4], we can obtain the following lemma.

Lemma 3.4. *The following equalities hold*:

$$c_a = c_a^* = c_a^{**} > 0.$$

For the mountain pass level c_a for I_a, we have the following estimate of upper boundedness.

Lemma 3.5 ([49]). *The following inequality holds*:

$$0 < c_a < \frac{s}{3} S_s^{\frac{3}{2s}},$$

if one of the following conditions is satisfied

(i) $s > \frac{3}{4} : p \in (\frac{4s}{3-2s}, 2_s^*)$ *and any* $\lambda > 0$;

(ii) $s > \frac{3}{4} : p \in (\frac{4s+2t}{s+t}, \frac{4s}{3-2s}]$ *and any* $\lambda > 0$ *large enough*;

(iii) $\frac{1}{2} < s \leq \frac{3}{4} : p \in (\frac{4s+2t}{s+t}, 2_s^*)$ *and any* $\lambda > 0$.

Proof. The strategy is coming from [12]. For the sake of completeness, we give the details here.

We define

$$u_\varepsilon(x) = \psi(x) U_\varepsilon(x), \quad x \in \mathbb{R}^3,$$

where $U_\varepsilon(x) = \varepsilon^{-\frac{3-2s}{2}} u^*(\frac{x}{\varepsilon})$, $u^*(x) = \frac{\tilde{u}(x/S_s^{\frac{1}{2s}})}{\|\tilde{u}\|_{2_s^*}}$, $\tilde{u}(x) = \kappa(\mu_0^2 + |x - x_0|^2)^{-\frac{3-2s}{2}}$ (see [46, Sec. 4]), $\kappa \in \mathbb{R}\setminus\{0\}$, $\mu_0 > 0$ and $x_0 \in \mathbb{R}^3$ are fixed constants, and $\psi \in C_0^\infty(\mathbb{R}^3)$ such that $0 \leq \psi \leq 1$ in $\mathbb{R}^3$, $\psi \equiv 1$ in B_r and $\psi \equiv 0$ in $\mathbb{R}^3\setminus B_{2r}$. From [46, Propositions 21 and 22], we know that

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon(x)|^2 dx \leq S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}), \tag{3.4}$$

$$\int_{\mathbb{R}^3} |u_\varepsilon(x)|^{2_s^*} dx = S_s^{\frac{3}{2s}} + O(\varepsilon^3), \tag{3.5}$$

and

$$\int_{\mathbb{R}^3} |u_\varepsilon(x)|^r dx = \begin{cases} O(\varepsilon^{\frac{3(2-r)+2sr}{2}}), & r > \dfrac{3}{3-2s}, \\ O(\varepsilon^{\frac{3(2-r)+2sr}{2}}|\log\varepsilon|), & r = \dfrac{3}{3-2s}, \\ O(\varepsilon^{\frac{3-2s}{2}r}), & r < \dfrac{3}{3-2s}. \end{cases} \tag{3.6}$$

Define

$$g(\theta) = \frac{\theta^{4s+2t-3}}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 dx - \frac{\theta^{(s+t)2_s^*-3}}{2_s^*}\int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*} dx \quad \text{for } \theta \geq 0.$$

By a direct calculation, we have that $g(\theta)$ attains its maximum at

$$\theta_0 = \left(\frac{2_s^*(4s+2t-3)}{2((s+t)2_s^*-3)} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 dx}{\int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*} dx} \right)^{\frac{1}{(s+t)2_s^*-4s-2t}}$$

$$= \left(\frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 dx}{\int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*} dx} \right)^{\frac{1}{(s+t)2_s^*-4s-2t}}.$$

Moreover, by (3.4) and (3.5), using the elementary inequality $(\alpha+\beta)^q \leq \alpha^q + q(\alpha+\beta)^{q-1}\beta$ which holds for $q \geq 1$ and $\alpha, \beta \geq 0$, and $\frac{2_s^*(4s+2t-3)}{2((s+t)2_s^*-3)} = 1$, $\frac{4s+2t-3}{(s+t)2_s^*-4s-2t} = \frac{3-2s}{2s}$, we deduce that

$$\begin{aligned} \max_{\theta \geq 0} g(\theta) = g(\theta_0) &= \frac{1}{2}\left(\frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 dx}{\int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*} dx} \right)^{\frac{4s+2t-3}{(s+t)2_s^*-4s-2t}} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 dx \\ &\quad - \frac{1}{2_s^*}\left(\frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 dx}{\int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*} dx} \right)^{\frac{(s+t)2_s^*-3}{(s+t)2_s^*-4s-2t}} c\int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*} dx \\ &= \frac{s}{3} \frac{\|(-\Delta)^{\frac{s}{2}} u_\varepsilon\|_2^{\frac{(s+t)2_s^*-3}{(s+t)2_s^*-4s-2t}}}{\|u_\varepsilon\|_{2_s^*}^{\frac{2_s^*(4s+2t-3)}{(s+t)2_s^*-4s-2t}}} \\ &\leq \frac{s}{3} \frac{\left(S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s})\right)^{\frac{(s+t)2_s^*-3}{(s+t)2_s^*-4s-2t}}}{\left(S_s^{\frac{3}{2s}} + O(\varepsilon^3)\right)^{\frac{4s+2t-3}{(s+t)2_s^*-4s-2t}}} \end{aligned}$$

$$\leq \frac{s}{3}\frac{(S_s^{\frac{3}{2s}})^{\frac{(s+t)2_s^*-3}{(s+t)2_s^*-4s-2t}}+O(\varepsilon^{3-2s})}{(S_s^{\frac{3}{2s}}+O(\varepsilon^3))^{\frac{4s+2t-3}{(s+t)2_s^*-4s-2t}}}$$

$$\leq \frac{s}{3}S_s^{\frac{3}{2s}}+O(\varepsilon^{3-2s}). \tag{3.7}$$

Since $I_a((u_\varepsilon)_\theta) \to -\infty$ as $\theta \to +\infty$, by a standard argument, there exists $\theta_\varepsilon > 0$ such that

$$0 < c_a \leq \max_{\theta\geq 0} I_a((u_\varepsilon)_\theta) = I_a((u_\varepsilon)_{\theta_\varepsilon}), \tag{3.8}$$

which implies that $\theta_\varepsilon \geq \theta_1 > 0$ for some constant θ_1. On the other hand, from (3.4)–(3.6), for any $\varepsilon > 0$, we have that

$$0 < c_a \leq I_a((u_\epsilon)_{\theta_\epsilon}) \leq C_1\theta_\varepsilon^{4s+2t-3} + C_2\theta_\varepsilon^{2s+2t-3} - C_3\theta_\varepsilon^{(s+t)2_s^*-3},$$

which implies that there exists $\theta_2 > 0$ such that $\theta_\varepsilon \leq \theta_2$ and thus $0 < \theta_1 \leq \theta_\varepsilon \leq \theta_2$ for any $\varepsilon > 0$.

Now, by (3.4)–(3.8), we deduce that

$$\begin{aligned}
I_a((u_\varepsilon)_{\theta_\varepsilon}) &\leq \frac{s}{3}S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) + \frac{\theta_\varepsilon^{2s+2t-3}a}{2}\int_{\mathbb{R}^3} u_\varepsilon^2 dx + \frac{\theta_\varepsilon^{4s+2t-3}}{4}\int_{\mathbb{R}^3}\phi_{u_\varepsilon}^t u_\varepsilon^2 dx \\
&\quad - \lambda\frac{\theta_\varepsilon^{(s+t)p-3}}{p}\int_{\mathbb{R}^3}|u_\varepsilon|^p dx - \frac{\theta_\varepsilon^{(s+t)2_s^*-3}}{2_s^*}\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}dx \\
&\leq \frac{s}{3}S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) + \frac{{\theta_2}^{2s+2t-3}a}{2}\int_{\mathbb{R}^3} u_\varepsilon^2 dx + \frac{{\theta_2}^{4s+2t-3}}{4}\int_{\mathbb{R}^3}\phi_{u_\varepsilon}^t u_\varepsilon^2 dx \\
&\quad - \lambda\frac{{\theta_1}^{(s+t)p-3}}{p}\int_{\mathbb{R}^3}|u_\varepsilon|^p dx \\
&\leq \frac{s}{3}S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) + \frac{{\theta_2}^{2s+2t-3}a}{2}\int_{\mathbb{R}^3} u_\varepsilon^2 dx \\
&\quad + C{\theta_2}^{4s+2t-3}\left(\int_{\mathbb{R}^3}|u_\varepsilon|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{3}} - \lambda\frac{{\theta_1}^{(s+t)p-3}}{p}\int_{\mathbb{R}^3}|u_\varepsilon|^p dx.
\end{aligned}$$

Next, we separate three cases:

Case 1. $s > \frac{3}{4} \Leftrightarrow \frac{3}{3-2s} > 2$. In this case, we have

$$\int_{\mathbb{R}^3} u_\varepsilon^2 dx = O(\varepsilon^{3-2s}).$$

Therefore,

$$I_a((u_\varepsilon)_{\theta_\varepsilon}) \leq \frac{s}{3}S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) + C\left(\int_{\mathbb{R}^3}|u_\varepsilon|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{3}} - C\int_{\mathbb{R}^3}|u_\varepsilon|^p dx.$$

Moreover, we deduce that

$$\lim_{\varepsilon\to 0}\frac{\left(\int_{\mathbb{R}^3}|u_\varepsilon(x)|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{3}}}{\varepsilon^{3-2s}} \le \begin{cases} \lim\limits_{\varepsilon\to 0}\dfrac{O(\varepsilon^{4s+2t-3})}{\varepsilon^{3-2s}}=0, & \dfrac{12}{3+2t}>\dfrac{3}{3-2s},\\ \lim\limits_{\varepsilon\to 0}\dfrac{O(\varepsilon^{4s+2t-3}|\log\varepsilon|^{\frac{3+2t}{3}})}{\varepsilon^{3-2s}}=0, & \dfrac{12}{3+2t}=\dfrac{3}{3-2s},\\ \lim\limits_{\varepsilon\to 0}\dfrac{O(\varepsilon^{6-4s})}{\varepsilon^{3-2s}}=0, & \dfrac{12}{3+2t}<\dfrac{3}{3-2s}, \end{cases}$$

and we also have $2s-\frac{3-2s}{2}p<0$ if $\frac{4s}{3-2s}<p<2_s^*$, then we deduce that

$$\lim_{\varepsilon\to 0}\lambda\frac{\int_{\mathbb{R}^3}|u_\varepsilon(x)|^p dx}{\varepsilon^{3-2s}} = \begin{cases} \lim\limits_{\varepsilon\to 0}\lambda\dfrac{O(\varepsilon^{3-\frac{3-2s}{2}p})}{\varepsilon^{3-2s}}=+\infty, & \dfrac{4}{3-2s}<p<2_s^*,\\ \lim\limits_{\varepsilon\to 0}\lambda\dfrac{O(\varepsilon^{3-\frac{3-2s}{2}p})}{\varepsilon^{3-2s}}, & \dfrac{3}{3-2s}<p\le\dfrac{4s}{3-2s},\\ \lim\limits_{\varepsilon\to 0}\lambda\dfrac{O(\varepsilon^{3-\frac{3-2s}{2}p}|\log\varepsilon|)}{\varepsilon^{3-2s}}, & p=\dfrac{3}{3-2s},\\ \lim\limits_{\varepsilon\to 0}\lambda\dfrac{O(\varepsilon^{\frac{3-2s}{2}p})}{\varepsilon^{3-2s}}, & \dfrac{4s+2t}{s+t}<p<\dfrac{3}{3-2s}, \end{cases}$$

where we can choosing λ large enough such that the above three limits equal to $+\infty$.

Case 2. $s=\frac{3}{4}\Leftrightarrow\frac{3}{3-2s}=2$. In this case, we have

$$\int_{\mathbb{R}^3}u_\varepsilon^2dx=O(\varepsilon^{2s}|\log\varepsilon|).$$

Therefore,

$$I_a((u_\varepsilon)_{\theta_\varepsilon})\le\frac{s}{3}S_s^{\frac{3}{2s}}+O(\varepsilon^{2s}|\log\varepsilon|)+C\left(\int_{\mathbb{R}^3}|u_\varepsilon|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{3}}-C\int_{\mathbb{R}^3}|u_\varepsilon|^p dx.$$

Moreover, since $\frac{12}{3+2t}>\frac{3}{3-2s}$, then we have

$$\lim_{\varepsilon\to 0}\frac{\left(\int_{\mathbb{R}^3}|u_\varepsilon(x)|^{\frac{12}{3+2t}}dx\right)^{\frac{3+2t}{3}}}{\varepsilon^{2s}|\log\varepsilon|}\le\lim_{\varepsilon\to 0}\frac{O(\varepsilon^{4s+2t-3})}{\varepsilon^{2s}|\log\varepsilon|}=0,$$

and also

$$\lim_{\varepsilon\to 0}\lambda\frac{\int_{\mathbb{R}^3}|u_\varepsilon(x)|^p dx}{\varepsilon^{2s}|\log\varepsilon|}\le\lim_{\varepsilon\to 0}\lambda\frac{O(\varepsilon^{3-\frac{3-2s}{2}p})}{\varepsilon^{2s}|\log\varepsilon|}=+\infty,$$

in view of $\frac{4s+2t}{s+t}<p<2_s^*$ and for any $\lambda>0$.

Case 3. $s < \frac{3}{4} \Leftrightarrow \frac{3}{3-2s} < 2$. In this case, we have

$$\int_{\mathbb{R}^3} u_\varepsilon^2 dx = O(\varepsilon^{2s}).$$

Therefore,

$$I_a((u_\varepsilon)_{\theta_\varepsilon}) \leq \frac{s}{3} S_s^{\frac{3}{2s}} + O(\varepsilon^{2s}) + C\left(\int_{\mathbb{R}^3} |u_\varepsilon|^{\frac{12}{3+2t}} dx\right)^{\frac{3+2t}{3}} - C\int_{\mathbb{R}^3} |u_\varepsilon|^p dx.$$

Since $\frac{12}{3+2t} > \frac{3}{3-2s}$ and $\frac{4s+2t}{s+t} < p < 2_s^*$, we have

$$\lim_{\varepsilon\to 0} \frac{\left(\int_{\mathbb{R}^3} |u_\varepsilon(x)|^{\frac{12}{3+2t}} dx\right)^{\frac{3+2t}{3}}}{\varepsilon^{2s}} = \lim_{\varepsilon\to 0} \frac{O(\varepsilon^{4s+2t-3})}{\varepsilon^{2s}} = 0$$

and for any $\lambda > 0$

$$\lim_{\varepsilon\to 0} \lambda \frac{\int_{\mathbb{R}^3} |u_\varepsilon(x)|^p dx}{\varepsilon^{2s}} = \lim_{\varepsilon\to 0} \lambda \frac{O(\varepsilon^{3-\frac{3-2s}{2}p})}{\varepsilon^{2s}} = +\infty.$$

Therefore, Cases 1–3 imply that

$$0 < c_a \leq I_a((u_\varepsilon)_{\theta_\varepsilon}) < \frac{s}{3} S_s^{\frac{3}{2s}}.$$

Thus we complete the proof. □

Lemma 3.6. *There is a sequence* $\{x_n\} \subset \mathbb{R}^3$ *and* $R > 0, \beta > 0$ *such that*

$$\int_{B_R(x_n)} u_n^2 \geq \beta,$$

where $\{u_n\}$ *is the sequence given in* (3.2).

Proof. It is easy to see from Lemma 3.5 that $\{u_n\}$ in (3.2) is bounded in $H^s(\mathbb{R}^3)$. Suppose by contradiction that the lemma does not hold. Then by the Vanishing Theorem [45] it follows that as $n \to \infty$,

$$\int_{\mathbb{R}^3} |u_n|^r dx \to 0 \quad \text{for all } 2 \leq r < 2_s^*,$$

and then

$$\int_{\mathbb{R}^3} \phi_{u_n}^t |u_n|^2 dx \to 0.$$

Using $\langle (I_a'(u_n), u_n)\rangle = o(1)$, we get

$$\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u_n|^2 + a u_n^2) dx - \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx = o(1).$$

By $I_a(u_n) \to c_a$, we have

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u_n|^{2_s^*} dx = c_a + o(1).$$

Let $\ell \geq 0$ be such that

$$\|u_n\|^2 \to \ell,$$

and

$$\int_{\mathbb{R}^3} |u|^{2_s^*} dx \to \ell.$$

It is easy to check that $\ell > 0$, otherwise $\|u_n\| \to 0$ as $n \to \infty$ which contradicts $c_a > 0$.

By the definition of S_s, we have that

$$S_s \|u_n\|_{2_s^*}^2 \leq \|u_n\|^2,$$

which implies that $\ell \geq S_s^{\frac{3}{2s}}$. Therefore, we get

$$c_a = \frac{s}{3}\ell \geq \frac{s}{3} S_s^{\frac{3}{2s}},$$

which contradicts to Lemma 3.5. □

Lemma 3.7 ([49]). *Let $\{u_n\} \subset M_a$ be a minimizing sequence for c_a which is given by Lemma* 3.4. *Then there exists $\{y_n\} \subset \mathbb{R}^3$ such that for any $\varepsilon > 0$, there exists an $R > 0$ satisfying*

$$\int_{\mathbb{R}^3 \setminus B_R(y_n)} \left(\int_{\mathbb{R}^3} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{3+2s}} dy + a u_n^2 \right) dx \leq \varepsilon.$$

We have the following theorem.

Theorem 3.1. *Problem* (3.1) *has a positive ground state solution $u \in H^s(\mathbb{R}^3)$.*

Proof. By Lemma 3.2 we see that the functional I_a possesses the mountain pass structure. Let $\{u_n\}$ be a sequence given in (3.2) and c_a be the mountain pass value for I_a, respectively. By Lemma 3.7, there exists $\{y_n\} \subset \mathbb{R}^3$ such that for any $\varepsilon > 0$, there exists an $R > 0$ satisfying

$$\int_{\mathbb{R}^3 \setminus B_R(y_n)} \left(\int_{\mathbb{R}^3} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{3+2s}} dy + a u_n^2 \right) dx \leq \varepsilon. \tag{3.9}$$

Define $\widetilde{u_n}(x) = u_n(x - y_n) \in H^s(\mathbb{R}^3)$, then $\phi_{\widetilde{u_n}}^t = \phi_{u_n}^t(\cdot - y_n)$ and thus $\widetilde{u_n} \in M_a$. This means that $\widetilde{u_n}$ is also a minimizing sequence for c_a. Hence, by (3.9), we have for any $\varepsilon > 0$, there exists an $R > 0$ such that

$$\int_{\mathbb{R}^3 \setminus B_R(0)} \left(\int_{\mathbb{R}^3} \frac{|\widetilde{u_n}(x) - \widetilde{u_n}(y)|^2}{|x-y|^{3+2s}} dy + a \widetilde{u_n}^2 \right) dx \leq \varepsilon. \tag{3.10}$$

Since $\widetilde{u_n}$ is bounded in $H^s(\mathbb{R}^3)$, up to a subsequence, we may assume that there is a $\widetilde{u} \in H^s(\mathbb{R}^3)$ such that as $n \to \infty$,

$$\begin{aligned} \widetilde{u_n} &\rightharpoonup \widetilde{u} \quad \text{in } H^s(\mathbb{R}^3), \\ \widetilde{u_n} &\to \widetilde{u} \quad \text{in } L^r_{\text{loc}}(\mathbb{R}^3), \quad 2 \leq r < 2_s^*, \\ \widetilde{u_n}(x) &\to \widetilde{u}(x) \quad \text{a.e. in } \mathbb{R}^3. \end{aligned} \tag{3.11}$$

By Fatou's Lemma and (3.10), we get

$$\int_{\mathbb{R}^3\setminus B_R(0)}\left(\int_{\mathbb{R}^3}\frac{|\widetilde{u}(x)-\widetilde{u}(y)|^2}{|x-y|^{3+2s}}dy+a\tilde{u}^2\right)dx\le\varepsilon. \tag{3.12}$$

By (3.10)–(3.12), and Sobolev's Imbedding Theorem, we have that for any $r\in[2,2_s^*)$ and any $\varepsilon>0$, there exists a $C>0$ such that

$$\begin{aligned}\int_{\mathbb{R}^3}|\widetilde{u_n}-\widetilde{u}|^r dx&=\int_{B_R(0)}|\widetilde{u_n}-\widetilde{u}|^r dx+\int_{\mathbb{R}^3\setminus B_R(0)}|\widetilde{u_n}-\widetilde{u}|^r dx\\&\le\varepsilon+C(\|\widetilde{u_n}\|_{H^s(\mathbb{R}^3\setminus B_R(0))}+\|\widetilde{u}\|_{H^s(\mathbb{R}^3\setminus B_R(0))})\le C\varepsilon.\end{aligned}$$

Hence we have proved that

$$\widetilde{u_n}\to\widetilde{u}\quad\text{in } L^r(\mathbb{R}^3),\ \ 2\le r<2_s^*. \tag{3.13}$$

Since $\widetilde{u_n}\in M_a$, by Lemma 3.6, $\widetilde{u}$ is nontrivial.

Finally, we show that $\widetilde{u_n}\to\widetilde{u}$ in $H^s(\mathbb{R}^3)$. By Lemma 2.3 and (3.13), we deduce that

$$\phi^t_{\widetilde{u_n}}\rightharpoonup\phi^t_{\widetilde{u}}\quad\text{in }\mathcal{D}^{t,2}(\mathbb{R}^3),$$

and thus

$$\int_{\mathbb{R}^3}\phi^t_{\widetilde{u_n}}\widetilde{u_n}^2dx\to\int_{\mathbb{R}^3}\phi^t_{\widetilde{u}}\widetilde{u}^2dx. \tag{3.14}$$

Set $\widetilde{v_n}=\widetilde{u_n}-\widetilde{u}$, by (3.11), we have that

$$\|\widetilde{u_n}\|^2-\|\widetilde{u}\|^2=\|\widetilde{v_n}\|^2+o_n(1),$$

which implies that

$$\|(-\Delta)^{\frac{s}{2}}\widetilde{u_n}\|_2^2-\|(-\Delta)^{\frac{s}{2}}\widetilde{u}\|_2^2=\|(-\Delta)^{\frac{s}{2}}\widetilde{v_n}\|_2^2+o_n(1). \tag{3.15}$$

By Lemma 2.4 and (3.11) we have that

$$\|\widetilde{u_n}\|_{2_s^*}^{2_s^*}-\|\widetilde{u}\|_{2_s^*}^{2_s^*}=\|\widetilde{v_n}\|_{2_s^*}^{2_s^*}+o_n(1). \tag{3.16}$$

Hence, from (3.11), (3.14)–(3.16), it follows that

$$\begin{aligned}c_a-I_a(\widetilde{u})&=I_a(\widetilde{u_n})-I_a(\widetilde{u})+o_n(1)\\&=\frac{1}{2}\|(-\Delta)^{\frac{s}{2}}\widetilde{v_n}\|_2^2-\frac{1}{2_s^*}\|\widetilde{v_n}\|_{2_s^*}^{2_s^*}+o_n(1).\end{aligned} \tag{3.17}$$

Note that $I_a(\widetilde{u})\ge 0$, hence, by Lemma 3.5 and (3.17) we have that

$$\frac{1}{2}\|(-\Delta)^{\frac{s}{2}}\widetilde{v_n}\|_2^2-\frac{1}{2_s^*}\|\widetilde{v_n}\|_{2_s^*}^{2_s^*}+o_n(1)=c_a-I_a(\widetilde{u})<\frac{s}{3}S_s^{\frac{3}{2s}}. \tag{3.18}$$

On the other hand, it follows from (3.11) that

$$\|(-\Delta)^{\frac{s}{2}}\widetilde{v_n}\|_2^2-\|\widetilde{v_n}\|_{2_s^*}^{2_s^*}=o_n(1).$$

We may assume that

$$\lim_{n\to\infty}\|(-\Delta)^{\frac{s}{2}}\widetilde{v_n}\|_2^2=\lim_{n\to\infty}\|\widetilde{v_n}\|_{2_s^*}^{2_s^*}=\ell\ge 0. \tag{3.19}$$

If $\ell > 0$, from the definition of S_s, we have that

$$S_s \leq \frac{\|(-\Delta)^{\frac{s}{2}}\widetilde{v_n}\|_2^2}{\|\widetilde{v_n}\|_{2_s^*}^2}$$

which implies that $\ell \geq S_s^{\frac{3}{2s}}$. Therefore, we get

$$\lim_{n\to\infty}\left\{\frac{1}{2}\|(-\Delta)^{\frac{s}{2}}\widetilde{v_n}\|_2^2 - \frac{1}{2_s^*}\|\widetilde{v_n}\|_{2_s^*}^{2_s^*}\right\} = \frac{s}{3}\ell \geq \frac{s}{3}S_s^{\frac{3}{2s}}$$

which contradicts (3.18). Hence, $\ell = 0$, that is $\widetilde{u_n} \to \widetilde{u}$ in $H^s(\mathbb{R}^3)$ and so we conclude that $\widetilde{u} \in M_a$ and $I_a(\widetilde{u}) = c_a$.

Next we prove that the solution u is positive. Put $u^\pm = \max\{\pm u, 0\}$ the positive (negative) part of u. We note that we get a ground state solution u of the equation

$$(-\Delta)^s u + au + \phi_u^t u = \lambda(u^+)^{p-1} + (u^+)^{2_s^*} \quad \text{in } \mathbb{R}^3. \tag{3.20}$$

Using u^- as a test function in (3.20) we obtain

$$\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u \cdot (-\Delta)^{\frac{s}{2}}u^- dx + \int_{\mathbb{R}^3} a|u^-|^2 dx + \int_{\mathbb{R}^3}\phi_u^t (u^-)^2 dx = 0. \tag{3.21}$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u \cdot (-\Delta)^{\frac{s}{2}}u^- dx &= \frac{1}{2}C(s)\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{3+2s}}dxdy \\
&\geq \frac{1}{2}C(s)\left[\int_{\{u>0\}\times\{u<0\}}\frac{(u(x)-u(y))(-u^-(y))}{|x-y|^{3+2s}}dxdy\right. \\
&\quad + \int_{\{u<0\}\times\{u<0\}}\frac{(u^-(x)-u^-(y))^2}{|x-y|^{3+2s}}dxdy \\
&\quad \left. + \int_{\{u<0\}\times\{u>0\}}\frac{(u(x)-u(y))u^-(x)}{|x-y|^{3+2s}}dxdy\right] \\
&\geq 0.
\end{aligned}$$

Thus, it follows from (3.21) and Lemma 2.3(i), we have $u^- = 0$ and $u \geq 0$. Moreover, if $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^3$, then $(-\Delta)^s u(x_0) = 0$ and by [19, Lemma 3.2], we have

$$(-\Delta)^s u(x_0) = -\frac{C(s)}{2}\int_{\mathbb{R}^3}\frac{u(x_0+y)+u(x_0-y)-2u(x_0)}{|y|^{3+2s}}dy,$$

therefore,

$$\int_{\mathbb{R}^3}\frac{u(x_0+y)+u(x_0-y)}{|y|^{3+2s}}dy = 0,$$

yielding $u \equiv 0$, a contradiction. Therefore, u is a positive solution of the system (3.1) and the proof is completed. □

Let Ω_a be the set of ground state solutions U of (3.1) satisfying $U(0) = \max_{x\in\mathbb{R}^3} U(x)$. Then, we obtain the following compactness of Ω_a.

Lemma 3.8. *For each $a > 0$, Ω_a is compact in $H^s(\mathbb{R}^3)$.*

Proof. It is easy to see that Ω_a is bounded in $H^s(\mathbb{R}^3)$. Then for any sequence $\{U_k\} \subset \Omega_a$, up to a subsequence, we assume that there is a $U_0 \in H^s(\mathbb{R}^3)$ such that

$$U_k \rightharpoonup U_0 \quad \text{in } H^s(\mathbb{R}^3),$$

and U_0 satisfies

$$(-\Delta)^s U_0 + aU_0 + \phi_{U_0}^t U_0 = \lambda U_0^{p-1} + U_0^{2_s^*-1} \quad \text{in } \mathbb{R}^3,\ \ U_0 \geq 0.$$

Next, we will show that U_0 is nontrivial. Indeed, similar to the proof in Lemma 2.5, we can claim that, up to a subsequence,

$$U_k \to U_0 \quad \text{in } L^{2_s^*}_{\text{loc}}(\mathbb{R}^3).$$

By Lemma 2.7, we check that

$$\|U_k\|_{L^\infty_{\text{loc}}(\mathbb{R}^3)} \leq C.$$

In view of [34] and Schauder's estimate, we see that there exists $\alpha \in (0,1)$ such that $\|U_k\|_{C^{2,\alpha}_{\text{loc}}(\mathbb{R}^3)} \leq C$ and the Arzela–Ascoli's Theorem show that

$$U_k(0) \to U_0(0) \quad \text{as } k \to \infty.$$

Since $(-\Delta)^s U_k(0) \geq 0$, from (3.1), we can check that there exists $C_0 > 0$ such that $U_k(0) \geq C_0 > 0$, hence $U_0(0) \geq C_0 > 0$, which means that U_0 is nontrivial. Similar to Theorem 3.1, we get $I_a(U_0) = c_a$ and $U_k \to U_0$ in $H^s(\mathbb{R}^3)$, which completes the proof that Ω_a is compact in $H^s(\mathbb{R}^3)$. □

4. Proof of Theorem 1.1

Observe that (1.1) can be rewritten as

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x)u + \phi u = \lambda |u|^{p-2}u + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^s \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{4.1}$$

and the corresponding energy functional for (4.1) is

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + V(\varepsilon x)u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^3} |u|^p dx \\ &\quad - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx, \end{aligned}$$

for $u \in H_\varepsilon$, where $H_\varepsilon := \{u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx < \infty\}$ endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + V(\varepsilon x)u^2) dx\right)^{\frac{1}{2}}.$$

Now, for each $\varepsilon > 0$, we define

$$\chi_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in \dfrac{\Lambda}{\varepsilon}, \\ \varepsilon^{-1} & \text{if } x \notin \dfrac{\Lambda}{\varepsilon}, \end{cases}$$

and

$$Q_\varepsilon(u) = \left(\int_{\mathbb{R}^3} \chi_\varepsilon u^2 - 1\right)_+^2.$$

Note that this type of penalization was first introduced in [14] which will act as a penalization to force the concentration phenomena to occur inside Λ. Finally, set $J_\varepsilon : H_\varepsilon \to \mathbb{R}$ be given by

$$J_\varepsilon(u) = I_\varepsilon(u) + Q_\varepsilon(u).$$

It is standard to show that $J_\varepsilon \in C^1(H_\varepsilon, \mathbb{R})$. To find solutions of (4.1) which concentrate around the local minimum of V in Λ as $\varepsilon \to 0$, we shall search critical points of J_ε for which Q_ε is zero.

Let $c_{V_0} = I_{V_0}(U)$ for $U \in \Omega_{V_0}$ and $\delta := \frac{1}{10}\text{dist}\{\mathcal{M}, \mathbb{R}^3 \backslash \Lambda\}$, we fix $\beta \in (0, \delta)$ and a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $0 \leq \varphi(x) \leq 1$ and

$$\begin{cases} \varphi(x) = 1 & \text{for } |x| \leq \beta, \\ \varphi(x) = 0 & \text{for } |x| > 2\beta, \\ |\nabla \varphi| \leq \dfrac{C}{\beta}. \end{cases}$$

We will find a solution of (4.1) near the set

$$X_\varepsilon := \left\{\varphi\left(x - \frac{x'}{\varepsilon}\right) U\left(x - \frac{x'}{\varepsilon}\right) : x' \in \mathcal{M}^\beta, U \in \Omega_{V_0}\right\}$$

for sufficiently small $\varepsilon > 0$, where

$$\mathcal{M}^\beta := \left\{y \in \mathbb{R}^3 : \inf_{z \in \mathcal{M}} |y - z| \leq \beta\right\}.$$

Similarly, for $A \subset H_\varepsilon$, we use the notation

$$A^a := \left\{u \in H_\varepsilon : \inf_{v \in A} \|u - v\| \leq a\right\}.$$

For $U^* \in \Omega_{V_0}$ arbitrary but fixed, we define $W_{\varepsilon,\theta} := \theta^{s+t}\varphi(\varepsilon x)U^*(\theta x)$, we will show that J_ε possesses the mountain pass geometry.

Denote $U_\theta^* := \theta^{s+t}U^*(\theta x)$, we have as $\theta \to \infty$

$$\begin{aligned} I_{V_0}(U_\theta^*) = \frac{\theta^{4s+2t-3}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}U^*|^2dx + \frac{\theta^{2s+2t-3}}{2}\int_{\mathbb{R}^3}V_0(U^*)^2dx \\ + \frac{\theta^{4s+2t-3}}{4}\int_{\mathbb{R}^3}\phi_{U^*}^t(U^*)^2dx - \frac{\theta^{(s+t)p-3}}{p}\lambda\int_{\mathbb{R}^3}|U^*|^pdx \\ - \frac{\theta^{(s+t)2_s^*-3}}{2_s^*}\int_{\mathbb{R}^3}|U^*|^{2_s^*}dx \to -\infty, \end{aligned}$$

then there exists a $\theta_0 > 0$ such that $I_{V_0}(U_{\theta_0}^*) < -3$.

We can easily check that $Q_\varepsilon(W_{\varepsilon,\theta_0}) = 0$, then

$$\begin{aligned} J_\varepsilon(W_{\varepsilon,\theta_0}) = I_\varepsilon(W_{\varepsilon,\theta_0}) &= \frac{1}{2}\|W_{\varepsilon,\theta_0}\|^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{W_{\varepsilon,\theta_0}}^tW_{\varepsilon,\theta_0}^2dx - \frac{\lambda}{p}\int_{\mathbb{R}^3}|W_{\varepsilon,\theta_0}|^pdx \\ &\quad - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|W_{\varepsilon,\theta_0}|^{2_s^*}dx \\ &\overset{\tilde{x}=\theta_0x}{=} \frac{\theta_0^{4s+2t-3}}{2}\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}\varphi\left(\frac{\varepsilon\tilde{x}}{\theta_0}\right)U^*(\tilde{x})\right|^2d\tilde{x} \\ &\quad + \frac{\theta_0^{2s+2t-3}}{2}\int_{\mathbb{R}^3}V_0\left(\frac{\varepsilon\tilde{x}}{\theta_0}\right)\varphi^2\left(\frac{\varepsilon\tilde{x}}{\theta_0}\right)(U^*(\tilde{x}))^2d\tilde{x} \\ &\quad + \frac{\theta_0^{4s+2t-3}}{4}\int_{\mathbb{R}^3}\phi^t_{\varphi\left(\frac{\varepsilon\tilde{x}}{\theta_0}\right)U^*(\tilde{x})}\varphi^2\left(\frac{\varepsilon\tilde{x}}{\theta_0}\right)(U^*(\tilde{x}))^2d\tilde{x} \\ &\quad - \frac{\theta_0^{(s+t)p-3}}{p}\lambda\int_{\mathbb{R}^3}\varphi^p\left(\frac{\varepsilon\tilde{x}}{\theta_0}\right)|U^*(\tilde{x})|^pd\tilde{x} \\ &\quad - \frac{\theta_0^{(s+t)2_s^*-3}}{2_s^*}\int_{\mathbb{R}^3}\varphi^{2_s^*}\left(\frac{\varepsilon\tilde{x}}{\theta_0}\right)|U^*(\tilde{x})|^{2_s^*}d\tilde{x} \\ &= I_{V_0}(U_{\theta_0}^*) + o(1) < -2 \end{aligned}$$

for $\varepsilon > 0$ small. Using Sobolev's Imbedding Theorem, we have

$$\begin{aligned} J_\varepsilon(u) \geq I_\varepsilon(u) \geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{p}\int_{\mathbb{R}^3}|u|^pdx - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx \\ \geq \frac{1}{2}\|u\|^2 - C_1\lambda\|u\|^p - C_2\|u\|^{2_s^*} \\ > 0 \end{aligned}$$

for $\|u\|$ small since $p > 2$.

Hence, we can define the mountain pass value of J_ε as

$$c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon}\sup_{h\in[0,1]}J_\varepsilon(\gamma(h)),$$

where the set of paths is defined as

$$\Gamma_\varepsilon := \{\gamma \in C([0,1], H_\varepsilon) : \gamma(0) = 0 \text{ and } \gamma(1) = W_{\varepsilon,\theta_0}\}.$$

We start with the following lemma.

Lemma 4.1.

$$\lim_{\varepsilon\to 0} c_\varepsilon = c_{V_0}.$$

Proof. We split the proof into two steps.

Step 1.

$$\limsup_{\varepsilon\to 0} c_\varepsilon \le c_{V_0}.$$

Denote $W_{\varepsilon,0} = \lim_{\theta\to 0} W_{\epsilon,\theta}$ in H_ε sense, then $W_{\varepsilon,0} = 0$. Thus, setting $\gamma(h) := W_{\varepsilon,h\theta_0}(0 \le h \le 1)$, we have $\gamma(h) \in \Gamma_\varepsilon$, then

$$c_\varepsilon \le \max_{h\in[0,1]} J_\varepsilon(\gamma(h)) = \max_{\theta\in[0,\theta_0]} J_\varepsilon(W_{\varepsilon,\theta}),$$

and we just need to verify that

$$\limsup_{\varepsilon\to 0} \max_{\theta\in[0,\theta_0]} J_\varepsilon(W_{\varepsilon,\theta}) \le c_{V_0}.$$

Indeed, similar to the argument above we have that

$$\begin{aligned}\max_{\theta\in[0,\theta_0]} J_\varepsilon(W_{\varepsilon,\theta}) &= \max_{\theta\in[0,\theta_0]} I_{V_0}(U_\theta^*) + o(1) \le \max_{\theta\in[0,\infty)} I_{V_0}(U_\theta^*) + o(1)\\ &= I_{V_0}(U^*) + o(1) = c_{V_o} + o(1).\end{aligned}$$

Step 2.

$$\liminf_{\varepsilon\to 0} c_\varepsilon \ge c_{V_0}.$$

Assuming to the contrary that $\liminf_{\varepsilon\to 0} c_\varepsilon < c_{V_0}$, then, there exist $\delta_0 > 0$, $\varepsilon_n \to 0$ and $\gamma_n \in \Gamma_\varepsilon$ satisfying

$$J_{\varepsilon_n}(\gamma_n(h)) < c_{V_0} - \delta_0$$

for $h \in [0,1]$. We can fix an ε_n such that

$$\frac{1}{2} V_0 \varepsilon_n (1 + (1 + c_{V_0})^{\frac{1}{2}}) < \min\{\delta_0, 1\}. \tag{4.2}$$

Since $I_{\varepsilon_n}(\gamma_n(0)) = 0$ and $I_{\varepsilon_n}(\gamma_n(1)) \le J_{\varepsilon_n}(\gamma_n(1)) = J_{\varepsilon_n}(W_{\varepsilon_n,\theta_0}) < -2$, we can find an $h_n \in (0,1)$ such that $I_{\varepsilon_n}(\gamma_n(h)) \ge -1$ for $h \in [0, h_n]$ and $I_{\varepsilon_n}(\gamma_n(h_n)) = -1$. Then, for any $h \in [0, h_n]$,

$$Q_{\varepsilon_n}(\gamma_n(h)) = J_{\varepsilon_n}(\gamma_n(h)) - I_{\varepsilon_n}(\gamma_n(h)) \le c_{V_0} - \delta_0 + 1,$$

which implies that

$$\int_{\mathbb{R}^3\setminus(\Lambda/\varepsilon_n)} \gamma_n^2(h)dx \le \varepsilon_n(1 + (1 + c_{V_0})^{\frac{1}{2}}) \quad \text{for } h \in [0, h_n].$$

Then, for $h \in [0, h_n]$,

$$\begin{aligned} I_{\varepsilon_n}(\gamma_n(h)) &= I_{V_0}(\gamma_n(h)) + \frac{1}{2}\int_{\mathbb{R}^3}(V(\varepsilon_n x) - V_0)\gamma_n^2(h)dx \\ &\geq I_{V_0}(\gamma_n(h)) + \frac{1}{2}\int_{\mathbb{R}^3\setminus(\Lambda/\varepsilon_n)}(V(\varepsilon_n x) - V_0)\gamma_n^2(h)dx \\ &\geq I_{V_0}(\gamma_n(h)) - \frac{1}{2}V_0\varepsilon_n(1 + (1 + c_{V_0})^{\frac{1}{2}}), \end{aligned}$$

then

$$\begin{aligned} I_{V_0}(\gamma_n(h)) &\leq I_{\varepsilon_n}(\gamma_n(h)) + \frac{1}{2}V_0\varepsilon_n(1 + (1 + c_{V_0})^{\frac{1}{2}}) \\ &= -1 + \frac{1}{2}V_0\varepsilon_n(1 + (1 + c_{V_0})^{\frac{1}{2}}) < 0 \end{aligned}$$

and we have

$$\max_{h\in[0,h_n]} I_{V_0}(\gamma_n(h)) \geq c_{V_0}.$$

Hence, we deduce that

$$\begin{aligned} c_{V_0} - \delta_0 &\geq \max_{h\in[0,1]} J_{\varepsilon_n}(\gamma_n(h)) \geq \max_{h\in[0,1]} I_{\varepsilon_n}(\gamma_n(h)) \geq \max_{h\in[0,h_n]} I_{\varepsilon_n}(\gamma_n(h)) \\ &\geq \max_{h\in[0,h_n]} I_{V_0}(\gamma_n(h)) - \frac{1}{2}V_0\varepsilon_n(1 + (1 + c_{V_0})^{\frac{1}{2}}), \end{aligned}$$

that is

$$0 < \delta_0 \leq \frac{1}{2}V_0\varepsilon_n(1 + (1 + c_{V_0})^{\frac{1}{2}}),$$

which contradicts (4.2). □

Lemma 4.1 implies that

$$\lim_{\varepsilon\to 0}\left(\max_{h\in[0,1]} J_\varepsilon(\gamma_\varepsilon(h)) - c_\varepsilon\right) = 0,$$

where

$$\gamma_\varepsilon(h) = W_{\varepsilon,h\theta_0} \quad \text{for } h \in [0,1].$$

Denote

$$\tilde{c}_\varepsilon := \max_{h\in[0,1]} J_\varepsilon(\gamma_\varepsilon(h)).$$

We see that

$$c_\varepsilon \leq \tilde{c}_\varepsilon \quad \text{and} \quad \lim_{\varepsilon\to 0} c_\varepsilon = \lim_{\varepsilon\to 0}\tilde{c}_\varepsilon = c_{V_0}.$$

In order to state the next lemma, we need some notations. For each $R > 0$, we regard $H_0^s(B_R(0))$ as a subspace of H_ε. Namely, for any $u \in H_0^s(B_R(0))$, we extend u by defining $u(x) = 0$ for $|x| > R$, then $\|\cdot\|$ is equivalent to the standard norm of

$H_0^s(B_R(0))$ for each $R>0, \varepsilon>0$. Using $\|\cdot\|_{H_\varepsilon}$, for each $T\in(H_0^s(B_R(0)))^{-1}$, we define

$$\|T\|_{*,\varepsilon,R} := \sup\{Tu : u\in H_0^s(B_R(0)), \|u\|_{H_\varepsilon}\le 1\}.$$

Note also that $\|\cdot\|_{*,\varepsilon,R}$ is equivalent to the standard norm of $(H_0^s(B_R(0)))^{-1}$.

We use the notation

$$J_\varepsilon^a := \{u\in E : J_\varepsilon \le a\}$$

and fix a $R_0>0$ such that $\Lambda\subset B_{R_0}(0)$.

Inspired by [22, 55], we have the following lemma and this lemma is a key for the proof of Theorem 1.1.

Lemma 4.2. *There exists a $d_0>0$ such that for any $\{\varepsilon_i\},\{R_{\varepsilon_i}\},\{u_{\varepsilon_i}\}$ with*

$$\begin{cases} \lim_{i\to\infty}\varepsilon_i = 0, \quad R_{\varepsilon_i}\ge \dfrac{R_0}{\varepsilon_i}, \quad u_{\varepsilon_i}\in X_{\varepsilon_i}^{d_0}\cap H_0^s(B_{R_{\varepsilon_i}}(0)), \\ \lim_{i\to\infty} J_{\varepsilon_i}(u_{\varepsilon_i})\le c_{V_0} \quad \text{and} \quad \lim_{i\to\infty}\|J'_{\varepsilon_i}(u_{\varepsilon_i})\|_{*,\varepsilon_i,R_{\varepsilon_i}} = 0, \end{cases} \tag{4.3}$$

then there exist, up to a subsequence, $\{y_i\}\subset\mathbb{R}^3, x_0\in\mathcal{M}, U\in\Omega_{V_0}$ such that

$$\lim_{i\to\infty}|\varepsilon_i y_i - x_0| = 0 \quad \text{and} \quad \lim_{i\to\infty}\|u_{\varepsilon_i} - \varphi(\varepsilon_i x-\varepsilon_i y_i)U(x-y_i)\|_{H_{\varepsilon_i}} = 0.$$

If we drop R_{ε_i} and replace (4.3) by

$$\lim_{i\to\infty}\varepsilon_i = 0, \quad u_{\varepsilon_i}\in X_{\varepsilon_i}^{d_0}, \quad \lim_{i\to\infty} J_{\varepsilon_i}(u_{\varepsilon_i})\le c_{V_0} \quad \text{and} \quad \lim_{i\to\infty}\|J'_{\varepsilon_i}(u_{\varepsilon_i})\|_{*,\varepsilon_i,R_{\varepsilon_i}} = 0,$$

then the same conclusion still holds.

Proof. Since the second assertion can be prove in the same way to the first one, we only treat the first case. For notational simplicity, we write ε for ε_i and still use ε after taking a subsequence. By the definition of $X_\varepsilon^{d_0}$ and the compactness of Ω_{V_0} and $\mathcal{M}^\beta$, we see that there exist $U_0\in\Omega_{V_0}, \{x_\varepsilon\}\subset\mathcal{M}^\beta$ such that for $\varepsilon>0$ small,

$$\left\|u_\varepsilon - \varphi(\varepsilon x - x_\varepsilon)U_0\left(x-\frac{x_\varepsilon}{\varepsilon}\right)\right\|_{H_\varepsilon}\le 2d_0 \quad \text{and} \quad x_\varepsilon\to x_0\in\mathcal{M}^\beta(\varepsilon\to 0). \tag{4.4}$$

We divide the proof into several steps.

Step 1. We claim that

$$\lim_{\varepsilon\to 0}\sup_{y\in A_\varepsilon}\int_{B_1(y)}|u_\varepsilon|^{2_s^*}dx = 0, \tag{4.5}$$

where $A_\varepsilon = B_{\frac{3\beta}{\varepsilon}}(\frac{x_\varepsilon}{\varepsilon})\backslash B_{\frac{\beta}{2\varepsilon}}(\frac{x_\varepsilon}{\varepsilon})$.

If the claim is true, by Lemma 2.2, we see that

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon}|u_\varepsilon|^{2_s^*}dx = 0, \tag{4.6}$$

where $B_\varepsilon = B_{\frac{2\beta}{\varepsilon}}(\frac{x_\varepsilon}{\varepsilon}) \backslash B_{\frac{\beta}{\varepsilon}}(\frac{x_\varepsilon}{\varepsilon})$. Indeed, since

$$\sup_{y\in A_\varepsilon} \int_{B_1(y)} |u_\varepsilon|^{2_s^*} dx \geq \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_\varepsilon \cdot \chi_{A_\varepsilon^1}|^{2_s^*} dx,$$

where $A_\varepsilon^1 = B_{\frac{3\beta}{\varepsilon}-1}(\frac{x_\varepsilon}{\varepsilon}) \backslash B_{\frac{\beta}{2\varepsilon}+1}(\frac{x_\varepsilon}{\varepsilon})$, then

$$\lim_{\varepsilon\to 0} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_\varepsilon \cdot \chi_{A_\varepsilon^1}|^{2_s^*} dx = 0.$$

By Lemma 2.2, we have

$$\int_{\mathbb{R}^3} |u_\varepsilon \cdot \chi_{A_\varepsilon^1}|^{2_s^*} dx \to 0 \quad \text{as } \varepsilon \to 0.$$

Since $B_\varepsilon \subset A_\varepsilon^1$ for $\varepsilon > 0$ small, (4.6) holds.

Next, we will prove (4.5). Assume by contradiction that there is a $r > 0$ such that

$$\lim_{\varepsilon\to 0} \sup_{y\in A_\varepsilon} \int_{B_1(y)} |u_\varepsilon|^{2_s^*} dx = 2r > 0,$$

then there exists $y_\varepsilon \in A_\varepsilon$ such that for $\varepsilon > 0$ small, $\int_{B_1(y_\varepsilon)} |u_\varepsilon|^{2_s^*} dx \geq r > 0$. Note that $y_\varepsilon \in A_\varepsilon$ and there exists $x^* \in \mathcal{M}^{4\beta} \subset \Lambda$ such that $\varepsilon y_\varepsilon \to x^*$ as $\varepsilon \to 0$. Let $v_\varepsilon(x) = u_\varepsilon(x + y_\varepsilon)$, then, for $\varepsilon > 0$ small,

$$\int_{B_1(0)} |v_\varepsilon|^{2_s^*} dx \geq r > 0, \tag{4.7}$$

and up to a subsequence, $v_\varepsilon \rightharpoonup v$ in $H^s(\mathbb{R}^3)$ and v satisfies

$$(-\Delta)^s v + V(x^*)v + \phi_v^t v = \lambda v^{p-1} + v^{2_s^*-1} \quad \text{in } \mathbb{R}^3, v \geq 0.$$

Case 1. If $v \neq 0$, then

$$\begin{aligned} c_{V(x^*)} &\leq I_{V(x^*)}(v) - \frac{1}{4s+2t-3} G_{V(x^*)}(v) \\ &= \frac{s}{4s+2t-3} V(x^*) \int_{\mathbb{R}^3} v^2 dx + \frac{p(s+t)-4s-2t}{4s+2t-3} \frac{\lambda}{p} \int_{\mathbb{R}^3} v^p dx \\ &\quad + \frac{2_s^*(s+t)-4s-2t}{(4s+2t-3)2_s^*} \int_{\mathbb{R}^3} v^{2_s^*} dx, \end{aligned}$$

and hence we have

$$\begin{aligned} &\|V\|_{L^\infty(\bar{\Lambda})} \int_{\mathbb{R}^3} v^2 dx + \frac{p(s+t)-4s-2t}{sp} \lambda \int_{\mathbb{R}^3} v^p dx + \frac{2_s^*(s+t)-4s-2t}{s2_s^*} \int_{\mathbb{R}^3} v^{2_s^*} dx \\ &\quad \geq V(x^*) \int_{\mathbb{R}^3} v^2 dx + \frac{p(s+t)-4s-2t}{sp} \lambda \int_{\mathbb{R}^3} v^p dx \\ &\qquad + \frac{2_s^*(s+t)-4s-2t}{s2_s^*} \int_{\mathbb{R}^3} v^{2_s^*} dx \\ &\quad \geq \frac{4s+2t-3}{s} c_{V(x^*)} \geq \frac{4s+2t-3}{s} c_{V_0}. \end{aligned}$$

Therefore, for sufficiently large R,

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\left\{\|V\|_{L^\infty(\bar{\Lambda})}\int_{B_R(y_\varepsilon)} u_\varepsilon^2 dx + \frac{p(s+t)-4s-2t}{sp}\lambda\int_{B_R(y_\varepsilon)} u_\varepsilon^p dx\right.\\
&\qquad\left. + \frac{2_s^*(s+t)-4s-2t}{s2_s^*}\int_{B_R(y_\varepsilon)} u_\varepsilon^{2_s^*} dx\right\}\\
&\quad = \liminf_{\varepsilon\to 0}\left\{\|V\|_{L^\infty(\bar{\Lambda})}\int_{B_R(0)} v_\varepsilon^2 dx + \frac{p(s+t)-4s-2t}{sp}\lambda\int_{B_R(0)} v_\varepsilon^p dx\right.\\
&\qquad\left. + \frac{2_s^*(s+t)-4s-2t}{s2_s^*}\int_{B_R(0)} v_\varepsilon^{2_s^*} dx\right\}\\
&\quad \geq \left\{\|V\|_{L^\infty(\bar{\Lambda})}\int_{B_R(0)} v^2 dx + \frac{p(s+t)-4s-2t}{sp}\lambda\int_{B_R(0)} v^p dx\right.\\
&\qquad\left. + \frac{2_s^*(s+t)-4s-2t}{s2_s^*}\int_{B_R(0)} v^{2_s^*} dx\right\}\\
&\quad \geq \frac{1}{2}\left\{\|V\|_{L^\infty(\bar{\Lambda})}\int_{\mathbb{R}^3} v^2 dx + \frac{p(s+t)-4s-2t}{sp}\lambda\int_{\mathbb{R}^3} v^p dx\right.\\
&\qquad\left. + \frac{2_s^*(s+t)-4s-2t}{s2_s^*}\int_{\mathbb{R}^3} v^{2_s^*} dx\right\}\\
&\quad \geq \frac{4s+2t-3}{2s}c_{V_0} > 0.
\end{aligned}$$

On the other hand, by Sobolev's Imbedding Theorem and (4.4),

$$\begin{aligned}
&\|V\|_{L^\infty(\bar{\Lambda})}\int_{B_R(y_\varepsilon)} u_\varepsilon^2 dx + \frac{p(s+t)-4s-2t}{sp}\lambda\int_{B_R(y_\varepsilon)} u_\varepsilon^p dx\\
&\qquad + \frac{2_s^*(s+t)-4s-2t}{s2_s^*}\int_{B_R(y_\varepsilon)} u_\varepsilon^{2_s^*} dx\\
&\quad \leq C_1 d_0 + C_2\int_{B_R(y_\varepsilon)}\left|\varphi(\varepsilon x - x_\varepsilon)U_0\left(x-\frac{x_\varepsilon}{\varepsilon}\right)\right|^2 dx\\
&\qquad + C_3\lambda\int_{B_R(y_\varepsilon)}\left|\varphi(\varepsilon x - x_\varepsilon)U_0\left(x-\frac{x_\varepsilon}{\varepsilon}\right)\right|^p dx\\
&\qquad + C_4\int_{B_R(y_\varepsilon)}\left|\varphi(\varepsilon x - x_\varepsilon)U_0\left(x-\frac{x_\varepsilon}{\varepsilon}\right)\right|^{2_s^*} dx\\
&\quad \leq C_1 d_0 + C_2\int_{B_R(y_\varepsilon - \frac{x_\varepsilon}{\varepsilon})}|U_0(x)|^2 dx + C_3\lambda\int_{B_R(y_\varepsilon - \frac{x_\varepsilon}{\varepsilon})}|U_0(x)|^p dx
\end{aligned}$$

$$+ C_4 \int_{B_R(y_\varepsilon - \frac{x_\varepsilon}{\varepsilon})} |U_0(x)|^{2_s^*} dx$$
$$= Cd_0 + o(1), \tag{4.8}$$

where $o(1) \to 0$ as $\varepsilon \to 0$, and we have used the fact that $|y_\varepsilon - \frac{x_\varepsilon}{\varepsilon}| \geq \frac{\beta}{2\varepsilon}$. This leads to a contradiction if d_0 is small enough.

Case 2. If $v = 0$, that is, $v_\varepsilon \rightharpoonup 0$ in $H^s(\mathbb{R}^3)$, then $v_\varepsilon \to 0$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3)$ for $r \in [2, 2_s^*)$. Thus, by (4.7) and Sobolev's Imbedding Theorem, there exists $C > 0$ (independent of ε) such that for $\varepsilon > 0$ small,

$$\int_{B_1(0)} |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 \geq C r^{\frac{2}{2_s^*}} > 0. \tag{4.9}$$

Now, we claim that

$$\lim_{\varepsilon \to 0} \sup_{\varphi \in C_0^\infty(B_2(0)), \|\varphi\|=1} |\langle \rho_\varepsilon, \varphi \rangle| = 0, \tag{4.10}$$

where $\rho_\varepsilon = (-\Delta)^s v_\varepsilon - v_\varepsilon^{2_s^*-1} \in (H^s(\mathbb{R}^3))^{-1}$. For $\varepsilon > 0$ small enough, it is easy to check

$$\int_{\mathbb{R}^3} \chi_\varepsilon(x) u_\varepsilon(x) \varphi(x - y_\varepsilon) dx \equiv 0$$

uniformly for any $\varphi \in C_0^\infty(B_2(0))$. Thus for any $\varphi \in C_0^\infty(B_2(0))$ with $\|\varphi\| = 1$,

$$\begin{aligned} \langle \rho_\varepsilon, \varphi \rangle &= \langle I'_\varepsilon(u_\varepsilon), \varphi(x - y_\varepsilon) \rangle - \int_{\mathbb{R}^3} V(\varepsilon x) u_\varepsilon(x) \varphi(x - y_\varepsilon) dx \\ &\quad - \int_{\mathbb{R}^3} \phi^t_{u_\varepsilon} u_\varepsilon(x) \varphi(x - y_\varepsilon) dx + \lambda \int_{\mathbb{R}^3} (u_\varepsilon(x))^{p-1} \varphi(x - y_\varepsilon) dx \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

In view of the fact that $\|J'_\varepsilon(u_\varepsilon)\|_{*,\varepsilon,R_\varepsilon} \to 0$, $\operatorname{supp} \varphi \subset B_2(0)$, $\sup_{x \in B_2(0)} V(\varepsilon x + \varepsilon y_\varepsilon) \leq C$ uniformly for all $\varepsilon > 0$ small, $v_\varepsilon \to 0$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3)$ for $r \in [2, 2_s^*)$ and Lemma 2.3, we have

$$|J_1| \leq \|I'_\varepsilon(u_\varepsilon)\|_{*,\varepsilon,R_\varepsilon} \|\varphi(x - y_\epsilon)\| \to 0,$$

$$|J_2| \leq \sup_{x \in B_2(0)} V(\varepsilon x + \varepsilon y_\varepsilon) \left(\int_{B_2(0)} |v_\varepsilon|^2 dx \right)^{\frac{1}{2}} \left(\int_{B_2(0)} |\varphi|^2 dx \right)^{\frac{1}{2}} \to 0,$$

$$|J_3| \leq \left(\int_{\mathbb{R}^3} |\phi^t_{v_\varepsilon}|^{2_t^*} dx \right)^{\frac{1}{2_t^*}} \left(\int_{B_2(0)} |v_\varepsilon|^{\frac{3}{t}} dx \right)^{\frac{t}{3}} \left(\int_{B_2(0)} |\varphi|^2 dx \right)^{\frac{1}{2}} \to 0$$

and

$$|J_4| \leq \lambda \left(\int_{B_2(0)} |v_\varepsilon|^p dx \right)^{\frac{p-1}{p}} \left(\int_{B_2(0)} |\varphi|^p dx \right)^{\frac{1}{p}} \to 0$$

as $\varepsilon \to 0$ uniformly for all $\varphi \in C_0^\infty(B_2(0))$ with $\|\varphi\| = 1$, we see that the claim (4.10) is true.

In view of Lemma 2.5, we see from (4.7), (4.9) and (4.10) that there exist $\tilde{y}_\varepsilon \in \mathbb{R}^3$ and $\sigma_\varepsilon > 0$ with $\tilde{y}_\varepsilon \to \tilde{y} \in \overline{B_1(0)}$, $\sigma_\varepsilon \to 0$ as $\varepsilon \to 0$ such that

$$w_\varepsilon(x) := \sigma_\varepsilon^{\frac{3-2s}{2}} v_\varepsilon(\sigma_\varepsilon x + \tilde{y}_\varepsilon) \rightharpoonup w \quad \text{in } \mathcal{D}^{s,2}(\mathbb{R}^3)$$

and $w \geq 0$ is a nontrivial solution of

$$(-\Delta)^s u = u^{2_s^*-1}, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^3). \tag{4.11}$$

It is well known that

$$w(x) = \frac{C\delta^{\frac{3-2s}{2}}}{(\delta^2 + |x|^2)^{\frac{3-2s}{2}}} \tag{4.12}$$

for some $\delta > 0$ and

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} w|^2 dx = \int_{\mathbb{R}^3} w^{2_s^*} dx = S^{\frac{3}{2s}}, \tag{4.13}$$

then there exists a $R > 0$ such that

$$\int_{B_R(0)} w^{2_s^*} dx \geq \frac{1}{2} \int_{\mathbb{R}^3} w^{2_s^*} dx = \frac{1}{2} S^{\frac{3}{2s}} > 0.$$

On the other hand,

$$\begin{aligned} \int_{B_R(0)} w^{2_s^*} dx &\leq \liminf_{\varepsilon \to 0} \int_{B_R(0)} w_\varepsilon^{2_s^*} dx = \liminf_{\varepsilon \to 0} \int_{B_{\sigma_\varepsilon R}(\tilde{y}_\varepsilon)} v_\varepsilon^{2_s^*} dx \\ &= \liminf_{\varepsilon \to 0} \int_{B_{\sigma_\varepsilon R}(\tilde{y}_\varepsilon + y_\varepsilon)} v_\varepsilon^{2_s^*} dx \leq \liminf_{\varepsilon \to 0} \int_{B_2(y_\varepsilon)} u_\varepsilon^{2_s^*} dx, \end{aligned} \tag{4.14}$$

where we have used the facts that $\sigma_\varepsilon \to 0$ and $\tilde{y}_\varepsilon \to \tilde{y} \in \overline{B_1(0)}$ as $\varepsilon \to 0$.

Similar to (4.8) we can check that (4.14) leads to a contradiction for $d_0 > 0$ small. Hence (4.5) holds.

It follows from (4.6) and the interpolation inequality that

$$\lim_{\varepsilon \to 0} \int_{B_\varepsilon} |u_\varepsilon|^s dx = 0 \quad \text{for all } s \in (2, 2_s^*]. \tag{4.15}$$

Step 2. Let $u_{\varepsilon,1} = \varphi(\varepsilon x - x_\varepsilon) u_\varepsilon(x)$, $u_{\varepsilon,2} = (1 - \varphi(\varepsilon x - x_\varepsilon)) u_\varepsilon(x)$. Direct computations show that

$$\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_\varepsilon|^2 dx \geq \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_{\varepsilon,1}|^2 dx + \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_{\varepsilon,2}|^2 dx + o(1),$$

$$\int_{\mathbb{R}^3} V(\varepsilon x) |u_\varepsilon|^2 dx \geq \int_{\mathbb{R}^3} V(\varepsilon x) |u_{\varepsilon,1}|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x) |u_{\varepsilon,2}|^2 dx,$$

$$\int_{\mathbb{R}^3} \phi_{u_\varepsilon}^t |u_\varepsilon|^2 dx \geq \int_{\mathbb{R}^3} \phi_{u_{\varepsilon,1}}^t |u_{\varepsilon,1}|^2 dx + \int_{\mathbb{R}^3} \phi_{u_{\varepsilon,2}}^t |u_{\varepsilon,2}|^2 dx,$$

$$\int_{\mathbb{R}^3}|u_\varepsilon|^s dx \geq \int_{\mathbb{R}^3}|u_{\varepsilon,1}|^s dx + \int_{\mathbb{R}^3}|u_{\varepsilon,2}|^s dx \quad \text{for all } s \in (2, 2_s^*],$$
$$Q_\varepsilon(u_{\varepsilon,1}) = 0, \quad Q_\varepsilon(u_{\varepsilon,2}) = Q_\varepsilon(u_\varepsilon) \geq 0.$$

Hence we get

$$J_\varepsilon(u_\varepsilon) \geq I_\varepsilon(u_{\varepsilon,1}) + I_\varepsilon(u_{\varepsilon,2}) + o(1). \tag{4.16}$$

Next, we claim that $\|u_{\varepsilon,2}\| \to 0$ as $\varepsilon \to 0$. By (4.4), we have

$$\begin{aligned}
\|u_{\varepsilon,2}\|_{H_\varepsilon} &\leq \left\|u_{\varepsilon,1} - \varphi(\varepsilon x - x_\varepsilon)U_0\left(x - \frac{x_\varepsilon}{\varepsilon}\right)\right\|_{H_\varepsilon} + 2d_0 \\
&= \left\|u_{\varepsilon,1} - \varphi(\varepsilon x - x_\varepsilon)U_0\left(x - \frac{x_\varepsilon}{\varepsilon}\right)\right\|_{H_\varepsilon(B_{\frac{2\beta}{\varepsilon}}(\frac{x_\varepsilon}{\varepsilon}))} + 2d_0 \\
&\leq \|u_{\varepsilon,2}\|_{H_\varepsilon(B_{\frac{2\beta}{\varepsilon}}(\frac{x_\varepsilon}{\varepsilon}))} + 4d_0 \\
&= \|u_{\varepsilon,2}\|_{H_\varepsilon(B_{\frac{2\beta}{\varepsilon}}(\frac{x_\varepsilon}{\varepsilon})\setminus B_{\frac{\beta}{\varepsilon}}(\frac{x_\varepsilon}{\varepsilon}))} + 4d_0 \\
&\leq C\|u_\varepsilon\|_{H_\varepsilon(B_{\frac{2\beta}{\varepsilon}}(\frac{x_\varepsilon}{\varepsilon})\setminus B_{\frac{\beta}{\varepsilon}}(\frac{x_\varepsilon}{\varepsilon}))} + 4d_0 \\
&\leq C\left\|\varphi(\varepsilon x - x_\varepsilon)U_0\left(x - \frac{x_\varepsilon}{\varepsilon}\right)\right\|_{H_\varepsilon(B_{\frac{2\beta}{\varepsilon}}(\frac{x_\varepsilon}{\varepsilon})\setminus B_{\frac{\beta}{\varepsilon}}(\frac{x_\varepsilon}{\varepsilon}))} + cd_0 \\
&\leq C\left\|U_0\left(x - \frac{x_\varepsilon}{\varepsilon}\right)\right\|_{H^s(B_{\frac{2\beta}{\varepsilon}}(\frac{x_\varepsilon}{\varepsilon})\setminus B_{\frac{\beta}{\varepsilon}}(\frac{x_\varepsilon}{\varepsilon}))} + cd_0 \\
&\leq C\|U_0\|_{H^s(B_{\frac{2\beta}{\varepsilon}}(\frac{x_\varepsilon}{\varepsilon})\setminus B_{\frac{\beta}{\varepsilon}}(\frac{x_\varepsilon}{\varepsilon}))} + Cd_0 \\
&= Cd_0 + o(1),
\end{aligned}$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Hence we have $\limsup_{\varepsilon\to 0}\|u_{\varepsilon,2}\|_{H_\varepsilon} \leq cd_0$.

Since $\langle J'_\varepsilon(u_\varepsilon), u_{\varepsilon,2}\rangle \to 0$ as $\varepsilon \to 0$ and $\langle Q'_\varepsilon(u_\varepsilon), u_{\varepsilon,2}\rangle = \langle Q'_\varepsilon(u_{\varepsilon,2}), u_{\varepsilon,2}\rangle \geq 0$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^3}(-\Delta)^{s/2}u_\varepsilon(-\Delta)^{s/2}u_{\varepsilon,2}dx + \int_{\mathbb{R}^3}V(\varepsilon x)u_\varepsilon u_{\varepsilon,2}dx + \int_{\mathbb{R}^3}\phi^t_{u_\varepsilon}u_\varepsilon u_{\varepsilon,2}dx \\
&\qquad + \langle Q'_\varepsilon(u_{\varepsilon,2}), u_{\varepsilon,2}\rangle \\
&\quad = \lambda\int_{\mathbb{R}^3}u_\varepsilon^{p-1}u_{\varepsilon,2}dx + \int_{\mathbb{R}^3}u_\varepsilon^{2_s^*-1}u_{\varepsilon,2}dx + o(1),
\end{aligned}$$

then we deduce from (4.15) and Sobolev's Imbedding Theorem that

$$\|u_{\varepsilon,2}\|^2_{H_\varepsilon} \leq C\|u_{\varepsilon,2}\|^{2_s^*}_{H_\varepsilon} + o(1).$$

Taking $d_0 > 0$ small, we have $\|u_{\varepsilon,2}\|_{H_\varepsilon} = o(1)$. From (4.16), it holds that

$$J_\varepsilon(u_\varepsilon) \geq I_\varepsilon(u_{\varepsilon,1}) + o(1). \tag{4.17}$$

Step 3. Let $\tilde{w}_\varepsilon(x) = u_{\varepsilon,1}(x + \frac{x_\varepsilon}{\varepsilon}) = \varphi(\varepsilon x)u_\varepsilon(x + \frac{x_\varepsilon}{\varepsilon})$, up to a subsequence, there exists a $\tilde{w} \in H^s(\mathbb{R}^3)$ such that

$$\tilde{w}_\varepsilon \rightharpoonup \tilde{w} \quad \text{in } H^s(\mathbb{R}^3) \quad \text{and} \quad \tilde{w}_\varepsilon(x) \to \tilde{w}(x) \quad \text{a.e. in } \mathbb{R}^3. \tag{4.18}$$

We claim that

$$\tilde{w}_\varepsilon \to \tilde{w} \quad \text{in } L^{2_s^*}(\mathbb{R}^3). \tag{4.19}$$

In view of Lemma 2.2, assuming to the contrary that there exists a $r > 0$ such that

$$\liminf_{\varepsilon\to 0} \sup_{z\in\mathbb{R}^3} \int_{B_1(z)} |\tilde{w}_\varepsilon - \tilde{w}|^{2_s^*} dx = 2r > 0.$$

Then, for $\varepsilon > 0$ small, there exists $z_\varepsilon \in \mathbb{R}^3$ such that

$$\int_{B_1(z_\varepsilon)} |\tilde{w}_\varepsilon - \tilde{w}|^{2_s^*} dx \geq r > 0. \tag{4.20}$$

Case 1. $\{z_\varepsilon\}$ is bounded, that is, $|z_\varepsilon| \leq \alpha$ for $\alpha > 0$. Then, for $\varepsilon > 0$ small,

$$\int_{B_1(z_\varepsilon)} |\tilde{v}_\varepsilon|^{2_s^*} dx \geq r > 0, \tag{4.21}$$

where $\tilde{v}_\varepsilon = \tilde{w}_\varepsilon - \tilde{w}$ and $\tilde{v}_\varepsilon \rightharpoonup 0$ in $H^s(\mathbb{R}^3)$. As the similar argument in Step 1, we can get that there exists $C > 0$ (independent of ε) such that for $\varepsilon > 0$ small,

$$\int_{B_{\alpha+1}(0)} |(-\Delta)^{\frac{s}{2}} \tilde{v}_\varepsilon|^2 dx \geq Cr^{\frac{2}{2_s^*}} > 0, \tag{4.22}$$

and

$$\lim_{\varepsilon\to 0} \sup_{\tilde{\varphi}\in C_0^\infty(B_{\alpha+2}(0)),\|\tilde{\varphi}\|=1} |\langle \tilde{\rho}_\varepsilon, \tilde{\varphi}\rangle| = 0, \tag{4.23}$$

where $\tilde{\rho}_\varepsilon = (-\Delta)^s \tilde{v}_\varepsilon - \tilde{v}_\varepsilon^{2_s^*-1} \in (H^s(\mathbb{R}^3))^{-1}$ and we have used the Brezis–Lieb splitting properties (Lemma 2.4).

In view of Lemma 2.5, we see from (4.21)–(4.23) that there exist $\tilde{z}_\varepsilon \in \mathbb{R}^3$ and $\delta_\varepsilon > 0$ with $\tilde{z}_\varepsilon \to \tilde{z} \in \overline{B_{\alpha+1}(0)}$, $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$ such that

$$\hat{w}_\varepsilon(x) := \delta_\varepsilon^{\frac{3-2s}{2}} \tilde{v}_\varepsilon(\delta_\varepsilon x + \tilde{z}_\varepsilon) \rightharpoonup \hat{w} \quad \text{in } \mathcal{D}^{s,2}(\mathbb{R}^3)$$

and $\hat{w} \geq 0$ is a nontrivial solution of (4.11) which satisfies (4.12).

Since

$$\begin{aligned}
\int_{\mathbb{R}^3} |\hat{w}|^{2_s^*} dx &\leq \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^3} |\hat{w}_\varepsilon|^{2_s^*} dx = \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^3} |\tilde{v}_\varepsilon|^{2_s^*} dx \\
&= \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^3} |\tilde{w}_\varepsilon|^{2_s^*} dx - \int_{\mathbb{R}^3} |\tilde{w}|^{2_s^*} dx \\
&\leq \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*} dx,
\end{aligned} \tag{4.24}$$

then by (4.4) and Sobolev's Imbedding Theorem, we get

$$\int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*} \leq Cd_0 + \int_{\mathbb{R}^3} \left|\varphi(\epsilon x - x_\epsilon) U_0\left(x - \frac{x_\varepsilon}{\varepsilon}\right)\right|^{2_s^*} \leq Cd_0 + \int_{\mathbb{R}^3} U_0^{2_s^*},$$

and combining with (4.24), it holds that

$$\int_{\mathbb{R}^3} |\hat{w}|^{2_s^*} \leq Cd_0 + \int_{\mathbb{R}^3} U_0^{2_s^*}. \tag{4.25}$$

Thus

$$\begin{aligned} c_{V_0} &= I_{V_0}(U_0) - \frac{1}{4s+2t-3} G_{V_0}(U_0) \\ &= \frac{s}{4s+2t-3} V_0 \int_{\mathbb{R}^3} v^2 dx + \frac{p(s+t)-4s-2t}{4s+2t-3} \frac{\lambda}{p} \int_{\mathbb{R}^3} v^p dx \\ &\quad + \frac{2_s^*(s+t)-4s-2t}{(4s+2t-3)2_s^*} \int_{\mathbb{R}^3} v^{2_s^*} dx, \\ &\geq \frac{2_s^*(s+t)-4s-2t}{(4s+2t-3)2_s^*} S^{\frac{3}{2s}} - Cd_0, \end{aligned}$$

where we have used (4.13) and (4.25). Since $\frac{2_s^*(4s+2t-3)}{2(2_s^*(s+t)-3)} = 1$, letting $d_0 \to 0$, we have

$$c_{V_0} \geq \frac{s}{3} S^{\frac{3}{2s}},$$

which contradicts Lemma 3.5.

Case 2. $\{z_\varepsilon\}$ is unbounded. Without loss of generality, we may assume that $\lim_{\varepsilon\to 0} |z_\varepsilon| = \infty$. Then, by (4.20)

$$\liminf_{\varepsilon\to 0} \int_{B_1(z_\varepsilon)} |\tilde{w}_\varepsilon|^{2_s^*} dx \geq r > 0, \tag{4.26}$$

that is

$$\liminf_{\varepsilon\to 0} \int_{B_1(z_\varepsilon)} \left|\varphi(\varepsilon x) u_\varepsilon\left(x + \frac{x_\varepsilon}{\varepsilon}\right)\right|^{2_s^*} dx \geq r > 0.$$

Since $\varphi(x) = 0$ for $|x| \geq 2\beta$, we see that $|z_\varepsilon| \leq 3\beta/\varepsilon$ for $\varepsilon > 0$ small. If $|z_\varepsilon| \geq \beta/2\varepsilon$, then $z_\varepsilon \in B_{\frac{3\beta}{\varepsilon}}(0)\backslash B_{\frac{\beta}{2\varepsilon}}(0)$ and by Step 1, we get

$$\begin{aligned} \liminf_{\varepsilon\to 0} \int_{B_1(z_\varepsilon)} |\tilde{w}_\varepsilon|^{2_s^*} dx &\leq \liminf_{\varepsilon\to 0} \sup_{z\in B_{\frac{3\beta}{\varepsilon}}(0)\backslash B_{\frac{\beta}{2\varepsilon}}(0)} \int_{B_1(z)} \left|u_\varepsilon\left(x + \frac{x_\varepsilon}{\varepsilon}\right)\right|^{2_s^*} dx \\ &= \lim_{\varepsilon\to 0} \sup_{z\in A_\varepsilon} \int_{B_1(z)} |u_\varepsilon|^{2_s^*} = 0, \end{aligned}$$

which contradicts (4.26). Thus $|z_\varepsilon| \leq \frac{\beta}{2\varepsilon}$ for $\varepsilon > 0$ small. Assume that $\varepsilon z_\varepsilon \to z_0 \in \overline{B_{\frac{\beta}{2}}(0)}$ and $\bar{w}_\varepsilon(x) := \tilde{w}_\varepsilon(x+z_\varepsilon) \rightharpoonup \bar{w}(x)$ in $H^s(\mathbb{R}^3)$. If $\bar{w} \neq 0$, then we can see that $\bar{w}$

satisfies

$$(-\Delta)^s\bar{w} + V(x_0+z_0)\bar{w} + \phi^t_{\bar{w}} = \lambda\bar{w}^{p-1} + \bar{w}^{2^*_s-1} \quad \text{in } H^s(\mathbb{R}^3),\ \ \bar{w} \geq 0.$$

Similarly as in Step 1, we get a contradiction if $d_0 > 0$ is small enough. Thus $\bar{w} \equiv 0$, that is, $\bar{w}_\varepsilon \rightharpoonup 0$ in $H^s(\mathbb{R}^3)$. By (4.26), we have

$$\liminf_{\varepsilon\to 0} \int_{B_1(0)} |\bar{w}_\varepsilon|^{2^*_s} dx \geq r > 0, \tag{4.27}$$

and similarly as in Step 1, we can check that there exists $C > 0$ (independent of ε) such that for $\varepsilon > 0$ small,

$$\int_{B_1(0)} |(-\Delta)^{\frac{s}{2}}\bar{w}_\varepsilon|^2 \geq C r^{\frac{2}{2^*_s}} > 0, \tag{4.28}$$

and

$$\lim_{\varepsilon\to 0} \sup_{\bar{\varphi}\in C_0^\infty(B_2(0)),\|\bar{\varphi}\|=1} |\langle \bar{\rho}_\varepsilon, \bar{\varphi}\rangle| = 0, \tag{4.29}$$

where $\bar{\rho}_\varepsilon = (-\Delta)^s\bar{w}_\varepsilon - \bar{w}_\varepsilon^{2^*_s-1} \in (H^s(\mathbb{R}^3))^{-1}$. By Lemma 2.5 again, we see from (4.27)–(4.29) that there exist $\tilde{x}_\varepsilon \in \mathbb{R}^3$ and $\xi_\varepsilon > 0$ with $\tilde{x}_\varepsilon \to \tilde{x} \in \overline{B_1(0)}$, $\xi_\varepsilon \to 0$ as $\varepsilon \to 0$ such that

$$w^*_\varepsilon(x) := \xi_\varepsilon^{\frac{3-2s}{2}} \bar{w}_\varepsilon(\xi_\varepsilon x + \tilde{x}_\varepsilon) \rightharpoonup w^* \quad \text{in } \mathcal{D}^{s,2}(\mathbb{R}^3)$$

and $w^* \geq 0$ is a nontrivial solution of (4.11) which satisfies (4.12). Therefore,

$$\lim_{\varepsilon\to 0} \sup_{z\in\mathbb{R}^3} \int_{B_1(z)} |\tilde{w}_\varepsilon - \tilde{w}|^{2^*_s} dx = 0.$$

By Lemma 2.2, (4.19) holds. Similar to (4.15), using the interpolation inequality for L^p norms, we have

$$\tilde{w}_\varepsilon \to \tilde{w} \quad \text{in } L^r(\mathbb{R}^3),\ \ r \in (2, 2^*_s]. \tag{4.30}$$

In view of (4.17) and recall that $\tilde{w}_\varepsilon(x) = u_{\varepsilon,1}(x + \frac{x_\varepsilon}{\varepsilon})$, we have

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\tilde{w}_\varepsilon|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x + x_\varepsilon)\tilde{w}_\varepsilon^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{\tilde{w}_\varepsilon}\tilde{w}_\varepsilon^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^3} |\tilde{w}_\varepsilon|^p dx \\ &\quad - \frac{1}{2^*_s}\int_{\mathbb{R}^3} |\tilde{w}_\varepsilon|^{2^*_s} dx \\ &\leq c_{V_0} + o(1). \end{aligned}$$

By Lemma 2.3, (4.18) and (4.30), we get

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\tilde{w}|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x_0)\tilde{w}^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{\tilde{w}}\tilde{w}^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^3} |\tilde{w}|^p dx \\ &\quad - \frac{1}{2^*_s}\int_{\mathbb{R}^3} |\tilde{w}|^{2^*_s} dx \\ &\leq c_{V_0} + o(1). \end{aligned}$$

Then we get that

$$I_{V(x_0)}(\tilde{w}) \leq c_{V_0}. \tag{4.31}$$

Since $\langle J'_\varepsilon(u_\varepsilon), u_{\varepsilon,1}\rangle \to 0$, $\|u_{\varepsilon,2}\|_{H_\varepsilon} \to 0$ as $\varepsilon \to 0$ and $\langle Q'_\varepsilon(u_\varepsilon), u_{\varepsilon,1}\rangle \equiv 0$ and together with the fact that $\tilde{w}_\varepsilon(x) = u_{\varepsilon,1}(x + \frac{x_\varepsilon}{\varepsilon})$, we get

$$\begin{aligned} &\int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{s}{2}}\tilde{w}_\varepsilon|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x + x_\varepsilon)\tilde{w}_\varepsilon^2\right) dx + \int_{\mathbb{R}^3} \phi^t_{\tilde{w}_\varepsilon} \tilde{w}^2 dx \\ &\quad = \lambda \int_{\mathbb{R}^3} |\tilde{w}_\varepsilon|^p dx + \int_{\mathbb{R}^3} |\tilde{w}_\varepsilon|^{2_s^*} dx + o(1). \end{aligned}$$

Thus

$$\begin{aligned} &\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\tilde{w}|^2 dx + \int_{\mathbb{R}^3} V(x_0)\tilde{w}^2 dx + \int_{\mathbb{R}^3} \phi^t_{\tilde{w}}\tilde{w}^2 dx \\ &\quad \leq \lim_{\varepsilon \to 0} \left\{ \int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{s}{2}}\tilde{w}_\varepsilon|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x + x_\varepsilon)\tilde{w}_\varepsilon^2\right) dx + \int_{\mathbb{R}^3} \phi^t_{\tilde{w}_\varepsilon}\tilde{w}_\varepsilon^2 dx \right\} \\ &\quad = \lim_{\varepsilon \to 0} \left\{ \lambda \int_{\mathbb{R}^3} |\tilde{w}_\varepsilon|^p dx + \int_{\mathbb{R}^3} |\tilde{w}_\varepsilon|^{2_s^*} dx \right\} \\ &\quad = \lambda \int_{\mathbb{R}^3} |\tilde{w}|^p dx - \int_{\mathbb{R}^3} |\tilde{w}|^{2_s^*} dx \\ &\quad = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\tilde{w}|^2 dx + \int_{\mathbb{R}^3} V(x_0)\tilde{w}^2 dx + \int_{\mathbb{R}^3} \phi^t_{\tilde{w}}\tilde{w}^2 dx, \end{aligned}$$

hence, as $\varepsilon \to 0$,

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\tilde{w}_\varepsilon|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x + x_\varepsilon)\tilde{w}_\varepsilon^2 dx \to \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\tilde{w}|^2 dx + \int_{\mathbb{R}^3} V(x_0)\tilde{w}^2 dx. \tag{4.32}$$

In view of (4.4), (4.30) and the fact that$\|u_{\varepsilon,2}\|_{H_\varepsilon} \to 0$ as $\varepsilon \to 0$, taking $d_0 > 0$ small, we can check that $\tilde{w} \neq 0$. Thus, we have

$$I_{V(x_0)}(\tilde{w}) \geq c_{V_0}. \tag{4.33}$$

Since $x_0 \in \mathcal{M}^\beta \subset \Lambda$, (4.31) and (4.33) imply that $V(x_0) = V_0$ and $x_0 \in \mathcal{M}$. At this point, it is clear that there exists a $U \in \Omega_{V_0}$ and $z_0 \in \mathbb{R}^3$ such that $\tilde{w} = U(x - z_0)$. Since

$$\int_{\mathbb{R}^3} V(x_0)\tilde{w}_\varepsilon^2 dx \leq \int_{\mathbb{R}^3} V(\varepsilon x + x_\varepsilon)\tilde{w}_\varepsilon^2 dx,$$

by (4.32), we have

$$\tilde{w}_\varepsilon \to \tilde{w} \quad \text{in } H^s(\mathbb{R}^3),$$

which implies that

$$\left\| u_\varepsilon - \varphi(\varepsilon x - (x_\varepsilon + \varepsilon z_0))U\left(x - \left(\frac{x_\varepsilon}{\varepsilon} + z_0\right)\right)\right\|_{H_\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0.$$

And we recall that $x_\varepsilon \to x_0 \in \mathcal{M}$ as $\varepsilon \to 0$, this completes the proof. ◻

Lemma 4.3. *Let d_0 be the number given in Lemma 4.2, then for any $d \in (0, d_0)$, there exist $\varepsilon_d > 0, \rho_d > 0$ and $\omega_d > 0$ such that*

$$\|J'_\varepsilon(u)\|_{*,\varepsilon,R} \geq \omega_d > 0$$

for all $u \in J_\varepsilon^{c_{V_0}+\rho_d} \cap (X_\varepsilon^{d_0} \backslash X_\varepsilon^d) \cap H_0^s(B_R(0))$ with $\varepsilon \in (0, \varepsilon_d)$ and $R \geq R_0/\varepsilon$.

Proof. If the conclusion does not hold, there exist $d \in (0, d_0), \{\varepsilon_i\}, \{\rho_i\}$ with $\varepsilon_i, \rho_i \to 0$, $R_{\varepsilon_i} \geq R_0/\varepsilon_i$ and $u_i \in J_{\varepsilon_i}^{c_{V_0}+\rho_i} \cap (X_{\varepsilon_i}^{d_0} \backslash X_{\varepsilon_i}^d) \cap H_0^s(B_{R_{\varepsilon_i}}(0))$ such that

$$\|J'_{\varepsilon_i}(u_i)\|_{*,\varepsilon_i,R_{\varepsilon_i}} \to 0 \quad \text{as } i \to \infty.$$

By Lemma 4.2(i), we can find $\{y_i\} \subset \mathbb{R}^3, x_0 \in \mathcal{M}, U \in \Omega_{V_0}$ such that

$$\|u_i - \varphi(\varepsilon_i x - \varepsilon_i y_i)U(x - y_i)\|_{H_{\varepsilon_i}} = 0 \quad \text{and} \quad \lim_{i\to\infty} |\varepsilon_i y_i - x_0| = 0,$$

which implies that $u_i \in X_{\varepsilon_i}^d$ for sufficiently large i. This contradicts $u_i \notin X_{\varepsilon_i}^d$. □

Lemma 4.4 ([22]). *There exists $T_0 > 0$ with the following property: for any $\delta > 0$ small, there exist $\alpha_\delta > 0$ and $\varepsilon_\delta > 0$ such that if $J_\varepsilon(\gamma_\varepsilon(h)) \geq c_{V_0} - \alpha_\delta$ and $\varepsilon \in (0, \varepsilon_\delta)$, then $\gamma_\varepsilon \in X_\varepsilon^{T_0\delta}$, where $\gamma_\varepsilon(h) := W_{\varepsilon,h\theta_0}, h \in [0, 1]$.*

For each $R > R_0/\varepsilon$, we see that $\gamma_\varepsilon(h) := W_{\varepsilon,h\theta_0} \in H_0^s(B_R(0))$ for each $h \in [0, 1]$, $X_\varepsilon \subset H_0^s(B_R(0))$. Define

$$c_{\varepsilon,R} := \inf_{\gamma\in\Gamma_{\varepsilon,R}} \max_{0\leq h\leq 1} J_\varepsilon(\gamma(h)),$$

where

$$\Gamma_{\varepsilon,R} := \{\gamma \in C([0, 1], H_0^s(B_R(0))) : \gamma(0) = 0, \gamma(1) = \gamma_\varepsilon(1) = W_{\varepsilon,\theta_0}\}.$$

Remark that $\gamma_\varepsilon(h) := W_{\varepsilon,h\theta_0} \in \Gamma_{\varepsilon,R}$, $c_\varepsilon \leq c_{\varepsilon,R} \leq \tilde{c}_\varepsilon$ and $J_\varepsilon^{\tilde{c}_\varepsilon} \cap X_\varepsilon \cap H_0^s(B_R(0)) \neq \emptyset$.

Choosing $\delta_1 > 0$ such that $T_0\delta_1 < d_0/4$ in Lemma 4.3 and fixing $d = d_0/4 := d_1$ in Lemma 4.2. The next lemma comes from [22] and details are omitted here.

Lemma 4.5. *There exists $\bar{\varepsilon} > 0$ such that for each $\varepsilon \in (0, \bar{\varepsilon}]$ and $R > R_0/\varepsilon$, there exists a sequence $\{v_{n,\varepsilon}^R\} \subset J_\varepsilon^{\tilde{c}_\varepsilon+\varepsilon} \cap X_\varepsilon^{d_0} \cap H_0^s(B_R(0))$ such that $J'_\varepsilon(v_{n,\varepsilon}^R) \to 0$ in $(H_0^s(B_R(0)))^{-1}$ as $n \to \infty$.*

Proof of Theorem 1.1. Step 1. By Lemma 4.5, there exists a $\bar{\varepsilon} > 0$ such that for each $\varepsilon \in (0, \bar{\varepsilon}]$ and $R > \frac{R_0}{\varepsilon}$, there exists a sequence $\{v_{n,\varepsilon}^R\} \subset J_\varepsilon^{\tilde{c}_\varepsilon+\varepsilon} \cap X_\varepsilon^{d_0} \cap H_0^s(B_R(0))$ such that $J'_\varepsilon(v_{n,\varepsilon}^R) \to 0$ in $(H_0^s(B_R(0)))^{-1}$ as $n \to \infty$.

Since $\{v_{n,\varepsilon}^R\}$ is bounded in $H_0^s(B_R(0))$, up to a subsequence, as $n \to \infty$, we have

$$v_{n,\varepsilon}^R \rightharpoonup v_\varepsilon^R \quad \text{in } H_0^s(B_R(0)),$$

$$v_{n,\varepsilon}^R \to v_\varepsilon^R \quad \text{in } L^r(B_R(0)), \quad 2 \leq r < 2_s^*,$$

$$v_{n,\varepsilon}^R(x) \to v_\varepsilon^R(x) \quad \text{a.e. in } B_R(0).$$

By similar arguments in the proof of Lemma 2.5, we can obtain that

$$v_{n,\varepsilon}^R \to v_\varepsilon^R \quad \text{in } L^{2_s^*}(B_R(0)) \quad \text{as } n \to \infty.$$

Then using the standard arguments, we can check that

$$v_{n,\varepsilon}^R \to v_\varepsilon^R \quad \text{in } H_0^s(B_R(0)),$$

then $v_\varepsilon^R \geq 0$ and satisfies

$$\begin{cases} (-\Delta)^s v_\varepsilon^R + V(\varepsilon x) v_\varepsilon^R + \phi_{v_\varepsilon^R}^t v_\varepsilon^R + 4\left(\int_{\mathbb{R}^3} \chi_\varepsilon (v_\varepsilon^R)^2 dx - 1\right)_+ \chi_\varepsilon v_\varepsilon^R \\ \qquad = \lambda (v_\varepsilon^R)^{p-1} + (v_\varepsilon^R)^{2_s^*-1} \quad \text{in } B_R(0), \\ v_\varepsilon^R = 0 \quad \text{on } \mathbb{R}^3 \backslash B_R(0) \end{cases} \tag{4.34}$$

and we can easily check that $v_\varepsilon^R \in J_\varepsilon^{\tilde{c_\varepsilon}+\varepsilon} \cap X_\varepsilon^{d_0}$ for $d_0 > 0$ small.

Step 2. We claim that there exists a $\bar{\varepsilon} > 0$ such that for each $\varepsilon \in (0, \bar{\varepsilon}]$ and $R > \frac{R_0}{\varepsilon}$,

$$\|v_\varepsilon^R\|_{L^\infty(\mathbb{R}^3)} \leq C. \tag{4.35}$$

Otherwise, there exist $\varepsilon_j \to 0$, $R_j > \frac{R_0}{\varepsilon_j}$ such that $\|v_{\varepsilon_j}^{R_j}\|_{L^\infty(\mathbb{R}^3)} \to \infty$ as $j \to \infty$. By Lemma 4.2(i), there exist, up to a subsequence, $\{y_j\} \subset \mathbb{R}^3$, $x_0 \in \mathcal{M}$, $U \in \Omega_{V_0}$ such that

$$\lim_{j\to\infty} |\varepsilon_j y_j - x_0| = 0 \quad \text{and} \quad \lim_{j\to\infty} \|v_{\varepsilon_j}^{R_j}(x) - \varphi(\varepsilon_j x - \varepsilon_j y_j) U(x - y_j)\|_{H_{\varepsilon_j}} = 0,$$

then by Sobolev's inequality,

$$\lim_{j\to\infty} \|v_{\varepsilon_j}^{R_j}(x + y_j) - \varphi(\varepsilon_j x) U(x)\|_{L^{2_s^*}(\mathbb{R}^3)} = 0,$$

which implies that as $j \to \infty$,

$$v_{\varepsilon_j}^{R_j}(x + y_j) \to U(x) \quad \text{in } L^{2_s^*}(\mathbb{R}^3).$$

Using Moser's iteration method [42] (see also [27 Lemma 4.1; 20 Proposition 5.1.1]), we have

$$\|v_{\varepsilon_j}^{R_j}(x + y_j)\|_{L^\infty(\mathbb{R}^3)} \leq C,$$

which leads to a contradiction.

Step 3. Next, we claim that $v_\varepsilon^R \to v_\varepsilon \in H_\varepsilon \cap J_\varepsilon^{\tilde{c_\varepsilon}+\varepsilon} \cap X_\varepsilon^{d_0}$ as $R \to \infty$ in H_ε sense for $\varepsilon > 0$ small but fixed.

In fact, since $\{v_\varepsilon^R\}$ is bounded in H_ε, we can assume that as $R \to \infty$,

$$\begin{aligned} v_\varepsilon^R &\rightharpoonup v_\varepsilon \quad \text{in } H_\varepsilon, \\ v_\varepsilon^R &\to v_\varepsilon \quad \text{in } L_{\text{loc}}^r(\mathbb{R}^3), \quad 2 \leq r < 2_s^*, \\ v_\varepsilon^R(x) &\to v_\varepsilon(x) \quad \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

By Lemma 3.7, (4.35) and Sobolev's Imbedding Theorem, we get

$$v_\varepsilon^R \to v_\varepsilon \quad \text{in } L^r(\mathbb{R}^3), \quad 2 \leq r \leq 2_s^* \quad \text{as } R \to \infty.$$

Now, using standard arguments, we can prove the claim.

Hence, $v_\varepsilon \in H_\varepsilon \cap J_\varepsilon^{\tilde{c_\varepsilon}+\varepsilon} \cap X_\varepsilon^{d_0}$ is a nontrivial solution of

$$(-\Delta)^s u + V(\varepsilon x)u + \phi_u^t u + 4\left(\int_{\mathbb{R}^3} \chi_\varepsilon u^2 dx - 1\right)_+ \chi_\varepsilon u = \lambda u^{p-1} + u^{2_s^*-1} \quad \text{in } \mathbb{R}^3.$$

Since Ω_{V_0} is compact in $H^s(\mathbb{R}^3)$, it is easy to see that $0 \notin X_\varepsilon^{d_0}$ for $d_0 >$ small. Thus $v_\varepsilon \neq 0$.

Step 4. For any sequence $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$. By Lemma 4.2(ii), there exist, up to a subsequence, $\{y_j\} \subset \mathbb{R}^3$, $x_0 \in \mathcal{M}$, $U \in \Omega_{V_0}$ such that

$$\lim_{j\to\infty} |\varepsilon_j y_j - x_0| = 0 \quad \text{and} \quad \lim_{j\to\infty} \|v_{\varepsilon_j}(x) - \varphi(\varepsilon_j x - \varepsilon_j y_j)U(x - y_j)\|_{H_{\varepsilon_j}} = 0, \tag{4.36}$$

which implies that as $j \to \infty$,

$$w_{\varepsilon_j}(x) := v_{\varepsilon_j}(x + y_j) \to U(x) \quad \text{in } L^{2_s^*}(\mathbb{R}^3).$$

Since w_{ε_j} satisfies the equation

$$(-\Delta)^s w_{\varepsilon_j} + w_{\varepsilon_j} = \Upsilon_{\varepsilon_j}, \quad x \in \mathbb{R}^3,$$

where

$$\Upsilon_{\varepsilon_j}(x) = w_{\varepsilon_j}(x) - V(\varepsilon_n(x + y_n))w_{\varepsilon_j}(x) - \phi_{w_{\varepsilon_j}}^t w_{\varepsilon_j}(x)$$
$$+ \lambda|w_{\varepsilon_j}|^{p-1} + |w_{\varepsilon_j}|^{2_s^*-1}, \quad x \in \mathbb{R}^3.$$

Putting $\Upsilon(x) = w(x) - V(x_0)w(x) - \phi_w^t w(x) + \lambda|w(x)|^{p-1} + |w(x)|^{2_s^*-1}$, by (4.35), we see that

$$\Upsilon_{\varepsilon_j} \to \Upsilon \quad \text{in } L^q(\mathbb{R}^3), \quad \forall\, q \in [2, +\infty),$$

and there exists a $C > 0$ such that

$$\|\Upsilon_{\varepsilon_j}\|_\infty \leq C.$$

From [21], we have that

$$w_{\varepsilon_j}(x) = \mathcal{G} * \Upsilon_{\varepsilon_j} = \int_{\mathbb{R}^3} \mathcal{G}(x - y)\Upsilon_{\varepsilon_j}(y)dy,$$

where $\mathcal{G}$ is the Bessel Kernel

$$\mathcal{G}(x) = \mathcal{F}^{-1}\left(\frac{1}{1 + |\xi|^{2s}}\right).$$

It is known from [21, Theorem 3.3] that $\mathcal{G}$ is positive, radially symmetric and smooth in $\mathbb{R}^3\backslash\{0\}$; there is $C > 0$ such that $\mathcal{G}(x) \leq \frac{C}{|x|^{3+2s}}$, and $\mathcal{G} \in L^q(\mathbb{R}^3), \forall\, q \in [1, \frac{3}{3-2s})$. Now argue as in the proof of [2, Lemma 2.6], we conclude that

$$w_{\varepsilon_j}(x) \to 0 \quad \text{as } |x| \to \infty, \tag{4.37}$$

uniformly in $\varepsilon_j \in \mathbb{N}$.

Step 5. According to [27], we get

$$w_{\varepsilon_j}(x) \leq \frac{C}{1 + |x|^{3+2s}}.$$

Thus

$$\varepsilon_j^{-1}\int_{\mathbb{R}^3\setminus\frac{\Lambda}{\varepsilon_j}} v_{\varepsilon_j}^2(x)dx=\varepsilon_j^{-1}\int_{\mathbb{R}^3\setminus\frac{\Lambda}{\varepsilon_j-y_j}} w_{\varepsilon_j}^2(x)dx\le\varepsilon_j^{-1}\int_{\mathbb{R}^3\setminus B_{\frac{\beta}{\varepsilon_j}}(0)} w_{\varepsilon_j}^2(x)dx\to 0,$$

that is $Q_{\varepsilon_j}(v_{\varepsilon_j})=0$ for ε_j small. Therefore, v_{ε_j} is a solution of (4.1). Set $u_\varepsilon(x)=v_\varepsilon(\frac{x}{\varepsilon})$, u_{ε_j} is a solution of (1.1).

Let P_j be a maximum point of w_{ε_j}, similar to the arguments in Theorem 3.1, we can check that there is $b>0$ such that $w_{\varepsilon_j}(P_j)>b$, by (4.37), $\{P_j\}$ must be bounded.

Since $u_{\varepsilon_j}=w_{\varepsilon_j}(\frac{x}{\varepsilon_j}-y_j)$, $x_j:=\varepsilon_jP_j+\varepsilon_jy_j$ is a maximum point of u_{ε_j}. From (4.36), $x_j\to x_0\in\mathcal{M}$ as $j\to\infty$. Since the sequence $\{\varepsilon_j\}$ is arbitrary, we have obtained the existence and concentration results in Theorem 1.1. Moreover, we have

$$\begin{aligned}
u_{\varepsilon_j}(x)&=w_{\varepsilon_j}\left(\frac{x}{\varepsilon_j}-y_j\right)\\
&\le\frac{C}{1+\left|\dfrac{x}{\varepsilon_j}-y_j\right|^{3+2s}}\\
&=\frac{C\varepsilon_j^{3+2s}}{\varepsilon_j^{3+2s}+|x-\varepsilon_jy_j|^{3+2s}}\\
&=\frac{C\varepsilon_j^{3+2s}}{\varepsilon_j^{3+2s}+|x-x_{\varepsilon_j}|^{3+2s}},\quad\forall x\in\mathbb{R}^3.
\end{aligned}$$

Thus, the proof of Theorem 1.1 is completed. □

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