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ORIGINAL PAPER

Multiplicity and concentration of nontrivial nonnegative solutions for a fractional Choquard equation with critical exponent

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Abstract

In present paper, we study the fractional Choquard equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = \varepsilon^{\mu-N}\left(\frac{1}{|x|^{\mu}} * F(u)\right) f(u) + |u|^{2_s^*-2}u$$

where $\varepsilon > 0$ is a parameter, $s \in (0,1)$, $N > 2s$, $2_s^* = \frac{2N}{N-2s}$ and $0 < \mu < \min\{2s, N-2s\}$. Under suitable assumption on V and f, we prove this problem has a nontrivial nonnegative ground state solution. Moreover, we relate the number of nontrivial nonnegative solutions with the topology of the set where the potential attains its minimum values and their's concentration behavior.

Keywords Fractional Choquard equation · Ground state · Lusternik–Schnirelmann theory

Mathematics Subject Classification 35P15 · 35P30 · 35R11

1 Introduction and the main results

In this paper, we are interested in the existence, multiplicity and concentration behavior of the semi-classical solutions of the fractional Choquard equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = \varepsilon^{\mu-N}\left(\frac{1}{|x|^{\mu}} * F(u)\right) f(u) + |u|^{2_s^*-2}u, \quad x \in \mathbb{R}^N \tag{1.1}$$

where $\varepsilon > 0$ is a parameter, $s \in (0,1)$, $N > 2s$, $2_s^* = \frac{2N}{N-2s}$, $0 < \mu < \min\{2s, N-2s\}$ and $F(u) = \int_0^u f(\tau)d\tau$. The fractional Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s\Psi(x) = C_{N,s}P.V.\int_{\mathbb{R}^N}\frac{\Psi(x)-\Psi(y)}{|x-y|^{N+2s}}dy, \quad \Psi \in \mathcal{S}(\mathbb{R}^N),$$

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where $P.V.$ stands for the Cauchy principal value, $C_{N,s}$ is a normalized constant, $\mathcal{S}(\mathbb{R}^N)$ is the Schwartz space of rapidly decaying functions, $s \in (0,1)$. As ε goes to zero in (1.1), the existence and asymptotic behavior of the solutions of the singularly perturbed equation (1.1) is known as the semi-classical problem. It was used to describe the transition between of Quantum Mechanics and Classical Mechanics.

Our motivation to study (1.1) mainly comes from the fact that solutions $u(x)$ of (1.1) corresponding to standing wave solutions $\Psi(x,t) = e^{-iEt/\varepsilon}u(x)$ of the following time-dependent fractional Schrödinger equation

$$i\varepsilon\frac{\partial\Psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s\Psi + (V(x)+E)\Psi - (K(x)*|G(\Psi)|)g(\Psi) \ \ (x,t) \in \mathbb{R}^N \times \mathbb{R} \tag{1.2}$$

where i is the imaginary unit, ε is related to the Planck constant. Equations of the type (1.2) was introduced by Laskin (see [22,23]) and come from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. With variational methods, this kind equation has been studied widely, we refer to [11,17,45] and the references therein.

When $s = 1$, the Eq. (1.1) turns out to be the Choquard equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{\mu-N}\left(\frac{1}{|x|^\mu} * F(u)\right) f(u) + |u|^{2^*-2}u \ \text{ in } \mathbb{R}^N. \tag{1.3}$$

The existence, multiplicity and concentration of solutions for (1.3) has been widely investigated. On one hand, some people have studied the classical problem, namely $\varepsilon = 1$ in (1.3). When $V = 1$ and $F(u) = \frac{|u|^q}{q}$, (1.3) covers in particular the Choquard–Pekar equation

$$-\Delta u + u = \left(\int_{\mathbb{R}^N} \frac{1}{|x|^\mu} * |u|^q dy\right) |u|^{q-2}u \ \text{ in } \mathbb{R}^N. \tag{1.4}$$

The case $N = 3$, $q = 2$ and $\mu = 1$ came from Pekar [36] in 1954 to describe the quantum mechanics of a polaron at rest. In 1976 Choquard used (1.4) to describe an electron trapped in its own hole, in a certain approximation to Hartree–Fock theory of one component plasma [24]. In this context (1.4) is also known as the nonlinear Schrödinger-Newton equation. By using critical point theory, Lions [26] obtained the existence of infinitely many radially symmetric solutions in $H^1(\mathbb{R}^N)$ and Ackermann [1] prove the existence of infinitely many geometrically distinct weak solutions for a general case. For the properties of the ground state solutions, Ma and Zhao [27] proved that every positive solution is radially symmetric and monotone decreasing about some point for the generalized Choquard equation (1.4) with $q \geq 2$. Later, Moroz and Van Schaftingen [29,30] eliminated this restriction and showed the regularity, positivity and radial symmetry of the ground states for the optimal range of parameters, and also derived that these solutions decay asymptotically at infinity.

On the other hand, some people have focused on the semiclassical problem, namely, $\varepsilon \to 0$ in (1.3). The question of the existence of semiclassical solutions for the non-local problem (1.3) has been posed in [6]. Note that if v is a solution of (1.3) for $x_0 \in \mathbb{R}^N$, then $u = v(\varepsilon x + x_0)$ verifies

$$-\Delta u + V(\varepsilon x + x_0)u = \left(\int_{\mathbb{R}^N} \frac{G(u(y))}{|x-y|^\mu} dy\right) g(u) \ \text{ in } \mathbb{R}^N, \tag{1.5}$$

which means some convergence of the family of solutions to a solution u_0 of the limit problem

$$-\Delta u + V(x_0)u = \left(\int_{\mathbb{R}^N} \frac{G(u(y))}{|x-y|^\mu} dy\right) g(u) \ \text{ in } \mathbb{R}^N. \tag{1.6}$$

For this case when $N = 3, \mu = 1$ and $G(u) = |u|^2$, Wei and Winter [43] constructed families of solutions by a Lyapunov–Schmidt-type reduction when $\inf V > 0$. This method of construction depends on the existence, uniqueness and non-degeneracy up to translations of the positive solution of the limiting equation (1.6), which is a difficult problem that has only been fully solved in the case when $N = 3, \mu = 1$ and $G(u) = |u|^2$. Moroz and Van Schaftingen [31] used variational methods to develop a novel non-local penalization technique to show that equation (1.3) with $G(u) = |u|^q$ has a family of solutions concentrated at the local minimum of V, with V satisfying some additional assumptions at infinity. In addition, Alves and Yang [5] investigated the multiplicity and concentration behaviour of solutions for a quasi-linear Choquard equation via the penalization method. Very recently, in an interesting paper, Alves et al. [3] study (1.4) with a critical growth, they consider the critical problem with both linear potential and nonlinear potential, and showed the existence, multiplicity and concentration behavior of solutions when the linear potential has a global minimum or maximum.

On the contrary, the results about fractional Choquard equation (1.1) are relatively few. Recently, d'Avenia, Siciliano and Squassina [15] studied the existence, regularity and asymptotic of the solutions for the following fractional Choquard equation

$$(-\Delta)^s u + \omega u = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^q}{|x-y|^\mu} dy \right) |u|^{q-2} u \ \text{ in } \mathbb{R}^N, \tag{1.7}$$

where $\omega > 0$, $\frac{2N-\mu}{N} < q < \frac{2N-\mu}{N-2s}$. Shen, Gao and Yang [39] obtained the existence of ground states for (1.7) with general nonlinearities by using variational methods. Chen and Liu [13] studied (1.7) with nonconstant linear potential and proved the existence of ground states without any symmetry property. For critical problem, Wang and Xiang [41] obtain the existence of infinitely many nontrivial solutions and the Brezis–Nirenberg type results can be founded in [34]. For other existence results we refer to [8,9,19,20,28,42,48] and the references therein.

For the concentration behavior of solutions, we note that the only works concerning the concentration behavior of solutions come from [44,46]. Assuming the global condition on $V \in C(\mathbb{R}^N, \mathbb{R})$:

(V_0) $0 < V_0 := \inf\limits_{x \in \mathbb{R}^N} V(x) < \liminf\limits_{|x| \to +\infty} V(x) := V_\infty < +\infty$,

which is firstly introduced by Rabinowitz [37] in the study of the nonlinear Schrödinger equations. By using the method of Nehari manifold developed by Szulkin and Weth [40], Zhang, Wang and Zhang in [46] obtained the multiplicity and concentration of positive solutions for the following fractional Choquard equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x) u = \varepsilon^{\mu-3} \left(\int_{\mathbb{R}^3} \frac{|u(y)|^{2^*_{\mu,s}} + F(u(y))}{|x-y|^\mu} dy \right) \left(|u|^{2^*_{\mu,s}-2} u + \frac{1}{2^*_{\mu,s}} f(u) \right) \ \text{in } \mathbb{R}^3, \tag{1.8}$$

where $\varepsilon > 0$, $0 < \mu < 3$, F is the primitive function of f. Different to the global condition (V_0), Yang in [44] establish the existence and concentration of positive solutions for the fractional Choquard equation (1.8) when the potential function $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies the following local conditions [16]:

(V_1) There is a constant $V_0 > 0$ such that $V_0 = \inf_{x \in \mathbb{R}^3} V(x)$.
(V_2) There is a bounded domain Ω such that

$$V_0 < \min_{\partial \Omega} V.$$

Note that in (1.8), the critical term is involved in the convolution-type nonlinearity, which is totally different from our problem (1.1). It is natural to ask how about the concentration behavior of solutions of (1.1) as $\varepsilon \to 0^+$? And how about the influence of the potential on the multiplicity of solutions? However, to the best of our knowledge, it seems that these two problems were not considered in literatures before. In this paper, we are concerned with the multiplicity and concentration property of nontrivial nonnegative solutions to (1.1), and we will give some answers to the above questions.

Concerning the continuous function $f \in C(\mathbb{R}, \mathbb{R})$, we assume that $f(t) = 0$ for $t < 0$ and satisfies the following conditions:

(f_1) $\lim_{t\to 0} \frac{f(t)}{t} = 0$;
(f_2) there exists $q \in (\frac{2N-\mu}{N}, \frac{2N-\mu}{N-2s})$ such that $\lim_{t\to\infty} \frac{f(t)}{t^{q-1}} = 0$;
(f_3) $\frac{f(t)}{t}$ is increasing for every $t > 0$;
(f_4) there exists $\sigma \in (q_N, \frac{2N-\mu}{N-2s})$, $c > 0$ such that $f(t) \geq ct^{\sigma-1}$ for all $t \in \mathbb{R}^+$, where $q_N = \max\{\frac{2N-2s}{N-2s}, \frac{N+2s}{N-2s}\}$.

Then we state our main result as follows.

Theorem 1.1 *Suppose* (V_0) *hold and* f *satisfies* (f_1)–(f_4). *Then there exists an* $\varepsilon^* > 0$ *such that for any* $\varepsilon \in (0, \varepsilon^*)$, *the problem* (1.1) *possesses a nontrivial nonnegative ground state solution.*

In order to describe the multiplicity, we first recall that, if Y is a closed subset of a topological space X, the Ljusternik–Schnirelmann category $cat_X Y$ is the least number of closed and contractible sets in X which cover Y. Then we have our second result as follows.

Theorem 1.2 *Suppose* (V_0) *hold and* f *satisfies* (f_1)–(f_4). *Then for any* $\delta > 0$, *there exists* $\varepsilon_\delta > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_\delta)$, *the problem* (1.1) *has at least* $cat_{\Lambda_\delta}(\Lambda)$ *nontrivial nonnegative solutions. Moreover, if* u_ε *denotes one of these solutions and* $x_\varepsilon \in \mathbb{R}^N$ *is its global maximum, then*

$$\lim_{\varepsilon\to 0} V(x_\varepsilon) = V_0,$$

where $\Lambda := \{x \in \mathbb{R}^N : V(x) = V_0\}$ *and* $\Lambda_\delta := \{x \in \mathbb{R}^N : d(x, \Lambda) \leq \delta\}$.

We shall use the method of Nehari manifold, concentration compactness principle and category theory to prove the main results. There are some difficulties in proving our theorems. The first difficulty is that the nonlinearity f is only continuous, we need to prove the new Brezis–Lieb type Lemma for this kind of nonlinearity. The second one is the lack of compactness of the embedding of $H^s(\mathbb{R}^N)$ into the space $L^{2_s^*}(\mathbb{R}^N)$. We shall borrow the idea in [3,12] to deal with the difficulties brought by the critical exponent. However, we require some new estimates, which are complicated because of the appearance of fractional Laplacian and the convolution-type nonlinearity.

This paper is organized as follows. In Sect. 2, besides describing the functional setting to study problem (1.1), we give some preliminary Lemmas which will be used later. In Sect. 3, we prove problem (1.1) has a ground state solution. Finally, we show the multiple of nontrivial nonnegative solutions and investigate its concentration behavior, which completes the proof Theorem 1.2.

Notation In this paper we make use of the following notations.

- For any $R > 0$ and for any $x \in \mathbb{R}^N$, $B_R(x)$ denotes the ball of radius R centered at x.

- $L^p(\mathbb{R}^N)$, $1 \le p < +\infty$ denotes the Lebesgue space with the norm $\|u\|_p = \|u\|_{L^p(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} |u|^p dx)^{\frac{1}{p}}$.
- The letters C, C_i stand for positive constants (possibly different from line to line).
- "$\rightarrow$" for the strong convergence and "$\rightharpoonup$" for the weak convergence.
- $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$ denote the positive part and the negative part of a function u, respectively.

2 Functional setting

Firstly, fractional Sobolev spaces are the convenient setting for our problem, so we will give some sketches of the fractional order Sobolev spaces and the complete introduction can be found in [17]. We recall that, for any $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$ is defined as follows:

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2\right) d\xi < \infty \right\},$$

whose norm is defined as

$$\|u\|^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \left(|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2\right) d\xi,$$

where $\mathcal{F}$ denotes the Fourier transform. We also define the homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as the completion of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} := \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy \right)^{\frac{1}{2}} = [u]_{H^s(\mathbb{R}^N)}.$$

The embedding $\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ is continuous and for any $s \in (0, 1)$, there exists a best constant $S_s > 0$ such that

$$S_s := \inf_{u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}}{\|u\|^2_{L^{2_s^*}(\mathbb{R}^N)}}.$$

According to [14], S_s is attained by

$$u_0(x) = C \left(\frac{b}{b^2 + |x - a|^2} \right)^{\frac{N-2s}{2}}, \quad x \in \mathbb{R}^N, \tag{2.1}$$

where $C \in \mathbb{R}$, $b > 0$ and $a \in \mathbb{R}^N$ are fixed parameters.

The fractional Laplacian, $(-\Delta)^s u$, of a smooth function $u : \mathbb{R}^N \to \mathbb{R}$, is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^N.$$

Also $(-\Delta)^s u$ can be equivalently represented [17] as

$$(-\Delta)^s u(x) = -\frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{N+2s}} dy, \quad \forall x \in \mathbb{R}^N$$

where

$$C(N, s) = \left(\int_{\mathbb{R}^N} \frac{(1 - cos\xi_1)}{|\xi|^{N+2s}} d\xi \right)^{-1}, \quad \xi = (\xi_1, \ldots, \xi_N).$$

Also, by the Plancherel formula in Fourier analysis, we have

$$[u]^2_{H^s(\mathbb{R}^N)} = \frac{2}{C(N,s)}\|(-\Delta)^{\frac{s}{2}}u\|^2_{L^2(\mathbb{R}^N)}.$$

For convenience, we will omit the normalization constant in the following. As a consequence, the norms on $H^s(\mathbb{R}^N)$ defined below

$$u \longmapsto \left(\int_{\mathbb{R}^N}|u|^2dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}};$$

$$u \longmapsto \left(\int_{\mathbb{R}^N}(|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2)d\xi\right)^{\frac{1}{2}};$$

$$u \longmapsto \left(\int_{\mathbb{R}^N}|u|^2dx + \|(-\Delta)^{\frac{s}{2}}u\|^2_{L^2(\mathbb{R}^N)}\right)^{\frac{1}{2}};$$

are equivalent.

Making the change of variable $x \mapsto \varepsilon x$, we can rewrite the equation (1.1) as the following equivalent form

$$(-\Delta)^s u + V(\varepsilon x)u = \left(\frac{1}{|x|^\mu} * F(u)\right) f(u) + |u|^{2^*_s-2}u \quad \text{in } \mathbb{R}^N. \tag{2.2}$$

If u is a solution of the equation (2.2), then $v(x) := u(\frac{x}{\varepsilon})$ is a solution of the equation (1.1). Thus, to study the equation (1.1), it suffices to study the equation (2.2). In view of the presence of potential $V(x)$, we introduce the subspace

$$H_\varepsilon = \left\{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)u^2dx < +\infty\right\},$$

which is a Hilbert space equipped with the inner product

$$(u,v)_{H_\varepsilon} = \int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}vdx + \int_{\mathbb{R}^N}V(\varepsilon x)uvdx,$$

and the norm

$$\|u\|^2_{H_\varepsilon} = \int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2dx + \int_{\mathbb{R}^N}V(\varepsilon x)u^2dx.$$

We denote $\|\cdot\|_{H_\varepsilon}$ by $\|\cdot\|_\varepsilon$ in the sequel for convenience. The energy functional corresponding to equation (2.2) is

$$E_\varepsilon(u) = \frac{1}{2}\|u\|^2_\varepsilon - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u)\right)F(u)dx - \frac{1}{2^*_s}\int_{\mathbb{R}^N}|u|^{2^*_s}dx.$$

Since we are interested in the nontrivial nonnegative solutions, we consider the following functional

$$J_\varepsilon(u) = \frac{1}{2}\|u\|^2_\varepsilon - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u^+)\right)F(u^+)dx - \frac{1}{2^*_s}\int_{\mathbb{R}^N}|u^+|^{2^*_s}dx.$$

Moreover, $J_\varepsilon(u) \in C^1(H^s, \mathbb{R}^N)$,

$$\begin{aligned}\langle J'_\varepsilon(u), \varphi\rangle = &\int\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}(\varphi(x)-\varphi(y))dxdy + \int_{\mathbb{R}^N}V(\varepsilon x)u\varphi dx\\ &- \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u^+)\right)f(u^+)\varphi dx - \int_{\mathbb{R}^N}|u^+|^{2^*_s-2}u\varphi dx.\end{aligned}$$

We collect the following useful result.

Lemma 2.1 [17] *Let $s \in (0,1)$ and $N > 2s$. Then there exists a sharp constant $C_* = C(N,s) > 0$ such that for any $u \in H^s(\mathbb{R}^N)$*

$$\|u\|^2_{L^{2^*_s}(\mathbb{R}^N)} \le C_*^{-1}[u]^2_{H^s(\mathbb{R}^N)}.$$

Moreover $H^s(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [2, 2^*_s]$ and compactly in $L^q_{loc}(\mathbb{R}^N)$ for any $q \in [2, 2^*_s)$.

Corollary 2.1 *The space H_ε is continuously embedded into $H^s(\mathbb{R}^N)$. Therefore, H_ε is continuously embedded into $L^r(\mathbb{R}^N)$ for any $r \in [2, 2^*_s]$ and compactly embedded into $L^r_{loc}(\mathbb{R}^N)$ for any $r \in [2, 2^*_s)$.*

Lemma 2.2 [38] *Let $N > 2s$, If $\{u_n\}$ is a bounded sequence in $H^s(\mathbb{R}^N)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx = 0$$

where $R > 0$, then $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (2, 2^*_s)$.

Lemma 2.3 [25] *Let $t, r > 1$ and $0 < \mu < N$ such that $\frac{1}{r} + \frac{\mu}{N} + \frac{1}{t} = 2$. Let $f \in L^r(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$. Then there exists a sharp constant $C(r, N, \mu, t) > 0$, independent of f and h, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu} dxdy \le C(r, N, \mu, t)\|f\|_{L^r(\mathbb{R}^N)}\|h\|_{L^t(\mathbb{R}^N)}.$$

Lemma 2.4 [35]*Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, $\varphi \in C_0^\infty(\mathbb{R}^N)$ and for each $r > 0$, $\varphi_r(x) = \varphi(\frac{x}{r})$. Then*

$$u\varphi_r \to 0 \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^N) \text{ as } r \to 0.$$

If, in addition, $\varphi \equiv 1$ in a neighbourhood of the origin, then

$$u\varphi_r \to u \quad \text{in } \mathcal{D}^{s,2}(\mathbb{R}^N) \quad \text{as } r \to +\infty.$$

3 Ground state solution

Lemma 3.1 *J_ε has a mountain pass geometry, that is*

(i) *There exists $\alpha, \rho > 0$ such that $J_\varepsilon(u) \ge \alpha$ for any $u \in H_\varepsilon$ which $\|u\|_\varepsilon = \rho$.*
(ii) *There exists $e \in H_\varepsilon$ with $\|e\|_\varepsilon > \rho$ such that $J_\varepsilon(e) < 0$.*

Proof In order to show this, we argue as in Lemma 2.2 in [7]. From (f_1) and (f_2), it follows that for any $\xi > 0$ there exists $C_\xi > 0$ such that

$$f(t) \le \xi|t| + C_\xi|t|^{q-1}, \quad F(t) \le \xi|t|^2 + C_\xi|t|^q. \tag{3.1}$$

By (3.1) and Lemma 2.3, we get

$$\begin{aligned} \left|\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u^+)\right)F(u^+)dx\right| &\le C\|F(u)\|_{L^t(\mathbb{R}^N)}\|F(u)\|_{L^t(\mathbb{R}^N)} \\ &\le C\left(\int_{\mathbb{R}^N}(|u|^2 + |u|^q)^t dx\right)^{\frac{2}{t}} \end{aligned} \tag{3.2}$$

where $t=\frac{2N}{2N-\mu}$. Since $q\in(\frac{2N-\mu}{N},\frac{2N-\mu}{N-2s})$, we can see that $tq\in(2,2_s^*)$, and from Corollary 2.1, we have

$$\left(\int_{\mathbb{R}^N}(|u|^2+|u|^q)^t dx\right)^{\frac{2}{t}}\le C(\|u\|_\varepsilon^2+\|u\|_\varepsilon^q)^2. \tag{3.3}$$

Taking into account (3.2) and (3.3) we can deduce that

$$\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F(u^+)\right)F(u^+)dx+\frac{1}{2_s^*}\int_{\mathbb{R}^N}|u^+|^{2_s^*}dx\le C(\|u\|_\varepsilon^4+\|u\|_\varepsilon^{2q}+\|u\|_\varepsilon^{2_s^*}). \tag{3.4}$$

As a consequence

$$J_\varepsilon(u)\ge\frac{1}{2}\|u\|_\varepsilon^2-C(\|u\|_\varepsilon^4+\|u\|_\varepsilon^{2q}+\|u\|_\varepsilon^{2_s^*}).$$

We can see that (i) holds.
Fix a positive function $u_0\in H_\varepsilon\setminus\{0\}$ and $u_0>0$, we set

$$h(t)=\frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F\left(\frac{tu_0}{\|u_0\|_\varepsilon}\right)\right)F\left(\frac{tu_0}{\|u_0\|_\varepsilon}\right)dx\quad\text{for}\ \ t>0.$$

By (f_3), we have

$$F(u)=\int_0^1 f(tu)udt=\int_0^1\frac{f(tu)}{tu}tu^2dt\le\int_0^1 f(u)tudt=\frac{1}{2}f(u)u\quad\text{for}\ \ u>0.$$

Hence,

$$\begin{aligned}h'(t)&=\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F\left(\frac{tu_0}{\|u_0\|_\varepsilon}\right)\right)f\left(\frac{tu_0}{\|u_0\|_\varepsilon}\right)\frac{u_0}{\|u_0\|_\varepsilon}dx\\&=\frac{4}{t}\int_{\mathbb{R}^N}\frac{1}{2}\left(\frac{1}{|x|^\mu}*F\left(\frac{tu_0}{\|u_0\|_\varepsilon}\right)\right)\frac{1}{2}f\left(\frac{tu_0}{\|tu_0\|_\varepsilon}\right)\frac{tu_0}{\|u_0\|_\varepsilon}dx\\&\ge\frac{4}{t}h(t).\end{aligned}\tag{3.5}$$

Integrating (3.5) on $[1,t\|u_0\|_\varepsilon]$ with $t>\frac{1}{\|u_0\|_\varepsilon}$, we find

$$h(t\|u_0\|_\varepsilon)\ge h(1)(t\|u_0\|_\varepsilon)^4$$

which gives

$$\frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F(tu_0)\right)F(tu_0)dx\ge\frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F\left(\frac{u_0}{\|u_0\|_\varepsilon}\right)\right)F\left(\frac{u_0}{\|u_0\|_\varepsilon}\right)dx\|u_0\|_\varepsilon^4t^4.$$

Therefore, we have

$$\begin{aligned}J_\varepsilon(tu_0)&=\frac{t^2}{2}\|u_0\|_\varepsilon^2-\frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F(tu_0)\right)F(tu_0)dx-\frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^N}|u_0|^{2_s^*}dx\\&\le C_1t^2-C_2t^4\end{aligned}$$

for $t>\frac{1}{\|u_0\|_\varepsilon}$. Taking $e=tu_0$ with t sufficiently large, we can see that (ii) holds. □

Lemma 3.2 *For each $u\in X_\varepsilon^+:=\{u\in H_\varepsilon:u^+(x)\ne0\}$ and $t>0$, set $h_u(t):=J_\varepsilon(tu)$.*

(i) *Then there exists an unique* $t_u > 0$ *such that* $h_u(t_u) = \max_{t\geq 0} h_u(t) = \max_{t\geq 0} J_\varepsilon(tu)$, $h'_u(t_u) = 0$, $h'_u(t) > 0$ *in* $(0, t_u)$, $h'_u(t) < 0$ *in* $(t_u, +\infty)$ *and* $tu \in \mathcal{N}_\varepsilon$ *if and only if* $t = t_u$, *where* $\mathcal{N}_\varepsilon = \{u \in X_\varepsilon^+ : \langle J'_\varepsilon(u), u\rangle = 0\}$.

(ii) *There is* $\kappa > 0$ *independent on* u, *such that* $t_u \geq \kappa$ *for all* $u \in \mathcal{S}_\varepsilon$, *where we denote by* $\mathcal{S}_\varepsilon$ *the unitary sphere in* H_ε. *Moreover, for any compact set* $E \subset \mathcal{S}_\varepsilon$, *there is a* $C_E > 0$ *such that* $t_u \leq C_E$ *for all* $u \in E$.

Proof (i) For every $u \in X_\varepsilon^+$, from Lemma 3.1 we know that $h_u(0) = 0$, $h_u(t) > 0$ for $t > 0$ small enough and $\lim_{t\to+\infty} h_u(t) = -\infty$. Hence, there exists a $t_u > 0$ such that $h_u(t_u) = \max_{t\geq 0} h_u(t)$ and $h'_u(t_u) = 0$. Notice that

$$h'_u(t) = 0 \Leftrightarrow tu \in \mathcal{N}_\varepsilon \Leftrightarrow \|u\|_\varepsilon^2 = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * \frac{F(tu^+)}{t}\right) f(tu^+)u^+dx + t^{2_s^*-2}\int_{\mathbb{R}^N} |u^+|^{2_s^*}dx.$$

From (f_3) we know $t \mapsto f(t)$ and $t \mapsto \frac{F(t)}{t}$ are increasing for all $t > 0$. Hence, we get the uniqueness of a such t_u and (i) is completed.
(ii) Let $u \in \mathcal{S}_\varepsilon$. By $t_u u \in \mathcal{N}_\varepsilon$ and (3.4) we have

$$t_u^2 = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(t_u u^+)\right) f(t_u u^+)t_u u^+dx + \int_{\mathbb{R}^N} |t_u u^+|^{2_s^*}dx \leq C\left(t_u^4 + t_u^{2q} + t_u^{2_s^*}\right).$$

So, there exists $\kappa > 0$ independent of u, such that $t_u \geq \kappa$. Let, $\alpha \in (2, 2_s^*)$, $\alpha \leq 4$ then $\frac{2}{\alpha} \geq \frac{1}{2}$. We can infer that

$$F(t) \leq \frac{1}{2} f(t)t \leq \frac{2}{\alpha} f(t)t, \quad \forall\, t \geq 0.$$

For any $v \in \mathcal{N}_\varepsilon$, we have

$$\begin{aligned} J_\varepsilon(v) &= J_\varepsilon(v) - \frac{1}{\alpha}\langle J'_\varepsilon(v), v\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|v\|_\varepsilon^2 - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(v^+)\right)\Big(F(v^+) \\ &\quad -\frac{2}{\alpha} f(v^+)v^+\Big)dx + \left(\frac{1}{\alpha} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^N} |v^+|^{2_s^*}dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|v\|_\varepsilon^2. \end{aligned} \tag{3.6}$$

If $E \subset \mathcal{S}_\varepsilon$ is a compact set and $u_n \subset E$ such that $t_{u_n} \to \infty$, up to a subsequence $u_n \to u$ in H_ε and $J_\varepsilon(t_{u_n}u_n) \to -\infty$. Taking $v_n = t_{u_n}u_n \in \mathcal{N}_\varepsilon$ in (3.6), we can see that

$$0 < \frac{1}{2} - \frac{1}{\alpha} \leq \frac{J_\varepsilon(t_{u_n}u_n)}{t_{u_n}^2} \leq 0 \quad \text{as} \quad n \to \infty,$$

which gives a contradiction. □

Define the mappings $\hat{n}_\varepsilon : H_\varepsilon\backslash\{0\} \to \mathcal{N}_\varepsilon$ and $n_\varepsilon := \mathcal{S}_\varepsilon \to \mathcal{N}_\varepsilon$ by set

$$\hat{n}_\varepsilon(u) := t_u u \quad \text{and} \quad n_\varepsilon := \hat{n}_\varepsilon|_{\mathcal{S}_\varepsilon}.$$

We can apply [40, Proposition 8, Proposition 9 and Corollary 10] to deduce the following Lemma.

Lemma 3.3 *Suppose that* (V_0) *and* (f_1)*–*(f_4), *then*

(a) *The mapping* $\hat{n}_\varepsilon$ *is continuous and* n_ε *is a homeomorphism between* $\mathcal{S}_\varepsilon$ *and* $\mathcal{N}_\varepsilon$. *Moreover,* $n_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon}$.

(b) *We define the maps* $\hat{\psi}_\varepsilon : H_\varepsilon\backslash\{0\} \to \mathbb{R}$ *by* $\hat{\psi}_\varepsilon(u) := J_\varepsilon(\hat{n}_\varepsilon(u))$. *Then* $\hat{\psi}_\varepsilon \in C^1(H_\varepsilon\backslash\{0\}, \mathbb{R})$ *and*

$$\langle \hat{\psi}_\varepsilon'(u), v\rangle = \frac{\|\hat{n}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon}\langle J_\varepsilon'(\hat{n}_\varepsilon(u), v)\rangle$$

for every $u \in H_\varepsilon\backslash\{0\}$ *and* $v \in H_\varepsilon$.

(c) *We define the maps* $\psi : \mathcal{S}_\varepsilon \to \mathbb{R}$ *by* $\psi_\varepsilon := \hat{\psi}|_{\mathcal{S}_\varepsilon}$. *Then* $\psi_\varepsilon \in C^1(\mathcal{S}_\varepsilon, \mathbb{R})$ *and* $\langle \psi_\varepsilon'(u), v\rangle = \|n_\varepsilon(u)\|_\varepsilon\langle J_\varepsilon'(n_\varepsilon(u)), v\rangle$ *for any* $v \in T_u\mathcal{S}_\varepsilon$.

(d) *If* $\{u_n\}$ *is a* $(PS)_d$ *sequence for* ψ_ε, *then* $\{n_\varepsilon(u_n)\}$ *is a* $(PS)_d$ *sequence for* J_ε. *Moreover, if* $\{u_n\} \subset \mathcal{N}_\varepsilon$ *is a bounded* $(PS)_d$ *sequence for* ψ_ε, *then* $\{n_\varepsilon^{-1}(u_n)\}$ *is a bounded* $(PS)_d$ *sequence for the functional* ψ_ε.

(e) u *is a critical point of* ψ_ε *if and only if* $n_\varepsilon(u)$ *is a nontrivial critical point for* J_ε. *Moreover, the corresponding critical values coincide and*

$$\inf_{u\in\mathcal{S}_\varepsilon} \psi_\varepsilon(u) = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u).$$

Remark 3.1 As in [40], we have the following minimax characterization:

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u) = \inf_{u\in H_\varepsilon\backslash\{0\}} \max_{t>0} J_\varepsilon(tu) = \inf_{u\in\mathcal{S}_\varepsilon} \max_{t>0} J_\varepsilon(tu).$$

Next, we give some properties of $(PS)_d$ sequence of J_ε.

Lemma 3.4 *Let* $\{u_n\} \subset H_\varepsilon$ *is a* $(PS)_d$ *sequence of* J_ε, *then* $\{u_n\}$ *is bounded in* $H^s(\mathbb{R}^N)$ *and* $\{\|u_n^-\|_\varepsilon\} = o_n(1)$.

Proof Let, $\alpha \in (2, 2_s^*)$, $\alpha \le 4$ then $\frac{2}{\alpha} \ge \frac{1}{2}$. We can infer that

$$F(t) \le \frac{1}{2}f(t)t \le \frac{2}{\alpha}f(t)t, \quad \forall t \ge 0.$$

Since $\{u_n\}$ is a $(PS)_d$ sequence of J_ε, we have

$$\begin{aligned}
d + 1 + \|u_n\|_\varepsilon &\ge J_\varepsilon(u_n) - \frac{1}{\alpha}\langle J_\varepsilon'(u_n), u_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u_n\|_\varepsilon^2 - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u_n^+)\right)\Big(F(u_n^+) \\
&\quad - \frac{2}{\alpha}f(u_n^+)u_n^+\Big)dx + \left(\frac{1}{\alpha} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^N}|u_n^+|^{2_s^*}dx \\
&\ge \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u_n\|_\varepsilon^2.
\end{aligned}$$

Therefore, we get that the sequence $\{u_n\}$ is bounded in H_ε. Next, we prove that $\|u_n^-\|_\varepsilon = o_n(1)$. Since $\langle J_\varepsilon'(u_n), u_n^-\rangle = o_n(1)$, by using $f(t) = 0$ for $t \le 0$ and $(x - y)(x^- - y^-) \ge |x^- - y^-|^2$ where $x^- = \min\{x, 0\}$, we can deduce that

$$\begin{aligned}
\|u_n^-\|_\varepsilon^2 &\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_n(x) - u_n(y))(u_n^-(x) - u_n^-(y))}{|x - y|^{N+2s}}dxdy + \int_{\mathbb{R}^N}V(\varepsilon x)u_nu_n^-dx \\
&= \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u_n^+)\right)f(u_n^+)u_n^-dx + \int_{\mathbb{R}^N}|u_n^+|^{2_s^*-2}u_n^+u_n^-dx + o_n(1) \\
&= o_n(1).
\end{aligned}$$

Therefore, we complete our proof. □

Lemma 3.5 *There exists a constant $r > 0$ such that $\|u\|_\varepsilon \geq r$ for all $\varepsilon \geq 0$ and $u \in \mathcal{N}_\varepsilon$*

Proof By using Lemma 2.3 and $(f_1) - (f_2)$, we can see that for any $u \in \mathcal{N}_\varepsilon$

$$\|u\|_\varepsilon^2 \leq C\left(\|u\|_\varepsilon^4 + \|u\|_\varepsilon^{2q} + \|u\|_\varepsilon^{2_s^*}\right)$$

then, there exists $r > 0$ such that

$$\|u\|_\varepsilon \geq r \quad \text{for all} \quad u \in \mathcal{N}_\varepsilon. \tag{3.7}$$

Hence, we deduce to the Lemma holds. □

When $V \equiv 1$, then $H^s(\mathbb{R}^N) = H_\varepsilon(\mathbb{R}^N)$. For $\tau > 0$ and $u \in H^s(\mathbb{R}^N)$, let

$$\begin{aligned} I_\tau(u) = &\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy + \frac{\tau}{2}\int_{\mathbb{R}^N} u^2 dx \\ &- \frac{1}{2_s^*}\int_{\mathbb{R}^N}|u^+|^{2_s^*}dx - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u^+)\right)F(u^+)dx, \\ &\mathcal{M}_\tau := \{u \in H^s(\mathbb{R}^N) : u^+ \neq 0, \langle I_\tau'(u), u\rangle = 0\}, \quad m_\tau := \inf_{\mathcal{M}_\tau} I_\tau. \end{aligned}$$

For m_τ, there also holds

$$m_\tau = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_\tau(\gamma(t)) = \inf_{u\in H^s(\mathbb{R}^N)}\sup_{t\geq 0} I_\tau(tu)$$

where $\Gamma = \{\gamma \in C([0,1], H^s(\mathbb{R}^N)) : \gamma(0) = 0, \quad I_\tau(\gamma(1)) < 0\}$.

Lemma 3.6 *For any $\tau > 0$, there exists $u \in H^s(\mathbb{R}^N)$ with $u^+ \neq 0$ such that*

$$\max_{t\geq 0} I_\tau(tu) < \frac{s}{N}S^{\frac{N}{2s}},$$

where $S := \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}}{\|u\|^2_{L^{2_s^*}(\mathbb{R}^N)}}$.

Proof Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi = 1$ in B_δ, $\varphi(x) = 0$ in $\mathbb{R}^N \setminus B_{2\delta}$. Denote

$$U_\varepsilon(x) = \varepsilon^{-\frac{N-2s}{2}} u^*\left(\frac{x}{\varepsilon}\right)$$

where $u^*(x) = \frac{\tilde{u}(x/S^{\frac{1}{2s}})}{\|\tilde{u}\|_{L^{2_s^*}(\mathbb{R}^N)}}$, $\tilde{u}(x/S^{\frac{1}{2s}}) = \frac{\alpha}{\left(1+|x/S^{\frac{1}{2s}}|^2\right)^{\frac{N-2s}{2}}}$ with $\alpha > 0$. We define

$$u_\varepsilon(x) := \varphi(x)U_\varepsilon(x)$$

then $u_\varepsilon \in H_\varepsilon$. From [17] and [34], we have the following estimations

$$[u_\varepsilon]_{H^s(\mathbb{R}^N)} \leq S^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) \tag{3.8}$$

$$\int_{\mathbb{R}^N}|u_\varepsilon(x)|^{2_s^*}dx = S^{\frac{N}{2s}} + O(\varepsilon^N) \tag{3.9}$$

$$\begin{cases} C_s\varepsilon^{2s} + O(\varepsilon^{N-2s}) & \text{if } N > 4s, \\ C_s\varepsilon^{2s}|ln\varepsilon| + O(\varepsilon^{2s}) & \text{if } N = 4s, \\ C_s\varepsilon^{N-2s} + O(\varepsilon^{N-2s}) & \text{if } N < 4s. \end{cases} \tag{3.10}$$

A standard argument shows that for any u_ε, there exists a unique t_ε such that $t_\varepsilon u_\varepsilon \in \mathcal{M}_\tau$ and $I_\tau(t_\varepsilon u_\varepsilon) = \max_{t\geq 0} I_\tau(tu_\varepsilon)$. As a consequence $m_\tau \leq I_\tau(t_\varepsilon u_\varepsilon)$ and

$$[u_\varepsilon]^2_{H^s(\mathbb{R}^N)} + \tau \int_{\mathbb{R}^N} u_\varepsilon^2 dx = t_\varepsilon^{-1} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(t_\varepsilon u_\varepsilon) \right) f(t_\varepsilon u_\varepsilon) u_\varepsilon dx + t_\varepsilon^{2_s^*-2} \int_{\mathbb{R}^N} |u_\varepsilon|^{2_s^*} dx.$$

As a consequence $t_\varepsilon \geq t_0$, where $t_0 > 0$ is independent of ε. Now, we estimate the convolution term. For $\varepsilon > 0$ small enough, it follows from (f_4) that

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(t_\varepsilon u_\varepsilon) \right) F(t_\varepsilon u_\varepsilon) dx &\geq C t_\varepsilon^{2\sigma} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * |u_\varepsilon|^\sigma \right) |u_\varepsilon|^\sigma dx \\
&\geq C t_0^{2\sigma} \int_{B_\delta} \int_{B_\delta} \frac{|u_\varepsilon(y)|^\sigma |u_\varepsilon(x)|^\sigma}{|x-y|^\mu} dxdy \\
&\geq C t_0^{2\sigma} \int_{B_\delta} \int_{B_\delta} \frac{C_1 \varepsilon^{\sigma(N-2s)}}{(\varepsilon^2 + |y|^2)^{\frac{\sigma(N-2s)}{2}} (\varepsilon^2 + |x|^2)^{\frac{\sigma(N-2s)}{2}}} dxdy \\
&\geq C t_0^{2\sigma} \int_{B_{\frac{\delta}{2}}} \int_{B_{\frac{\delta}{2}}} \frac{C_2 \varepsilon^{2N-\sigma(N-2s)}}{(1 + |x|^2)^{\frac{\sigma(N-2s)}{2}} (1 + |y|^2)^{\frac{\sigma(N-2s)}{2}}} dxdy \\
&\geq O(\varepsilon^{2N-\sigma(N-2s)}).
\end{aligned} \tag{3.11}$$

Set $g(t) := \frac{t^2}{2}([u_\varepsilon]^2_{H^s(\mathbb{R}^N)} + \tau \int_{\mathbb{R}^N} u_\varepsilon^2 dx) - \frac{t^{2_s^*}}{2_s^*} \int_{\mathbb{R}^N} |u_\varepsilon|^{2_s^*} dx$. If $N > 4s$, by a simple calculation, we get

$$\begin{aligned}
\max_{t\geq 0} g(t) &= \frac{s}{N} \left(\frac{[u_\varepsilon]^2_{H^s \mathbb{R}^N} + \tau \int_{\mathbb{R}^N} u_\varepsilon^2 dx}{\|u_\varepsilon\|^2_{2_s^*}} \right)^{\frac{N}{2s}} \\
&= \frac{s}{N} \left(\frac{S^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) + O(\varepsilon^{2s})}{(S^{\frac{N}{2s}} + O(\varepsilon^N))^{\frac{N-2s}{N}}} \right)^{\frac{N}{2s}} \\
&= \frac{s}{N} S^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) + O(\varepsilon^{2s}).
\end{aligned} \tag{3.12}$$

Nothing that $\sigma \in [q_N, \frac{2N-\mu}{N-2s})$, for $\varepsilon > 0$ small enough, using (3.11),(3.12) we can check

$$\begin{aligned}
\max_{t\geq 0} I_\tau(tu_\varepsilon) \leq \max_{t\geq 0} g(t) - \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(t_\varepsilon u_\varepsilon) \right) F(t_\varepsilon u_\varepsilon) dx \\
< \frac{s}{N} S^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) - O(\varepsilon^{2N-\sigma(N-2s)}) + O(\varepsilon^{2s}) \\
< \frac{s}{N} S^{\frac{N}{2s}}.
\end{aligned}$$

In similar way, we can check $N = 4s$ and $N < 4s$. □

Lemma 3.7 *Let $\{u_n\} \subset H_\varepsilon$ be a $(PS)_d$ sequence of J_ε with $d < \frac{s}{N} S^{\frac{N}{2s}}$ and $u_n \rightharpoonup 0$ in H_ε. Then one of the following conclusions holds:*

(a) *$u_n \to 0$ in H_ε;*
(b) *There exists a sequence $\{y_n\} \subset \mathbb{R}^N$ and positive constants r, β such that*

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n(x)|^2 dx > \beta.$$

Proof If (b) does not occur, then for all $R > 0$, up to a subsequence

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^2 dx = 0.$$

Since we know that $\{u_n\}$ is bounded in H_ε, we can use Lemma 2.2 to deduce that $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for any $r \in (2, 2_s^*)$. So, apply Hardy–Littlewood–Sobolev inequality, we know that

$$\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u_n^+)\right) F(u_n^+) dx = o_n(1). \tag{3.13}$$

Taking into account $\langle J_\varepsilon'(u_n), u_n\rangle = o_n(1)$ we can infer that

$$\|u_n\|_\varepsilon^2 = \|u_n\|_{L^{2_s^*}(\mathbb{R}^N)}^{2_s^*} + o_n(1).$$

Since $\{u_n\}$ is bounded, up to a subsequence, we have

$$\|u_n\|_\varepsilon^2 \to l \geq 0 \quad \text{and} \quad \|u_n\|_{L^{2_s^*}(\mathbb{R}^N)}^{2_s^*} \to l \geq 0.$$

If $l > 0$, then

$$\begin{aligned} S &\leq \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx}{(\|u_n\|_{L^{2_s^*}(\mathbb{R}^N)}^{2_s^*})^{\frac{2}{2_s^*}}} \\ &\leq \frac{\|u_n\|_\varepsilon^2}{(\|u_n\|_{L^{2_s^*}(\mathbb{R}^N)}^{2_s^*})^{\frac{2}{2_s^*}}} \\ &\to l^{\frac{2s}{N}} \end{aligned}$$

as $n \to \infty$, hence $l \geq S^{\frac{N}{2s}}$. Consequently, by (3.13), we have

$$\begin{aligned} d &= \lim_{n\to\infty} J_\varepsilon(u_n) \\ &= \lim_{n\to\infty} \left(\frac{1}{2}\|u_n\|_\varepsilon^2 - \frac{1}{2}\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u_n^+)\right) F(u_n^+) dx - \frac{1}{2_s^*}\int_{\mathbb{R}^N} |u_n^+|^{2_s^*} dx\right) \\ &= \frac{s}{N} l \\ &> \frac{s}{N} S^{\frac{N}{2s}} \end{aligned}$$

a contradiction, hence $l = 0$. Consequently, by the boundedness of $\{u_n\}$ in H_ε, we have $u_n \to 0$ in H_ε, so (a) holds. This completes the proof. □

Lemma 3.8 *Let $\{u_n\} \subset H_\varepsilon$ be a $(PS)_d$ sequence of J_ε with $d < m_{V_\infty}$ and $u_n \rightharpoonup 0$ in H_ε. Then $u_n \to 0$ in H_ε.*

Proof By Lemma 3.4 we can assume $u_n \geq 0$. For any subsequence of $\{u_n\}$ still denoted by $\{u_n\}$. Since $u_n \rightharpoonup 0$ in H_ε, up to a subsequence, we can assume

$$u_n \to 0 \text{ in } L_{loc}^r(\mathbb{R}^N) \quad r \in [2, 2_s^*) \quad \text{and} \quad u_n(x) \to 0 \quad \text{a.e.} \quad x \in \mathbb{R}^N.$$

If $u_n \nrightarrow 0$ in H_ε, by Lemma 3.2 we know for any $\{t_n\} \subset (0, +\infty)$ such that $\{t_n u_n\} \subset \mathcal{N}_{V_\infty}$.

Case 1 $\limsup_{n\to\infty} t_n \leq 1$. If does not occur for any $\delta > 0$, consider any subsequence of $\{t_n\}$ and satisfies the following

$$t_n \geq 1+\delta, \quad \forall \ n \in \mathbb{N}.$$

Since $\{u_n\}$ is a $(PS)_d$ sequence of J_ε, we can see that

$$[u_n]^2_{H^s(\mathbb{R}^N)} + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^2 dx = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u_n)\right) f(u_n)u_n dx + \int_{\mathbb{R}^N} |u_n|^{2_s^*} dx + o_n(1). \tag{3.14}$$

We observe that $\{t_n u_n\} \subset \mathcal{N}_{V_\infty}$, we have

$$t_n^2[u_n]^2_{H^s(\mathbb{R}^N)} + t_n^2 \int_{\mathbb{R}^N} V_\infty u_n^2 dx = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(t_n u_n)\right) f(t_n u_n) t_n u_n dx + \int_{\mathbb{R}^N} |t_n u_n|^{2_s^*} dx. \tag{3.15}$$

Taking into account (3.14) and (3.15) we can deduce that

$$\begin{aligned} \int_{\mathbb{R}^N} (V_\infty - V(\varepsilon x))|u_n|^2 dx &= \int_{\mathbb{R}^N} \left(\frac{\left(\frac{1}{|x|^\mu} * F(t_n u_n)\right) f(t_n u_n) u_n^2}{t_n u_n} - \frac{\left(\frac{1}{|x|^\mu} * F(u_n)\right) f(u_n) u_n^2}{u_n} \right) dx \\ &\quad + \int_{\mathbb{R}^N} \left(\frac{|t_n u_n|^{2_s^*}}{t_n^2} - |u_n|^{2_s^*} \right) dx + o_n(1). \end{aligned}$$

By (V_0), for any $\xi > 0$ there exists $R(\xi) := R > 0$ such that

$$V(\varepsilon x) \geq V_\infty - \xi, \quad |\varepsilon x| \geq R.$$

Notice that $u_n \to 0$ in $L^2(B_R(0))$ and the boundedness of $\{u_n\}$ in H_ε, we get

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(\frac{\left(\frac{1}{|x|^\mu} * F(t_n u_n)\right) f(t_n u_n) u_n^2}{t_n u_n} - \frac{\left(\frac{1}{|x|^\mu} * F(u_n)\right) f(u_n) u_n^2}{u_n} \right) dx \\ &\leq \int_{\mathbb{R}^N} (V_\infty - V(\varepsilon x))|u_n|^2 dx \\ &= \int_{B_R(0)} (V_\infty - V(\varepsilon x))|u_n|^2 dx + \int_{B_R^c(0)} (V_\infty - V(\varepsilon x))|u_n|^2 dx \\ &\leq V_\infty \int_{B_R(0)} |u_n|^2 dx + \xi \int_{B_R^c(0)} |u_n|^2 dx \\ &\leq o_n(1) + \frac{\xi}{V_0} \int_{B_R^c(0)} V(\varepsilon x)|u_n|^2 dx \\ &\leq o_n(1) + \frac{\xi}{V_0} \|u_n\|_\varepsilon^2 \leq o_n(1) + \xi C. \end{aligned}$$

If $u_n \nrightarrow 0$, there exists $\{y_n\} \subset \mathbb{R}^N$, $r, \delta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n(x)|^2 dx \geq \delta.$$

Let $\tilde{u}_n(x) = u_n(x + y_n)$ then there exists $\tilde{u}$, up to a subsequence, we have

$$\tilde{u}_n \rightharpoonup \tilde{u} \text{ in } H^s(\mathbb{R}^N), \quad \tilde{u}_n \to \tilde{u} \text{ in } L^t_{loc}(\mathbb{R}^N), \quad \tilde{u}_n(x) \to \tilde{u}(x) \quad \text{a.e.} \quad x \in \mathbb{R}^N.$$

Therefore, there exists $\Omega \subset B_r(0)$ such that $\tilde{u} > 0$ in Ω, then we can infer

$$\int_\Omega \left(\frac{\left(\frac{1}{|x|^\mu} * F((1+\delta)\tilde{u})\right) f((1+\delta)\tilde{u})}{(1+\delta)\tilde{u}} - \left(\frac{(\frac{1}{|x|^\mu} * F(\tilde{u}))f(\tilde{u})}{\tilde{u}} \right) \right) \tilde{u}^2 dx \leq \xi C + o_n(1).$$

Taking the limit as $n \to \infty$ and by applying Fatou's lemma we obtain

$$0 < \int_\Omega \int_\Omega \left(\frac{F((1+\delta)\tilde{u}(y))}{|x-y|^\mu} \frac{f((1+\delta)\tilde{u}(x))}{(1+\delta)\tilde{u}(x)} - \frac{F(\tilde{u}(y))}{|x-y|^\mu} \frac{f(\tilde{u}(x))}{\tilde{u}(x)} \right) \tilde{u}^2 dx dy \leq \xi C.$$

For any $\xi > 0$, this gives a contradiction. Therefore, $\limsup_{n\to\infty} t_n \leq 1$.

Case 2 $\limsup_{n\to\infty} t_n = 1$. Hence there exists a subsequence of $\{t_n\}$, still denoted by $\{t_n\}$ such that $t_n \to 1$. Clearly,

$$d + o_n(1) = J_\varepsilon(u_n) \geq J_\varepsilon(u_n) + m_{V_\infty} - I_{V_\infty}(t_n u_n).$$

Moreover,

$$\begin{aligned} J_\varepsilon(u_n) - I_{V_\infty}(t_n u_n) \geq & \frac{1-t_n^2}{2}[u_n]^2_{H^s(\mathbb{R}^N)} + \frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon x) - t_n^2 V_\infty)|u_n|^2 dx \\ & + \frac{1}{2_s^*}\int_{\mathbb{R}^N} (|t_n u_n|^{2_s^*} - |u_n|^{2_s^*}) dx \\ & + \frac{1}{2}\int_{\mathbb{R}^N} \left(\left(\frac{1}{|x|^\mu} * F(t_n u_n) \right) F(t_n u_n) - \left(\frac{1}{|x|^\mu} * F(u_n) \right) F(u_n) \right) dx. \end{aligned}$$

Since $\{u_n\}$ is bounded in H_ε, by using the Mean Value Theorem and $t_n \to 1$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} (V(\varepsilon x) - t_n^2 V_\infty)|u_n|^2 dx = & \int_{B_R(0)} (V(\varepsilon x) - t_n^2 V_\infty)|u_n|^2 dx \\ & + \int_{B_R^c(0)} (V(\varepsilon x) - t_n^2 V_\infty)|u_n|^2 dx \\ \geq & (V_0 - t_n^2 V_\infty) \int_{B_R(0)} |u_n|^2 dx - \xi \int_{B_R^c(0)} |u_n|^2 dx \\ & + V_\infty (1 - t_n^2) \int_{B_R^c(0)} |u_n|^2 dx \\ \geq & o_n(1) - \xi C. \end{aligned}$$

For any ξ and this gives a contradiction.

Case 3 $\limsup_{n\to\infty} t_n := t_0 < 1$. Then there exists a subsequence of $\{t_n\}$, still denoted by $\{t_n\}$ such that $t_n \to t_0$ and $t_n < 1$ for any $n \in \mathbb{N}$, we deduce that

$$\begin{aligned} m_{V_\infty} & \leq I_{V_\infty}(t_n u_n) \\ & = J_\varepsilon(t_n u_n) + \frac{t_n^2}{2}\int_{\mathbb{R}^N} (V_\infty - V(\varepsilon x))|u_n|^2 dx \\ & = J_\varepsilon(t_n u_n) + C\xi + o_n(1) \\ & = d + C\xi + o_n(1). \end{aligned}$$

This gives a contradiction. □

By similar argument as the Lemma 3.1 in [2] and Lemma 4.7 in [47], we have the following lemma.

Lemma 3.9 *Let $\{u_n\}$ be a sequence such that $u_n \rightharpoonup u$ in H_ε and $w_n := u_n - u$. Then, we have*

(i) $\int_{\mathbb{R}^N} |F(w_n) - F(u_n) + F(u)|^t dx = o_n(1)$ *where* $t = \frac{2N}{2N-\mu}$.

(ii) $\int_{\mathbb{R}^N} (\frac{1}{|x|^\mu} * F(u_n - u))F(u_n - u)dx - \int_{\mathbb{R}^N} (\frac{1}{|x|^\mu} * F(u_n))F(u_n)dx + \int_{\mathbb{R}^N} (\frac{1}{|x|^\mu} * F(u))F(u)dx = o_n(1)$.

(iii) $\forall\, \xi > 0$, *we have*

$$\int_{\mathbb{R}^N} |f(u_n - u) - f(u_n) + f(u)|^t |\varphi|^t dx \le C\xi \|\varphi\|_\varepsilon^t \le C\xi. \quad \forall \varphi \in H_\varepsilon(\mathbb{R}^N), \quad \|\varphi\|_\varepsilon = 1.$$

(iv)

$$\begin{aligned} &|\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u_n - u)\right) f(u_n - u)\varphi dx - \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u_n)\right) f(u_n)\varphi dx \\ &+ \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u)\right) f(u)\varphi dx| \le C\xi \|\varphi\|_\varepsilon. \end{aligned}$$

where, $\xi > 0$, $\varphi \in H_\varepsilon(\mathbb{R}^N)$.

Proof (i) By the Mean Value Theorem and (3.1), it follows that

$$\begin{aligned} |F(w_n) - F(u_n)| &= \left| \int_0^1 \left(\frac{d}{dt} F(u_n - tu) \right) dt \right| \\ &\le \int_0^1 |uf(u_n - tu)| dt \\ &\le \int_0^1 (\xi |u||u_n - tu| + C_\xi |u||u_n - tu|^{q-1}) dt \\ &\le \xi |u_n||u| + \xi |u|^2 + C_\xi |u_n|^{q-1}|u| + C_\xi |u|^q. \end{aligned}$$

By applying Young inequality with $\delta > 0$, we get

$$|F(w_n) - F(u_n)| \le \delta(|u_n|^2 + |u_n|^q) + C_\delta(|u|^2 + |u|^q)$$

which yields

$$\begin{aligned} |F(w_n) - F(u_n) + F(u)| &\le \delta(|u_n|^2 + |u_n|^q) + C_\delta(|u|^2 + |u|^q) + C(|u|^2 + |u|^q). \\ |F(w_n) - F(u_n) + F(u)|^t &\le 4^t \delta(|u_n|^{2t} + |u_n|^{qt}) + C(|u_n|^{2t} + |u_n|^{qt}) \\ &\le 4^t \delta(|u_n|^{2t} + |u_n|^{qt} - |u_n|^{2t} + |u_n|^{qt}) + C_1(|u_n|^{2t} + |u_n|^{qt}). \end{aligned}$$

Let

$$G_{\delta,n}(x) = \max\left\{ |F(w_n) - F(u_n) + F(u)|^t - 4^t \delta(|u_n|^{2t} + |u_n|^{qt} - |u|^{2t} - |u|^{qt}), 0 \right\}.$$

Then $G_{\delta,n} \to 0\ a.e.$ in $\mathbb{R}^N$ as $n \to \infty$ and $0 \le G_{\delta,n} \le C_1(|u|^{2t} + |u|^{qt}) \in L^1(\mathbb{R}^N)$. As a consequence of the Dominated Convergence Theorem, we have

$$\int_{\mathbb{R}^N} G_{\delta,n}(x) dx \to 0 \quad \text{as} \quad n \to \infty.$$

On the other hand, from the definition of $G_{\delta,n}$, we get

$$|F(w_n) - F(u_n) + F(u)|^t \le 4^t \delta(|u_n|^{2t} + |u_n|^{qt}) + G_{\delta,n}$$

which together with the boundedness of $\{u_n\}$ gives

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|F(w_n)-F(u_n)+F(u)|^t dx \le C\delta \quad \text{for some} \quad C>0.$$

As δ is arbitrary, we obtain

$$\int_{\mathbb{R}^N}|F(w_n)-F(u_n)+F(u)|^t dx = o_n(1).$$

(ii)

$$\begin{aligned}
&\int_{\mathbb{R}^N}(\frac{1}{|x|^\mu}*F(u_n-u))F(u_n-u)dx-\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F(u_n)\right)F(u_n)dx\\
&\quad+\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F(u)\right)F(u)dx\\
&=\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F(u_n-u)\right)(F(u_n-u)-F(u_n)\\
&\quad+F(u))dx+\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F(u_n)\right)(F(u_n-u)-F(u_n)+F(u))\,dx\\
&\quad+\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F(u)\right)(F(u_n-u)-F(u_n)+F(u))dx\\
&\quad-2\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F(u)\right)F(u_n-u)dx\\
&=:I_1+I_2+I_3+I_4.
\end{aligned}$$

By the boundedness of $\{u_n\}$ and $(f_1)-(f_2)$. we know that

$$\left(\int_{\mathbb{R}^N}|F(u_n-u)|^t dx\right)^{\frac{1}{t}}\le C.$$

From Lemma 2.3, we have

$$|I_1|\le\left(\int_{\mathbb{R}^N}|F(u_n-u)|^t dx\right)^{\frac{1}{t}}\left(\int_{\mathbb{R}^N}|F(u_n-u)-F(u_n)-F(u)|^t dx\right)^{\frac{1}{t}}\to 0.$$

Likewise, $I_2\to 0,\ I_3\to 0$. By the boundedness of $\{u_n\}$, we have $\{F(u_n-u)\}$ is bounded in $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$ and $F(u_n-u)\to 0$ $a.e.$ in $\mathbb{R}^N$. So, $F(u_n-u)\rightharpoonup 0$ in $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$. In view of $\frac{1}{|x|^\mu}*F(u)\in\left(L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)\right)^*$, we obtain

$$I_4=-2\int_{\mathbb{R}^N}(\frac{1}{|x|^\mu}*F(u))F(u_n-u)dx\to 0 \text{ as } n\to\infty.$$

Therefore, we can conclude (ii) holds.
(iii) By using (f_1) and (f_2), we know that for any $\xi>0$, there exists $N_0\in(0,1)$ and $N_1>2$ such that

$$\begin{aligned}
&|f(t)|\le\xi|t| \quad \text{in} \quad |t|\le 2N_0,\\
&|f(t)|\le\xi|t|^{q-1} \quad \text{in} \quad |t|\ge N_1-1,\\
&|f(t)|\le C_\xi|t|+\xi|t|^{q-1} \quad \text{for} \quad t\in\mathbb{R}.
\end{aligned}$$

Since f is a continuous function, we deduce that exists $\delta \in (0, N_0)$ such that

$$|f(t_1) - f(t_2)| \le N_0 \xi, \quad \forall |t_1| \le N_0 + N_1, |t_2| \le N_0 + N_1 \text{ and } |t_1 - t_2| \le \delta.$$

Taking into account $u \in H_\varepsilon$, we know that there exists $R_0 > 0$ such that

$$\left(\int_{B^c_{R_0}(0)} |u|^{2t} dx \right)^{\frac{1}{2}} < \xi, \quad \left(\int_{B^c_{R_0}(0)} |u|^{tq} dx \right)^{\frac{q-1}{q}} < \xi.$$

For any $\varphi \in H_\varepsilon$, $\|\varphi\|_\varepsilon = 1$, we have

$$\begin{aligned} \int_{B^c_{R_0}(0)} |f(u)\varphi|^{\frac{2N}{2N-\mu}} dx &\le \int_{B^c_{R_0}(0)} (\xi|u| + C_\xi |u|^{q-1})^{\frac{2N}{2N-\mu}} |\varphi|^{\frac{2N}{2N-\mu}} dx \\ &\le \int_{B^c_{R_0}(0)} (2^t \xi |u|^t + C|u|^{t(q-1)})|\varphi|^t dx \\ &\le 2^t \xi \left(\int_{B^c_{R_0}(0)} |u|^{2t} dx \right)^{\frac{1}{2}} \left(\int_{B^c_{R_0}(0)} |\varphi|^{2t} dx \right)^{\frac{1}{2}} \\ &\quad + C \left(\int_{B^c_{R_0}(0)} |u|^{tq} dx \right)^{\frac{q-1}{q}} \left(\int_{B^c_{R_0}(0)} |\varphi|^{qt} dx \right)^{\frac{1}{q}} \\ &\le C\xi \|\varphi\|^t_\varepsilon. \end{aligned}$$

Let $A_n := \left\{ B^c_{R_0}(0) : |u_n(x)| \le N_0 \right\}$, $B_n := \left\{ B^c_{R_0}(0) : |u_n(x)| \ge N_1 \right\}$, $C_n := \left\{ B^c_{R_0}(0) : N_0 < |u_n(x)| < N_1 \right\}$, then we have

$$\begin{aligned} &\int_{A_n \bigcap \{|u| \le \delta\}} |f(u_n - u) - f(u_n)|^t |\varphi|^t dx \\ &\le \xi \int_{A_n \bigcap \{|u| \le \delta\}} (|u_n - u| + |u_n|)^t |\varphi|^t dx \\ &\le 2^t \xi^t \left(\int_{A_n \bigcap \{|u| \le \delta\}} |u_n - u|^t |\varphi|^t dx + \int_{A_n \bigcap \{|u| \le \delta\}} |u_n|^t |\varphi|^t dx \right) \\ &\le 2^t \xi^t \left(\left(\int_{A_n \bigcap \{|u| \le \delta\}} |u_n - u|^{2t} dx \right)^{\frac{1}{2}} + \left(\int_{A_n \bigcap \{|u| \le \delta\}} |u_n|^{2t} dx \right)^{\frac{1}{2}} \right) \left(\int_{A_n \bigcap \{|u| \le \delta\}} |\varphi|^{2t} dx \right)^{\frac{1}{2}} \\ &\le \xi^t C \|\varphi\|^t_\varepsilon \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} &\int_{B_n \bigcap \{|u| \le \delta\}} |f(u_n - u) - f(u_n)|^t |\varphi|^t dx \\ &\le \xi^t \left(\int_{B_n \bigcap \{|u| \le \delta\}} (|u_n - u|^{q-1} + |u_n|^{q-1})^t |\varphi|^t dx \right) \\ &\le 2^t \xi^t \left(\int_{B_n \bigcap \{|u| \le \delta\}} |u_n - u|^{t(q-1)} |\varphi|^t dx + \int_{B_n \bigcap \{|u| \le \delta\}} |u_n|^{t(q-1)} |\varphi|^t dx \right) \end{aligned}$$

$$
\begin{aligned}
&\leq 2^t\xi^t\Bigg[\left(\int_{B_n\bigcap\{|u|\leq\delta\}}|u_n-u|^{tq}dx\right)^{\frac{q-1}{q}}\left(\int_{B_n\bigcap\{|u|\leq\delta\}}|\varphi|^{tq}dx\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{B_n\bigcap\{|u|\leq\delta\}}|u_n-u|^{tq}dx\right)^{\frac{q-1}{q}}\left(\int_{B_n\bigcap\{|u|\leq\delta\}}|\varphi|^{tq}dx\right)^{\frac{1}{q}}\Bigg]\\
&\leq 2^t\xi^t\Bigg[\left(\int_{B_n\bigcap\{|u|\leq\delta\}}|u_n-u|^{tq}dx\right)^{\frac{q-1}{q}}\\
&\quad+\left(\int_{B_n\bigcap\{|u|\leq\delta\}}|u_n-u|^{tq}dx\right)^{\frac{q-1}{q}}\Bigg]\left(\int_{B_n\bigcap\{|u|\leq\delta\}}|\varphi|^{tq}dx\right)^{\frac{1}{q}}\\
&\leq \xi^t C\|\varphi\|_\varepsilon^t,
\end{aligned}
\tag{3.17}
$$

and

$$
\begin{aligned}
&\int_{C_n\bigcap\{|u|\leq\delta\}}|f(u_n-u)-f(u_n)|^t|\varphi|^tdx\leq N_0^t\xi^t\int_{C_n}|\varphi|^tdx\\
&\leq N_0^t\xi^t|C_n|^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}|\varphi|^{2t}dx\right)^{\frac{1}{2}}\leq \xi^tC\|\varphi\|_\varepsilon.
\end{aligned}
\tag{3.18}
$$

Thus, putting together (3.16), (3.17) and (3.18) we get

$$
\int_{(B^c_{R_0}(0))\bigcap\{|u|\leq\delta\}}|f(u_n)-f(u_n-u)|^t|\varphi|^tdx\leq C\xi\|\varphi\|_\varepsilon.
$$

Moreover,

$$
\begin{aligned}
&\int_{(B^c_{R_0}(0))\bigcap\{|u|>\delta\}}|f(u_n)-f(u_n-u)|^t|\varphi|^tdx\\
&\leq C_\xi\int_{(B^c_{R_0}(0))\bigcap\{|u|>\delta\}}2^t(|u_n-u|^t|\varphi|^t+|u_n|^t|\varphi|^t)dx\\
&\quad+\xi\int_{(B^c_{R_0}(0))\bigcap\{|u|>\delta\}}2^t(|u_n-u|^{(q-1)t}|\varphi|^t+|u_n|^{(q-1)t}|\varphi|^t)dx\\
&\leq \xi C\|\varphi\|_\varepsilon+C_\xi 2^t\int_{(B^c_{R_0}(0))\bigcap\{|u|>\delta\}}(|u_n-u|^t+|u_n|^t)|\varphi|^tdx.
\end{aligned}
$$

In view of $u\in H_\varepsilon$ we know that $\left|(\mathbb{R}^N\setminus B_R(0))\bigcap\{|u|>\delta\}\right|\to 0$ as $R\to\infty$, then there exists $R_1>0$ such that $\left|(\mathbb{R}^N\setminus B_{R_1}(0))\bigcap\{|u|>\delta\}\right|<\xi$. We define $R_2=\max\{R_0,R_1\}$, we deduce that

$$
\begin{aligned}
&\int_{(B^c_{R_2}(0))\bigcap\{|u|>\delta\}}|u_n-u|^t|\varphi|^tdx\\
&\leq\left(\int_{(B^c_{R_2}(0))\bigcap\{|u|>\delta\}}|u_n-u|^{t\frac{2_s^*}{t}}dx\right)^{\frac{t}{2_s^*}}\left(\int_{(B^c_{R_2}(0))\bigcap\{|u|>\delta\}}|\varphi|^{t\frac{2_s^*}{t}}dx\right)^{\frac{t}{2_s^*}}
\end{aligned}
$$

$$\times\left(\int_{(B^c_{R_2}(0))\bigcap\{|u|>\delta\}} 1dx\right)^{\frac{4s-\mu}{2N-\mu}}$$
$$\le C\xi^{\frac{4s-\mu}{2N-\mu}}\|\varphi\|^t_\varepsilon.$$

In similar way, we can prove that the following inequality is true.

$$\int_{(B^c_{R_2}(0))\bigcap\{|u|>\delta\}} |u_n|^t|\varphi|^t dx \le C\xi^{\frac{2s-\mu}{2N-\mu}}\|\varphi\|^t_\varepsilon.$$

Hence,

$$\int_{B^c_{R_2}(0)} |f(w_n)+f(u)-f(u_n)|^t|\varphi|^t dx \le \xi C\|\varphi\|^t_\varepsilon.$$

It is easy to verify that

$$\int_{B_{R_2}(0)} |f(w_n)+f(u)-f(u_n)|^t|\varphi|^t dx \le C\xi\|\varphi\|_\varepsilon. \tag{3.19}$$

Since u_n is bounded, we have $u_n\to u$ $a.e.$ in $\mathbb{R}^N$, $u_n\to u$ in $L^p_{loc}(\mathbb{R}^N)$, $p\in[1,2^*_s)$. Let $l>q$, such that $lt\in(2,2^*_s)$, $\frac{l}{l-1}t(q-1)\in(2,2^*_s)$ and

$$1<\frac{l}{l-1}t\le\frac{q}{q-1}t=\left(1+\frac{1}{q-1}\right)t<\left(1+\frac{N}{N-\mu}\right)\cdot\frac{2N}{2N-\mu}<\frac{2N}{N-2s}=2^*_s.$$

Hence, we have

$$\begin{aligned}
&|f(u_n-u)-f(u_n)+f(u)|^{t\frac{l}{l-1}}\\
&\quad\le C(|u_n-u|+|u_n-u|^{q-1}+|u_n|+|u_n|^{q-1}+|u|+|u|^{q-1})^{t\frac{l}{l-1}}\\
&\quad\le C(|u_n-u|^{t\frac{l}{l-1}}+|u_n-u|^{t\frac{l}{l-1}(q-1)}+|u_n|^{t\frac{l}{l-1}}+|u_n|^{t\frac{l}{l-1}(q-1)}+|u|^{t\frac{l}{l-1}}+|u|^{t\frac{l}{l-1}(q-1)})\\
&\quad=: Ch_n.
\end{aligned}$$

Thus,

$$\begin{aligned}
&2\int_{B_{R_2}(0)} C(|u|^{\frac{l}{l-1}t}+|u|^{\frac{l}{l-1}t(q-1)})dx\\
&=\int_{B_{R_2}(0)}\lim_{n\to\infty}(Ch_n-|f(u_n-u)-f(u_n)+f(u)|^{\frac{l}{l-1}t})dx\\
&\le\lim_{n\to\infty}\left[C\int_{B_{R_2}(0)}h_n dx-\int_{B_{R_2}(0)}|f(u_n-u)-f(u_n)+f(u)|^{t\frac{l}{l-1}}dx\right]\\
&=\lim_{n\to\infty}C\int_{B_{R_2}(0)}h_n dx-\limsup_{n\to\infty}\int_{B_{R_2}(0)}|f(u_n-u)-f(u_n)+f(u)|^{t\frac{l}{l-1}}dx\\
&=2\int_{B_{R_2}(0)}C(|u|^{\frac{l}{l-1}t}+|u|^{\frac{l}{l-1}t(q-1)})dx\\
&\quad-\limsup_{n\to\infty}\int_{B_{R_2}(0)}|f(u_n-u)-f(u_n)+f(u)|^{t\frac{l}{l-1}}dx.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
0 &\leq \liminf_{n\to\infty} \int_{B_{R_2}(0)} |f(u_n-u)-f(u_n)+f(u)|^{t\frac{l}{l-1}}dx \\
&\leq \limsup_{n\to\infty} \int_{B_{R_2}(0)} |f(u_n-u)-f(u_n)+f(u)|^{t\frac{l}{l-1}}dx \\
&\leq 0.
\end{aligned}
$$

So, we obtain (3.19) holds. By applying Hölder inequality, for any $\xi > 0$, n large enough, we have

$$
\begin{aligned}
&\int_{B_{R_2}(0)} |f(u_n-u)-f(u_n)+f(u)|^t|\varphi|^t dx \\
&\leq \left(\int_{B_{R_2}(0)} |f(u_n-u)-f(u_n)+f(u)|^{t\frac{l}{l-1}}dx\right)^{\frac{l-1}{l}} \left(\int_{B_{R_2}(0)} |\varphi|^{tl}dx\right)^{\frac{1}{l}} \\
&< \xi\|\varphi\|_\varepsilon^t = \xi.
\end{aligned}
$$

As a consequence, $\int_{\mathbb{R}^N} |f(u_n-u)-f(u_n)+f(u)|^t|\varphi|^t dx \leq \xi\|\varphi\|_\varepsilon^t = \xi$.
(iv)

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u_n-u)\right) f(u_n-u)\varphi dx - \int_{\mathbb{R}^N}(\frac{1}{|x|^\mu} * F(u_n))f(u_n)\varphi dx \\
&\quad + \left(\frac{1}{|x|^\mu} * F(u)\right) f(u)\varphi dx \\
&= \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u_n-u)\right)(f(u_n-u)-f(u_n)+f(u))\varphi dx \\
&\quad + \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u)\right)(f(u)-f(u_n))\varphi dx \\
&\quad + \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * (F(u_n-u)-F(u_n)+F(u))\right) f(u)\varphi dx \\
&\quad - \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u_n-u)\right) f(u)\varphi dx \\
&= I_1 + I_2 + \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u)\right)(f(u)-f(u_n)+f(u_n-u))\varphi dx \\
&\quad - \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u)\right) f(u_n-u)\varphi dx - \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u_n-u)\right) f(u)\varphi dx.
\end{aligned}
$$

Clearly, we have

$$
|I_1| \leq \left(\int_{\mathbb{R}^N} |F(u_n-u)|^t dx\right)^{\frac{1}{t}} \left(\int_{\mathbb{R}^N} |f(u_n-u)-f(u_n)+f(u)|^t|\varphi|^t dx\right)^{\frac{1}{t}} \leq C\xi\|\varphi\|_\varepsilon.
$$

$$
\begin{aligned}
|I_2| &\leq \left(\int_{\mathbb{R}^N} |f(u_n)|^t|\varphi|^t dx\right)^{\frac{1}{t}} \left(\int_{\mathbb{R}^N} |F(u_n-u)-F(u_n)+F(u)|^t\right)^{\frac{1}{t}} \\
&\leq \left(\int_{\mathbb{R}^N} |f(u_n)|^{t\cdot\frac{l}{l-1}}dx\right)^{\frac{l-1}{lt}} \left(\int_{\mathbb{R}^N} |\varphi|^{lt}dx\right)^{\frac{1}{lt}} \xi \\
&\leq C\xi\|\varphi\|_\varepsilon.
\end{aligned}
$$

In similar way, we get $|I_3| \leq C\xi\|\varphi\|_\varepsilon$. Let us observe that,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u)\right) f(u_n-u)\varphi dx\right| \\
&\quad\leq C\int_{\mathbb{R}^N}(\frac{1}{|x|^\mu} * F(u))(|u_n-u|+|u_n-u|^{q-1})|\varphi|dx \\
&\quad\leq C\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u)\right)|u_n-u||\varphi|dx + C\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u)\right)|u_n-u|^{q-1}|\varphi|dx \\
&\quad\leq C\left(\int_{\mathbb{R}^N}|\varphi|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u)\right)^2|u_n-u|^2dx\right)^{\frac{1}{2}} \\
&\qquad + C\left(\int_{\mathbb{R}^N}|\varphi|^{qt}dx\right)^{\frac{1}{qt}}\left(\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u)\right)^{\frac{qt}{qt-1}}|u_n-u|^{\frac{(q-1)qt}{qt-1}}dx\right)^{\frac{qt-1}{qt}}.
\end{aligned}$$

Since $\frac{1}{|x|^\mu} * F(u) \in L^{\frac{2N}{\mu}}(\mathbb{R}^N)$, we have $(\frac{1}{|x|^\mu} * F(u))^2 \in L^{\frac{N}{\mu}}(\mathbb{R}^N)$ and $\frac{2N}{N-\mu} \in (2, 2_s^*)$. so, we have $|u_n-u|^2 \in L^{\frac{N}{N-\mu}}(\mathbb{R}^N)$. Since $|u_n-u|^2 \rightharpoonup 0$ in $L^{\frac{N}{N-\mu}}(\mathbb{R}^N)$, we deduce that

$$\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u)\right)^2|u_n-u|^2 \to 0.$$

Moreover, we have $\frac{qt}{qt-1}\cdot\frac{qt-1}{qt}\cdot\frac{2N}{\mu} = \frac{2N}{\mu}$, $\frac{(q-1)qt}{qt-1}\cdot\frac{2N(qt-1)}{(2N-\mu)qt-2N} = qt \in (2, 2_s^*)$ and $|u_n-u|^{\frac{(q-1)qt}{qt-1}} \rightharpoonup 0$ in $L^{\frac{2N(qt-1)}{(2N-\mu)qt-2N}}(\mathbb{R}^N)$, we get $(\frac{1}{|x|^\mu} * F(u))^{\frac{qt}{qt-1}} \in L^{\frac{2N(qt-1)}{qt\mu}}(\mathbb{R}^N)$, $|u_n-u|^{\frac{(q-1)qt}{qt-1}} \in L^{\frac{2N(qt-1)}{(2N-\mu)qt-2N}}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N}(\frac{1}{|x|^\mu} * F(u))^{\frac{qt}{qt-1}}|u_n-u|^{\frac{(q-1)qt}{qt-1}}dx \to 0$. Hence, we have

$$\left|\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u)\right) f(u_n-u)\varphi\right| \leq C\xi\|\varphi\|_\varepsilon.$$

In similar way, $|\int_{\mathbb{R}^N}(\frac{1}{|x|^\mu} * F(u_n-u))f(u)\varphi| \leq C\xi\|\varphi\|_\varepsilon$. Therefore (iv) holds. □

By using Brezis–Lieb Lemma [10,18] and Lemma 3.9, we have the following lemma.

Lemma 3.10 *Let $\{u_n\} \subset H_\varepsilon$ be a $(PS)_d$ sequence of J_ε with $u_n \rightharpoonup u$ in H_ε, then*

(i) $J_\varepsilon(w_n) = J_\varepsilon(u_n) - J_\varepsilon(u) + o_n(1)$,
(ii) $\|J'_\varepsilon(w_n)\| = o_n(1)$.

Proof (i) We note that

$$\begin{aligned}
&J_\varepsilon(u_n-u) - J_\varepsilon(u_n) + J_\varepsilon(u) \\
&\quad= \frac{1}{2}(\|u_n-u\|_\varepsilon^2 - \|u_n\|_\varepsilon^2 + \|u\|_\varepsilon^2) - \frac{1}{2}\left(\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F((u_n-u)^+)\right)F((u_n-u)^+)dx\right. \\
&\qquad\left. - \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u_n^+)\right)F(u_n^+)dx + \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(u)\right)F(u)dx\right) \\
&\qquad - \frac{1}{2_s^*}\int_{\mathbb{R}^N}(|(u_n-u)^+|^{2_s^*} - |u_n^+|^{2_s^*} + |u^+|^{2_s^*})dx.
\end{aligned}$$

By the Lemma 3.9 (ii), $u_n \rightharpoonup u$ in H_ε and Brezis–Lieb Lemma. we have (i) holds.

(ii) Recall that $\{u_n\}$ is a $(PS)_d$ sequence of J_ε, we have $\|J'_\varepsilon(u_n)\| = o_n(1)$, $J'_\varepsilon(u) = 0$. For any $\xi > 0$, n large enough, $\forall\, \varphi \in H_\varepsilon$ and $\|\varphi\|_\varepsilon = 1$, by the Lemma 3.9 (iv) we get

$$
\begin{aligned}
&|\langle J'_\varepsilon(u_n - u), \varphi\rangle| \\
&= \Bigg| \langle J'_\varepsilon(u_n), \varphi\rangle - \langle J'_\varepsilon(u), \varphi\rangle - \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F((u_n - u)^+)\right) F((u_n - u)^+)\varphi dx \\
&\quad - \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u_n^+)\right) f(u_n^+)\varphi dx \\
&\quad - \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u^+)\right) f(u^+)\varphi dx - \int_{\mathbb{R}^N} (|(u_n - u)^+|^{2_s^*-1} - |u_n^+|^{2_s^*-1} + |u^+|^{2_s^*-1})\varphi dx \Bigg| \\
&\le \|J'_\varepsilon(u_n)\| \|\varphi\|_\varepsilon + \Bigg| \int_{\mathbb{R}^N} (\frac{1}{|x|^\mu} * F((u_n - u)^+)) f((u_n - u)^+)\varphi dx \\
&\quad - \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u_n^+)\right) f(u_n^+)\varphi dx \\
&\quad + \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u^+)\right) f(u^+)\varphi dx \Bigg| + \int_{\mathbb{R}^N} \left| |(u_n - u)^+|^{2_s^*-1} - |u_n^+|^{2_s^*-1} + |u^+|^{2_s^*-1} \right| |\varphi| dx \\
&\le \xi \|\varphi\|_\varepsilon + C\xi \|\varphi\|_\varepsilon + \left(\int_{\mathbb{R}^N} ||u_n - u|^{2_s^*-1} - |u_n^+|^{2_s^*-1} + |u^+|^{2_s^*-1}|^{\frac{2_s^*}{2_s^*-1}} dx \right)^{\frac{2_s^*-1}{2_s^*}} \|\varphi\|_\varepsilon \\
&\le \xi \|\varphi\|_\varepsilon + C\xi \|\varphi\|_\varepsilon + \xi \|\varphi\|_\varepsilon.
\end{aligned}
$$

This completes the proof of (ii). □

Lemma 3.11 *J_ε satisfies the $(PS)_d$ condition at any level $d \le m_{V_\infty}$.*

Proof Let $u_n \subset H_\varepsilon$ be a $(PS)_d$ sequence of J_ε. Then, by Lemma 3.4 we know that $\{u_n\}$ is bounded in H_ε and we can assume $u_n \ge 0$. Hence, up to a subsequence, there is $u \in H_\varepsilon$ such that $u_n \rightharpoonup u \ge 0$ in H_ε, $u_n \to u$ in $L^r_{loc}(\mathbb{R}^N)$ for each $r \in [2, 2_s^*)$, $u_n(x) \to u(x)$ $a.e.$ in $\mathbb{R}^N$ and $J'_\varepsilon(u) = 0$. Set $w_n = u_n - u$, by Lemma 3.9 we have

$$J_\varepsilon(w_n) = J_\varepsilon(u_n) - J_\varepsilon(u) + o_n(1) = d - J_\varepsilon(u) + o_n(1) \quad \text{and} \quad J'_\varepsilon(w_n) = o_n(1).$$

Moreover, for any $\alpha \in (2, 2_s^*)$ and $\alpha \le 4$, we have

$$
\begin{aligned}
J_\varepsilon(u) &= J_\varepsilon(u) - \frac{1}{\alpha}\langle J'_{\varepsilon(u),u}\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\alpha}\right) \|u_n\|_\varepsilon^2 + \frac{1}{\alpha} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u_n)\right) f(u_n) u_n dx \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u_n)\right) F(u_n) dx \\
&\quad - \frac{1}{2_s^*} \int_{\mathbb{R}^N} |u_n|^{2_s^*} dx + \frac{1}{\alpha} \int_{\mathbb{R}^N} |u_n|^{2_s^*-2} u_n^2 dx \\
&\ge 0.
\end{aligned}
$$

By Lemma 3.6 we have $d - J_\varepsilon(u) \le d \le m_{V_\infty} < \frac{s}{N} S^{\frac{N}{2s}}$ and by Lemma 3.8 we know $u_n \to u$ in H_ε. Hence, the Lemma is proved. □

By Lemmas 3.3 and 3.11 we have the following lemma.

Lemma 3.12 *$J_\varepsilon|_{\mathcal{N}_\varepsilon}$ satisfies the $(PS)_d$ condition at any level $d < m_{V_\infty}$.*

Proof of Theorem 1.1 By Lemma 3.1 we know that functional J_ε satisfies the mountain pass geometry, then using a version of the mountain pass theorem, there exists a sequence $\{u_n\} \subset H_\varepsilon$ such that

$$\lim_{n\to\infty} J_\varepsilon = c_\varepsilon \quad \text{and} \quad (1 + \|u_n\|_\varepsilon)\|J_\varepsilon'\| = o_n(1).$$

For any $\tau \in \mathbb{R}$ with $V_0 < \tau < V_\infty$, we have $m_{V_0} < m_\tau < m_{V_\infty}$. By Lemma 3.6, $m_\tau < \frac{s}{N} S^{\frac{N}{2s}}$. Apply Lemma 3.4, Lemma 3.11 and Theorem 6.3.4 in [49], we obtain that m_τ is a critical value of I_τ with corresponding nontrivial nonnegative critical point $u \in H_\varepsilon$. For any $r > 0$, take $\eta_r \in C_0^\infty(\mathbb{R}^N, [0, 1])$ be such that

$$\eta_r = 1 \text{ if } |x| < r \quad \text{and} \quad \eta_r = 0 \text{ if } |x| > 2r.$$

Set $u_r := \eta_r u$, it is easy to verify that $u_r \in H_\varepsilon$ for each $r > 0$. By Lemma 3.2 there exists $t_r > 0$ such that $\tilde{u}_r := t_r u_r \in \mathcal{M}_\tau$. Consequently, there is $r_0 > 0$ such that $\tilde{u} := \tilde{u}_{r_0}$ satisfies $I_\tau(\tilde{u}) < m_{V_\infty}$. In fact, if this is false, then $I_\tau(\tilde{u}_r) = I_\tau(t_r u_r) \geq m_{V_\infty}$ for all $r > 0$. Notice that $u_r \to u$ in H_ε as $r \to +\infty$ and $u \in \mathcal{M}_\tau$. we can deduce that $t_r \to 1$ as $r \to +\infty$. Hence,

$$m_{V_\infty} \leq \liminf_{r\to+\infty} I_\tau(t_r u_r) = I_\tau(u) = m_\tau < m_{V_\infty},$$

which gives a contradiction, then $I_\tau(\tilde{u}) < m_{V_\infty}$. The invariance by translation, we may assume $V_0 = V(0) < \tau$ and $supp(\tilde{u})$ is compact. We use the continuity of V, there is an $\varepsilon^* > 0$ such that

$$V(\varepsilon x) < \tau, \quad \forall \varepsilon \in (0, \varepsilon^*) \quad \text{and} \quad x \in supp(\tilde{u}).$$

Hence,

$$J_\varepsilon(t\tilde{u}) \leq I_\tau(t\tilde{u}), \quad \forall \varepsilon \in (0, \varepsilon^*) \quad \text{and} \quad t \geq 0,$$

and

$$\max_{t\geq 0} J_\varepsilon(t\tilde{u}) \leq \max_{t\geq 0} I_\tau(t\tilde{u}) = I_\tau(\tilde{u}) < m_{V_\infty}, \quad \forall \;\; \varepsilon \in (0, \varepsilon^*).$$

Consequently,

$$c_\varepsilon < m_{V_\infty}, \quad \forall \;\; \varepsilon \in (0, \varepsilon^*).$$

Lemma 3.11 guarantees up to a subsequence such that $u_n \to u$ in H_ε, then $J_\varepsilon'(u) = 0$ and $J_\varepsilon(u) = c_\varepsilon$. Hence u is a ground nontrivial nonnegative solution of (2.2). This completes the proof of Theorem 1.1. □

4 Multiplicity results

4.1 Technical results

In this section, we focus our attention on the study of the multiplicity of solutions to (1.1). Since $V_0 > 0$, by Lemma 3.6, $m_{V_0} < \frac{s}{N} S^{\frac{N}{2s}}$. From the proof of Theorem 1.1 we know that m_{V_0} is a critical value of I_{V_0} with corresponding nontrivial nonnegative critical point

$w \in H^s(\mathbb{R}^N)$. Fix $\delta > 0$ and let $\eta \in C^\infty(\mathbb{R}^+, [0, 1])$ be a function such that $\eta(t) = 1$ if $0 \le t \le \frac{\delta}{2}$ and $\eta(t) = 0$ if $t \ge \delta$. For any $y \in \Lambda$, we define

$$\Psi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|)w\left(\frac{\varepsilon x - y}{\varepsilon}\right), \quad \forall x \in \mathbb{R}^N.$$

Then for small $\varepsilon > 0$, one has $\Psi_{\varepsilon,y} \in H_\varepsilon \backslash \{0\}$ for all $y \in \Lambda$. In fact, using the change of variable $z = x - \frac{y}{\varepsilon}$, one has

$$\begin{aligned}\int_{\mathbb{R}^N} V(\varepsilon x)\Psi^2_{\varepsilon,y}(x)dx &= \int_{\mathbb{R}^N} V(\varepsilon x)\eta^2(|\varepsilon x - y|)w^2\left(\frac{\varepsilon x - y}{\varepsilon}\right)dx \\ &= \int_{\mathbb{R}^N} V(\varepsilon z + y)\eta^2(|\varepsilon z|)w^2(z)dz \\ &\le C\int_{\mathbb{R}^N} w^2(z)dz < +\infty.\end{aligned}$$

Moreover, using the change of variable $x' = x - \frac{y}{\varepsilon}$, $z' = z - \frac{y}{\varepsilon}$, we have

$$\begin{aligned}\|(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,y}\|^2_{L^2(\mathbb{R}^N)} &= \frac{1}{2}C(s)\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\left|\eta(|\varepsilon x - y|)w\left(\frac{\varepsilon x-y}{\varepsilon}\right) - \eta(|\varepsilon z - y|)w\left(\frac{\varepsilon z-y}{\varepsilon}\right)\right|^2}{|x - z|^{N+2s}}dxdz \\ &= \frac{1}{2}C(s)\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\left|\eta(|\varepsilon x'|)w(x') - \eta(|\varepsilon z'|)w(z')\right|^2}{|x' - z'|^{N+2s}}dx'dz' \\ &= \|(-\Delta)^{\frac{s}{2}}\eta(|\varepsilon x|)w(x)\|^2_{L^2(\mathbb{R}^N)} = \|(-\Delta)^{\frac{s}{2}}\eta_\varepsilon w\|^2_{L^2(\mathbb{R}^N)},\end{aligned}$$

where $\eta_\varepsilon(x) = \eta(|\varepsilon x|)$. By Lemma 2.4, we see that $\eta_\varepsilon w \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ as $\varepsilon \to 0$, and hence $\Psi_{\varepsilon,y} \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ for $\varepsilon > 0$ small. Hence $\Psi_{\varepsilon,y} \in H_\varepsilon$. Now we proof $\Psi_{\varepsilon,y} \neq 0$. In fact,

$$\begin{aligned}\int_{\mathbb{R}^N} \Psi^2_{\varepsilon,y}(x)dx &= \int_{\mathbb{R}^N} \eta^2(|\varepsilon x - y|)w^2\left(\frac{\varepsilon x - y}{\varepsilon}\right)dx = \int_{|\varepsilon x - y|<\delta} \eta^2(|\varepsilon x - y|)w^2\left(\frac{\varepsilon x - y}{\varepsilon}\right)dx \\ &\ge \int_{|z|\le\frac{\delta}{2\varepsilon}} \eta^2(|\varepsilon z|)w^2(z)dz \ge \int_{B_0(\frac{\delta}{2\varepsilon})} w^2(z)dz \to \int_{\mathbb{R}^N} w^2(z)dz > 0\end{aligned}$$

as $\varepsilon \to 0$. Then $\Psi_{\varepsilon,y} \neq 0$ for small $\varepsilon > 0$. Therefore, there exists unique $t_\varepsilon > 0$ such that

$$\max_{t\ge0} I_\varepsilon(t\Psi_{\varepsilon,y}) = I_\varepsilon(t_\varepsilon\Psi_{\varepsilon,y}) \text{ and } t_\varepsilon\Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon.$$

We introduce the map $\Phi_\varepsilon : \Lambda \to \mathcal{N}_\varepsilon$ by setting

$$\Phi_\varepsilon(y) = t_\varepsilon\Psi_{\varepsilon,y}.$$

By construction, $\Phi_\varepsilon(y)$ has a compact support for any $y \in \Lambda$ and Φ_ε is a continuous map.

Lemma 4.1

$$\lim_{\varepsilon\to0} J_\varepsilon(\Phi_\varepsilon(y)) = m_{V_0} \quad \textit{uniformly in} \quad y \in \Lambda.$$

Proof Assume by contradiction, then there exists $\delta_0 > 0$, $\{y_n\} \subset \Lambda$ and $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - m_{V_0}| \ge \delta_0. \tag{4.1}$$

By using $\Phi_{\varepsilon_n} \in \mathcal{N}_{\varepsilon_n}$ and Lemma 3.5 we know that there is a $r_0 > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(\Phi_{\varepsilon_n}(y_n))\right) f(\Phi_{\varepsilon_n}(y_n))\Phi_{\varepsilon_n}(y_n)dx + \int_{\mathbb{R}^N} |\Phi_{\varepsilon_n}(y_n)|^{2_s^*} dx \\ = \|\Phi_{\varepsilon_n}(y_n)\|_{\varepsilon_n}^2 \\ \geq r_0\end{aligned} \tag{4.2}$$

which implies that $t_\varepsilon \nrightarrow 0$. Hence there exists a $T > 0$ such that $t_{\varepsilon_n} \geq T$. If $t_{\varepsilon_n} \to \infty$, we have

$$\begin{aligned}C\|w\|_\varepsilon^2 &\geq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon_n, y_n}|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon_n x)\Psi_{\varepsilon_n, y_n}^2 dx \\ &= t_{\varepsilon_n}^{-2} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(\Phi_{\varepsilon_n}(y_n))\right) f(\Phi_{\varepsilon_n}(y_n))\Phi_{\varepsilon_n}(y_n)dx + t_{\varepsilon_n}^{-2} \int_{\mathbb{R}^N} |\Phi_{\varepsilon_n}(y_n)|^{2_s^*} dx \\ &\geq t_{\varepsilon_n}^{-2} \int_{\mathbb{R}^N} |t_{\varepsilon_n} \Psi_{\varepsilon_n, y_n}|^{2_s^*} dx \\ &\geq t_{\varepsilon_n}^{-2} \int_{\mathbb{R}^N} |t_{\varepsilon_n} \eta(|\varepsilon_n x|) w(x)|^{2_s^*} dx \\ &\geq t_{\varepsilon_n}^{-2} \int_{|x| < \frac{\delta}{2\varepsilon_n}} |t_{\varepsilon_n} w(x)|^{2_s^*} dx \\ &\geq t_{\varepsilon_n}^{2_s^* - 2} \int_{\frac{\delta}{2} < |x| < \delta} |w(x)|^{2_s^*} dx \\ &\to +\infty\end{aligned}$$

for large n. This yield a contradiction, then $t_\varepsilon \to t_0 > 0$. Now we claim that $t_0 \to 1$. By using Lebesgue's theorem, we can verify that

$$\lim_{n\to\infty} \|\Phi_{\varepsilon_n}(y_n)\|_\varepsilon^2 = t_0^2 \|w\|_{V_0}^2,$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(\Phi_{\varepsilon_n}(y_n))\right) f(\Phi_{\varepsilon_n}(y_n))\Phi_{\varepsilon_n}(y_n)dx = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(t_0 w)\right) f(t_0 w)t_0 w dx,$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\Phi_{\varepsilon_n}(y_n)|^{2_s^*} dx = \int_{\mathbb{R}^N} |t_0 w|^{2_s^*} dx.$$

Therefore, from (4.2), we get

$$t_0^2 \|w\|_{V_0}^2 = \int_{\mathbb{R}^N} (\frac{1}{|x|^\mu} * F(t_0 w)) f(t_0 w) t_0 w dx + \int_{\mathbb{R}^N} |t_0 w|^{2_s^*} dx.$$

This show $t_0 w \in \mathcal{M}_{V_0}$. Noting that $w \in \mathcal{M}_{V_0}$, we see $t_0 = 1$, so claim is proved. Moreover, similar to the above arguments, we can get

$$\lim_{n\to\infty} J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = I_{V_0}(w) = m_{V_0}$$

which contradicts to (4.1). This completes the proof. □

Now, we are ready to introduce the barycenter map. For any $\delta > 0$, let $\rho = \rho(\delta) > 0$ such that $\Lambda_\delta \subset B_\rho(0)$. Define $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ as follow:

$$\Upsilon(x) = \begin{cases} x & \text{if } |x| < \rho, \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

We define the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ as follows

$$\beta_\varepsilon = \frac{\int_{\mathbb{R}^N} \Upsilon(\varepsilon x)|w(x)|^2 dx}{\int_{\mathbb{R}^N} |w(x)|^2 dx}.$$

Lemma 4.2

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \quad \textit{uniformly in} \quad y \in \Lambda.$$

Proof Assume by contradiction, then there exists $\delta_0 > 0$, $\{y_n\} \subset \Lambda$ and $\varepsilon_n \to 0^+$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0 > 0, \quad \forall n \in \mathbb{N}. \tag{4.3}$$

By using the definitions of β_{ε_n} and Φ_{ε_n}, we can see that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N} [\Upsilon(\varepsilon_n x + y_n) - y_n]|\eta(|\varepsilon_n x|)w(x)|^2 dx}{\int_{\mathbb{R}^N} |\eta(|\varepsilon_n x|)w(x)|^2 dx}.$$

Taking into account the Lebesgue dominant convergence theorem, we can infer that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \to 0$$

which contradicts (4.3). □

Lemma 4.3 *For any $\tau > 0$, let $\{u_n\} \subset \mathcal{M}_\tau$ with $I_\tau(u_n) \to m_\tau$. Then $\{u_n\}$ has a subsequence strongly convergent in $H^s(\mathbb{R}^N)$. In Particular, there exists a minimizer for m_τ.*

Proof From the proof of Lemmas 3.4 and 3.6, we know that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and $m_\tau < \frac{s}{N} S^{\frac{N}{2s}}$. By the Ekeland Variational principle, we may assume that $\{u_n\}$ is a $(PS)_{m_\tau}$ sequence of I_τ. Then, by Lemma 3.8, there exists $u \in H^s(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \to u$ in $H^s(\mathbb{R}^N)$. Moreover, u is a minimizer of m_τ. □

Lemma 4.4 *Let $\varepsilon_n \to 0$ and $u_n \in \mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(u_n) \to m_{V_0}$. Then there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot + y_n)$ has a convergent subsequence in $H^s(\mathbb{R}^N)$. Moreover, up to a subsequence, $\tilde{y}_n = \varepsilon_n y_n \to y \in \Lambda$.*

Proof Since $u_n \in \mathcal{N}_{\varepsilon_n}$ and $\lim_{n\to\infty} J_{\varepsilon_n}(u_n) = m_{V_0}$, by Lemma 3.4 we can see that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. By Lemma 3.5, we have $\|u_n\|_{\varepsilon_n} \nrightarrow 0$. we can argue as in Lemma 3.7 to obtain a sequence $\{y_n\}$ and constant $r > 0$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n(x)|^2 dx = \beta > 0. \tag{4.4}$$

Note, if this is false, then for any $r > 0$, we have

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^2 dx = 0.$$

By Lemma 2.2, we know that $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for $t \in [2, 2_s^*)$, we can argue as the proof of (3.2) and we deduce that

$$\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u_n) \right) f(u_n) u_n dx = o_n(1).$$

As the proof of Lemma 3.7, we can prove $\int_{\mathbb{R}^N} |u|^{2_s^*} dx = o_n(1)$. Since $u_n \in \mathcal{N}_{\varepsilon_n}$, we get $\|u_n\|_{\varepsilon_n} = o_n(1)$, which gives a contradiction. Hence, (4.4) holds. Now, we set $\tilde{u}_n =$

$u_n(\cdot + y_n)$. Since, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and (4.4), up to a subsequence, we have $\tilde{u}_n \rightharpoonup \tilde{u} \neq 0$ in $H^s(\mathbb{R}^N)$ and $\tilde{u}_n(x) \to \tilde{u}(x)$ $a.e.$ in $\mathbb{R}^N$. Fix $t_n > 0$ such that $t_n\tilde{u}_n \in \mathcal{M}_{V_0}$ and set $\tilde{y}_n = \varepsilon_n y_n$. Since $u_n \in \mathcal{N}_{\varepsilon_n}$, we can see that

$$\begin{aligned}
m_{V_0} &\leq I_{V_0}(t_n\tilde{u}_n)\\
&= \frac{1}{2}t_n^2[\tilde{u}_n]^2 + \frac{t_n^2}{2}\int_{\mathbb{R}^N} V_0\tilde{u}_n^2 dx - \frac{1}{2_s^*}\int_{\mathbb{R}^N} |t_n\tilde{u}_n^+|^{2_s^*}dx - \frac{1}{2}\left(\frac{1}{|x|^\mu} * F(t_n\tilde{u}_n^+)\right)F(t_n\tilde{u}_n^+)dx\\
&\leq J_{\varepsilon_n}(t_nu_n)\\
&\leq J_{\varepsilon_n}(u_n)\\
&= m_{V_0} + o_n(1),
\end{aligned}$$

which gives

$$\lim_{n\to\infty} I_{V_0}(t_n\tilde{u}_n) = m_{V_0} > 0.$$

By Lemma 4.3, up to subsequence, we get $t_n\tilde{u}_n := v_n \to v_0$ in $H^s(\mathbb{R}^N)$. Note that,

$$\beta = \liminf_{n\to\infty}\int_{B_r(y_n)} |u_n(x)|^2 dx = \liminf_{n\to\infty}\int_{B_r(0)} |\tilde{u}_n(x)|^2 dx \leq \liminf_{n\to\infty}\|\tilde{u}_n\|^2_{H^s(\mathbb{R}^N)}.$$

For large n, we have $0 < \frac{\beta}{2} < \|\tilde{u}_n\|^2_{H^s(\mathbb{R}^N)}$, then

$$0 < \frac{\beta}{2}t_n^2 < \|t_n\tilde{u}_n\|^2_{H^s(\mathbb{R}^N)} = \|v_n\|^2_{H^s(\mathbb{R}^N)} \leq C.$$

Hence $\{t_n\}$ is bounded, and we may assume that $t_n \to t^* > 0$. So, up to a subsequence, we have

$$v_n \to v_0 = t^*\tilde{u} \neq 0 \text{ in } H^s(\mathbb{R}^N), \quad \tilde{u}_n \to \frac{1}{t^*}v_0 = \tilde{u} \text{ in } H^s(\mathbb{R}^N).$$

In order to complete the proof of the lemma, we show that $\{\tilde{y}_n\}$ is bounded in $\mathbb{R}^N$. We argue by contradiction, up to a subsequence, we assume that $|\tilde{y}_n| \to \infty$. Notice that, up to subsequence, we have $v_n \to v_0 \neq 0$ in $H^s(\mathbb{R}^N)$. By Fatou's lemma we get

$$\begin{aligned}
m_{V_0} &= I_{V_0}(v_0)\\
&< I_{V_\infty}(v_0) - \frac{1}{2}\langle I'_{V_0}(v_0), v_0\rangle\\
&= \frac{1}{2}\int_{\mathbb{R}^N}(V_\infty v_0^2 - V_0v_0^2)dx - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(v_0^+)\right)F(v_0^+)dx\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * F(v_0^+)\right)f(v_0^+)v_0^+dx\\
&\quad - \frac{1}{2_s^*}\int_{\mathbb{R}^N}|v_0^+|^{2_s^*}dx + \frac{1}{2}\int_{\mathbb{R}^N}|v_0^+|^{2_s^*}dx\\
&\leq \liminf_{n\to\infty}(J_{\varepsilon_n}(v_n) - \frac{1}{2}\langle I'_{V_0}(v_n), v_n\rangle)\\
&= \liminf_{n\to\infty} J_{\varepsilon_n}(v_n)\\
&\leq \lim_{n\to\infty} J_{\varepsilon_n}(u_n) = m_{V_0}
\end{aligned}$$

which is a contradiction, so we get $\{\tilde{y}_n\}$ is bounded in $\mathbb{R}^N$. Therefore, up to subsequence, $\tilde{y}_n \to y \in \mathbb{R}^N$. If $y \in \mathbb{R}^N \backslash \Lambda$ then $V_0 < V(y)$. This is a contradiction. Hence, we can conclude that $y \in \Lambda$. □

Now, we introduce a subset $\tilde{\mathcal{N}}_\varepsilon$ of $\mathcal{N}_\varepsilon$ by setting

$$\tilde{\mathcal{N}}_\varepsilon = \{u \in \mathcal{N}_\varepsilon : J_\varepsilon(u) \le m_{V_0} + h(\varepsilon)\},$$

where $h(\varepsilon) := \max\limits_{y\in\Lambda} |J_\varepsilon(\Phi_\varepsilon(y)) - m_{V_0}|$. Then, we can use Lemma 4.1 to conclude that

$$\lim_{\varepsilon\to 0^+} h(\varepsilon) = 0.$$

Hence, for each $y \in \Lambda$ and $\varepsilon > 0$, we have $\Phi_\varepsilon(y) \in \tilde{\mathcal{N}}_\varepsilon$. By Lemma 4.4, we can prove the following Lemma.

Lemma 4.5 *For any $\delta > 0$, we have*

$$\lim_{\varepsilon\to 0} \sup_{u\in\tilde{\mathcal{N}}_\varepsilon} dist(\beta_\varepsilon(u), \Lambda_\delta) = 0.$$

Proof Let $\varepsilon_n \to 0$. For any $n \in \mathbb{N}$, there exists $\{u_n\} \subset \tilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\inf_{y\in\Lambda_\delta} |\beta_{\varepsilon_n}(u_n) - y| = \sup_{u\in\tilde{\mathcal{N}}} \inf_{y\in\Lambda_\delta} |\beta_{\varepsilon_n}(u) - y| + o_n(1).$$

Since $\{u_n\} \in \mathcal{N}_{\varepsilon_n}$, it follow that

$$m_{V_0} \le c_{\varepsilon_n} \le J_{\varepsilon_n(u_n)} \le m_{V_0} + h(\varepsilon_n).$$

Then, $J_{\varepsilon_n}(u_n) \to m_{V_0}$. By Lemma 4.4, there exists $\{y_n\} \in \mathbb{R}^N$ such that $\{\tilde{u}_n(\cdot) := u_n(\cdot + y_n)\}$ has a convergent subsequence in $H^s(\mathbb{R}^N)$ and $\tilde{y}_n := \varepsilon_n y_n \to y \in \Lambda$. Then,

$$\beta_{\varepsilon_n}(u_n) = \tilde{y}_n + \frac{\int_{\mathbb{R}^N} [\chi(\varepsilon_n x + \tilde{y}_n) - \tilde{y}_n]|\tilde{u}_n|^2 dx}{\int_{\mathbb{R}^N} |\tilde{u}_n| dx} \to y \in \Lambda.$$

The proof is completed. □

4.2 Proof of Theorem 1.2

Lemma 4.6 *Assume that (V) and (f_1)–(f_4) hold. Then, for any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that the problem (1.1) has at least $cat_{\Lambda_\delta}(\Lambda)$ nontrivial nonnegative solutions for all $\varepsilon \in (0, \varepsilon_\delta)$.*

Proof By Lemma 4.1 and the define of ψ_ε, we have

$$\lim_{\varepsilon\to 0} \psi_\varepsilon\left(n_\varepsilon^{-1}(\Phi_\varepsilon(y))\right) = \lim_{\varepsilon\to 0} J_\varepsilon(\Phi_\varepsilon(y)) = m_{V_0} \text{ uniformly in } y \in \Lambda.$$

Then, there exists $\varepsilon_1 > 0$ such that $\tilde{\mathcal{S}}_\varepsilon := \{u \in \mathcal{S}_\varepsilon : \psi_\varepsilon(u) \le m_{V_0} + h(\varepsilon)\} \ne 0$ for all $\varepsilon \in (0, \varepsilon_1)$.

Applying Lemmas 3.3, 4.1, 4.2 and 4.5, we can find some $\varepsilon_1 = \varepsilon_\delta > 0$ such that the following diagram

$$\Lambda \overset{\Phi_\varepsilon}{\to} \tilde{\mathcal{N}}_\varepsilon \overset{n_\varepsilon^{-1}}{\to} \tilde{\mathcal{S}}_\varepsilon \overset{n_\varepsilon}{\to} \tilde{\mathcal{N}}_\varepsilon \overset{\beta_\varepsilon}{\to} \Lambda_\delta$$

is well defined for any $\varepsilon \in (0, \varepsilon_1)$.By the proof of [7, Theorem5.1,Theorem5.2], we know that for $\varepsilon > 0$ small enough, we deduce from Lemma 3.12 that ψ_ε satisfies the PS condition in $\tilde{\mathcal{S}}_\varepsilon$. And ψ_ε has at least $cat_{\tilde{\mathcal{S}}_\varepsilon}(\tilde{\mathcal{S}}_\varepsilon)$ critical points on $\tilde{\mathcal{S}}_\varepsilon$. By Lemma 3.3 we conclude that J_ε admits at least $cat_{\Lambda_\delta}(\Lambda)$ critical points on $\mathcal{N}_\varepsilon$. □

Now, we use a Moser iteration argument [32] to study of behavior of the maximum points of the solutions.

Lemma 4.7 *Let $\varepsilon_n \to 0$ and $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ is a nontrivial nonnegative solution to (2.2). Then exists $y_n \in \mathbb{R}^N$ such that $v_n = u_n(\cdot + y_n)$ satisfies the following problem*

$$\begin{cases} (-\Delta)^s v_n + V_n(x) v_n = (\frac{1}{|x|^\mu} * F(v_n)) f(v_n) + |v_n|^{2_s^*-2} & in\ \mathbb{R}^N \\ v_n \in H^s(\mathbb{R}^N), \\ v_n \geq 0 & in\ \mathbb{R}^N, \end{cases} \tag{4.5}$$

where $V_n(x) = V(\varepsilon_n x + \varepsilon_n y_n)$, $\varepsilon_n y_n \to y \in \Lambda$ and there exists $C > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^N)} \leq C$ for all $n \in \mathbb{N}$. Furthermore,

$$\lim_{|x|\to\infty} v_n(x) = 0 \quad uniformly\ in \quad n \in \mathbb{N}.$$

Proof For any $L > 0$ and $\beta > 1$, let us define the function

$$r(v_n) = r_{L,\beta}(v_n) = v_n v_{L,n}^{2(\beta-1)} \in H^s(\mathbb{R}^N)$$

where $v_{L,n} = \min\{v_n, L\}$. Since r is an increasing function in $(0, +\infty)$, then we have

$$(a-b)(r(a) - r(b)) \geq 0 \quad \text{for any} \quad a, b \in \mathbb{R}^+.$$

Define the functions

$$H(t) = \frac{|t|^2}{2} \quad \text{and} \quad L(t) = \int_0^t (r'(\tau))^{\frac{1}{2}} d\tau.$$

For all $a, b \in \mathbb{R}$ such that $a > b$, by applying Jensen inequality we get

$$\begin{aligned} H'(a-b)(r(a)-r(b)) &= (a-b)(r(a)-r(b)) = (a-b)\int_b^a r'(t)dt \\ &= (a-b)\int_b^a (L'(t))^2 dt \geq (\int_b^a L'(t)dt)^2. \end{aligned}$$

In similar way, we can prove that the above inequality is true for all $a \leq b$. Therefore

$$H'(a-b)(r(a)-r(b)) \geq |L(a)-L(b)|^2 \quad \text{for any} \quad a, b \in \mathbb{R}. \tag{4.6}$$

By using (4.6), we have

$$|L(v_n)(x) - L(v_n)(y)|^2 \leq (v_n(x) - v_n(y)) \left(\left(v_n v_{L,n}^{2(\beta-1)} \right)(x) - \left(v_n v_{L,n}^{2(\beta-1)} \right)(y) \right). \tag{4.7}$$

Now, we take $r(v_n) = v_n v_{L,n}^{2(\beta-1)}$ as test-function in (4.5) and in view of (4.7), we obtain

$$\begin{aligned}
&[L(v_n)]^2 + \int_{\mathbb{R}^N} V_n(x)|v_n|^2 v_{L,n}^{2\beta-1} dx \\
&\quad \le \int\int_{\mathbb{R}^{2N}} \frac{v_n(x) - v_n(y)}{|x-y|^{N+2s}} \left(\left(v_n v_{L,n}^{2(\beta-1)} \right)(x) - \left(v_n v_{L,n}^{2(\beta-1)} \right)(y) \right) dxdy \\
&\quad\quad + \int_{\mathbb{R}^N} V_n(x)|v_n|^2 v_{L,n}^{2(\beta-1)} dx \\
&\quad = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(v_n) \right) f(v_n) v_n v_{L,n}^{2(\beta-1)} dx + \int_{\mathbb{R}^N} |v_n|^{2_s^*-2} v_n v_{L,n}^{2(\beta-1)} dx.
\end{aligned} \tag{4.8}$$

Since

$$L(v_n) \ge \frac{1}{\beta} v_n v_{L,n}^{2(\beta-1)}$$

and we can use Lemma 2.1 to deduce that

$$[L(v_n)]^2 \ge C\|L(v_n)\|^2_{L^{2_s^*}(\mathbb{R}^N)} \ge \left(\frac{1}{\beta}\right)^2 C\|v_n v_{L,n}^{(\beta-1)}\|^2_{L^{2_s^*}(\mathbb{R}^N)}. \tag{4.9}$$

On the other hand, since $\{v_n\}$ is bounded in $H^s(\mathbb{R}^N)$, there exists $C_0 > 0$ such that

$$\left\| \frac{1}{|x|^\mu} * F(v_n) \right\|_{L^\infty(\mathbb{R}^N)} < C_0. \tag{4.10}$$

Taking $\xi \in (0, V_0)$, and using (3.1), (4.9) and (4.10), we can see that (4.8) yields

$$\|v_n v_{L,n}^{\beta-1}\|^2_{L^{2_s^*}(\mathbb{R}^N)} \le C\beta^2 \left(\int_{\mathbb{R}^N} |v_n|^q v_{L,n}^{2(\beta-1)} dx + \int_{\mathbb{R}^N} |v_n|^{2_s^*} v_{L,n}^{2(\beta-1)} dx \right).$$

Set $q + 2\beta - 2 = 2_s^* \Rightarrow \beta = \frac{1}{2}(2_s^* + 2 - q) > 1$, then

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} |v_n v_{L,n}^{\beta-1}|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \\
&\quad \le C\beta^2 \left(\int_{\mathbb{R}^N} |v_n|^q v_{L,n}^{2\beta-1} dx + \int_{\mathbb{R}^N} |v_n|^{2_s^*-1} (v_n v_{L,n}^{2(\beta-1)}) dx \right) \\
&\quad \le C\beta^2 \left(\int_{\mathbb{R}^N} |v_n|^{2_s^*} dx + \int_{\{v_n \le R_0\}} |v_n|^{2_s^*-1} (v_n v_{L,n}^{2(\beta-1)}) dx + \int_{\{v_n > R_0\}} |v_n|^{2_s^*-2} (v_n v_{L,n}^{\beta-1})^2 dx \right).
\end{aligned}$$

By $\{u_n\}$ is bounded in H_s, there exists $R_0 > 0$ such that $\left(\int_{\{v_n > R_0\}} |v_n|^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}} \le \frac{1}{2C\beta^2}$. Hence, we can see that

$$\begin{aligned}
&\int_{\{v_n \le R_0\}} |v_n|^{2_s^*-q+1} |v_n|^{q-1} (v_n vL, n^{2(\beta-1)}) dx \\
&\quad + \left(\int_{\{v_n > R_0\}} |v_n|^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}} \left(\int_{\{v_n > R_0\}} (v_n v_{L,n}^{\beta-1})^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \\
&\quad \le R_0^{2_s^*-q+1} \int_{\mathbb{R}^N} |v_n|^{2_s^*} dx + \frac{1}{2C\beta^2} \left(\int_{\mathbb{R}^N} (v_n v_{L,n}^{\beta-1})^{2_s^*} dx \right)^{\frac{2}{2_s^*}}.
\end{aligned}$$

Therefore, we can deduce that

$$\left(\int_{\mathbb{R}^N} |v_n v_{L,n}^{\beta-1}|^{2_s^*} dx\right)^{\frac{2}{2_s^*}} \le 2C\beta^2 \left(1 + R_0^{2_s^*-q+1}\right) \int_{\mathbb{R}^N} |v_n|^{2_s^*} dx < C < +\infty. \tag{4.11}$$

Taking the limit in (4.11) as $L \to +\infty$ and Fatou lemma, we have $\left(\int_{\mathbb{R}^N} |v_n|^{2_s^*\beta} dx\right)^{\frac{2}{2_s^*}} \le C < +\infty$. So, $v_n \in L^{2_s^*\beta}(\mathbb{R}^N)$. For any $\beta > \frac{1}{2}(2_s^* + 2 - q) > 1$ and $\beta \le 1 + \frac{2_s^*}{2} \cdot \frac{2_s^*-q}{2}$ then $2 < q + 2\beta - 2 < 2_s^* + 2\beta - 2 \le 2_s^*(1 + \frac{2_s^*-q}{2})$. we can deduce that

$$\left(\int_{\mathbb{R}^N} |v_n|^{2_s^*\beta} dx\right)^{\frac{2}{2_s^*}} \le C\beta^2 \left(\int_{\mathbb{R}^N} |v_n|^{q+2\beta-2} dx + \int_{\mathbb{R}^N} |v_n|^{2_s^*+2\beta-2} dx\right) \le C_0 < +\infty.$$

Let $a = \frac{2_s^*(2_s^*-q)}{2(\beta-1)}$, $b = q + 2\beta - 2 - a$, $r = \frac{2_s^*}{a}$, $r' = \frac{2_s^*}{2_s^*-a}$, then $\frac{2_s^* b}{2_s^*-a} = 2_s^* + 2\beta - 2$. Taking into account Young inequality we have

$$\begin{aligned}\int_{\mathbb{R}^N} |v_n|^{q+2\beta-2} dx &\le \frac{a}{2_s^*} \int_{\mathbb{R}^N} |v_n|^{2_s^*} dx + \frac{2_s^* - a}{2_s^*} \int_{\mathbb{R}^N} |v_n|^{2_s^*+2\beta-2} dx \\ &\le C\left(1 + \int_{\mathbb{R}^N} |v_n|^{2_s^*+2\beta-2} dx\right).\end{aligned}$$

Therefore,

$$\left(\int_{\mathbb{R}^N} |v_n|^{2_s^*\beta} dx\right)^{\frac{2}{2_s^*}} \le C\beta^2 \left(1 + \int_{\mathbb{R}^N} |v_n|^{2_s^*+2\beta-2} dx\right).$$

We note to $\beta > 1$, we deduce that

$$\left(1 + \int_{\mathbb{R}^N} |v_n|^{2_s^*\beta} dx\right)^{\frac{2}{2_s^*}} \le C\beta^2 \left(1 + \int_{\mathbb{R}^N} |v_n|^{2_s^*+\beta-2} dx\right). \tag{4.12}$$

Now, we set $\beta = 1 + \frac{2_s^*}{2} \cdot \frac{2_s^*-q}{2}$, then observing that $2_s^* + 2\beta - 2 = 2_s^*(\frac{1}{2}(2_s^* + 2 - q))$. Iterating this process and recalling that $2_s^* + 2\beta_{i-1} - 2 = 2_s^*\beta_i$. Argue as [21]. Thus,

$$\beta_{i+1} - 1 = (\frac{2_s^*}{2})^i (\beta_1 - 1).$$

Replacing it in (4.12) we have

$$\left(1 + \int_{\mathbb{R}^N} |v_n|^{2_s^*\beta_{i+1}} dx\right)^{\frac{1}{2_s^*(\beta_{i+1}-1)}} \le (C\beta_{i+1}^2)^{\frac{1}{2(\beta_{i+1}-1)}} \left(1 + \int_{\mathbb{R}^N} v_n^{2\beta_i+2_s^*-2} dx\right)^{\frac{1}{2(\beta_i-1)}}.$$

Denoting $C_{i+1} = C\beta_{i+1}^2$ and $K_i := (1 + \int_{\mathbb{R}^N} v_n^{2\beta_i+2_s^*-2} dx)^{\frac{1}{2(\beta_i-1)}}$. We conclude that there exists a constant $C_0 > 0$ independent of i, such that

$$K_{i+1} \le \prod_{i=2}^{i+1} C_i^{\frac{1}{2(\beta_i-1)}} K_1 \le CK_1.$$

Therefore,

$$\|v_n(x)\|_{L^\infty(\mathbb{R}^N)} \le C_0 K_1 < \infty,$$

uniformly in $n \in \mathbb{N}$, thanks to $v_n \in L^{2_s^*\beta_1}(\mathbb{R}^N)$ and $\|v_n\|_{\varepsilon_n} \le C$. Arguing as in [4], we can prove that

$$\lim_{|x|\to\infty} v_n(x) = 0 \text{ uniformly in } n \in \mathbb{N}.$$

□

Now we consider $\varepsilon_n \to 0^+$ and take a sequence $u_n \in \tilde{\mathcal{N}_\varepsilon}$ of solutions of the problem (2.2) as above. There exists $\gamma > 0$ such that

$$\|u_n\|_{L^\infty(\mathbb{R}^N)} \ge \gamma \quad \text{uniformly in} \quad n \in \mathbb{N}. \tag{4.13}$$

Assume by contradiction, we have $\lim\limits_{n\to\infty} \|u_n\|_{L^\infty(\mathbb{R}^N)} = 0$. For any $\xi > 0$, there exists n_0 such that $\|u_n\|_{L^\infty(\mathbb{R}^N)} < \xi$ for any $n > n_0$. Since $u_n \in \tilde{\mathcal{N}_\varepsilon}$, we have

$$\begin{aligned}\|u_n\|_{\varepsilon_n}^2 =& \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u_n)\right) f(u_n)u_n dx + \int_{\mathbb{R}^N} |u_n|^{2_s^*} dx \\ \le& C\left(\int_{\mathbb{R}^N} (|u_n|^{2t} + |u_n|^{qt}) dx\right)^{\frac{2}{t}} + \int_{\mathbb{R}^N} |u_n|^{2_s^*} dx\end{aligned}$$

where $t = \frac{2N}{2N-\mu}$. Since $2t \in (2, 2_s^*)$ and $qt \in (2, 2_s^*)$, there exists $\sigma > 0$ small enough such that $(2t-\sigma) \in (2, 2_s^*)$ and $(qt-\sigma) \in (2, 2_s^*)$. Since we have that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$, we can deduce to

$$\begin{aligned}\|u_n\|_{\varepsilon_n} \le& C\left(\int_{\mathbb{R}^N} (|u_n|^{2t-\sigma}|u_n|^\sigma + |u_n|^{qt-\sigma}|u_n|^\sigma) dx\right)^{\frac{2}{t}} + \int_{\mathbb{R}^N} |u_n|^{2_s^*-\sigma}|u_n|^\sigma dx \\ \le& C\|u_n\|_{L^\infty(\mathbb{R}^N)}^{\sigma\cdot\frac{2}{t}} \left(\int_{\mathbb{R}^N} (|u_n|^{2t-\sigma} + |u_n|^{qt-\sigma}) dx\right)^{\frac{2}{t}} + \|u_n\|_{L^\infty(\mathbb{R}^N)}^\sigma \int_{\mathbb{R}^N} |u_n|^{2_s^*-\sigma} dx \\ <& C_1\xi^{\frac{2\sigma}{t}} + C_2\xi^\sigma.\end{aligned}$$

This implies that $\|u_n\|_{\varepsilon_n} \to 0$ $(n \to \infty)$. In similar way, we can deduce

$$\frac{1}{2}\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u_n)\right) F(u_n) dx + \frac{1}{2_s^*}\int_{\mathbb{R}^N} |u_n|^{2_s^*} dx \to 0 \ (n \to \infty),$$

then $J_{\varepsilon_n}(u_n) \to 0$ $(n \to \infty)$, this contradict with $J_{\varepsilon_n}(u_n) \to m_{V_0} > 0$. As a consequence, (4.13) holds. By Lemma 4.7, we have

$$\|v_n\|_{L^\infty(\mathbb{R}^N)} \le C, \quad \text{uniformly in} \quad n \in \mathbb{N},$$

and

$$\lim_{|x|\to\infty} v_n(x) = 0 \quad \text{uniformly in} \quad n \in \mathbb{N}.$$

There exists $R > 0$ such that $\|v_n\|_{L^\infty(B_R^c(0))} < \gamma$, then

$$\|u_n\|_{L^\infty(B_R^c(y_n))} < \gamma. \tag{4.14}$$

Hence

$$\|u_n\|_{L^\infty(B_R(y_n))} \ge \gamma. \tag{4.15}$$

Let p_n is the global maximum point of u_n, taking into account (4.14) and (4.15) we can get $p_n \in B_R(y_n)$. Hence, $p_n = y_n + q_n$ for some $q_n \in B_R(0)$. Then $\xi_{\varepsilon_n} = \varepsilon_n y_n + \varepsilon_n q_n$ is the maximum point of $u_n(\frac{x}{\varepsilon_n})$. Since $|q_n| < R$ for any $n \in \mathbb{N}$ and $\varepsilon_n y_n \to y_0 \in \Lambda$. Therefore,

$$\lim_{n\to\infty} V(\xi_{\varepsilon_n}) = V(y_0) = V_0,$$

which ends the proof of the Theorem 1.2. □

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