Research article

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# Multiplicity and concentration behaviour of solutions for a fractional Choquard equation with critical growth

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**Abstract:** In this paper, we study the singularly perturbed fractional Choquard equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = \varepsilon^{\mu-3}\Big(\int_{\mathbb{R}^3} \frac{|u(y)|^{2^\star_{\mu,s}} + F(u(y))}{|x-y|^{\mu}}dy\Big)\Big(|u|^{2^\star_{\mu,s}-2}u + \frac{1}{2^\star_{\mu,s}}f(u)\Big) \text{ in } \mathbb{R}^3,$$

where $\varepsilon > 0$ is a small parameter, $(-\Delta)^s$ denotes the fractional Laplacian of order $s \in (0,1)$, $0 < \mu < 3$, $2^\star_{\mu,s} = \frac{6-\mu}{3-2s}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality and fractional Laplace operator. $F$ is the primitive of $f$ which is a continuous subcritical term. Under a local condition imposed on the potential $V$, we investigate the relation between the number of positive solutions and the topology of the set where the potential attains its minimum values. In the proofs we apply variational methods, penalization techniques and Ljusternik-Schnirelmann theory.

**Keywords:** Variational method; fractional Choquard equation; critical growth

**MSC:** 35Q40; 35J50; 58E05

## 1 Introduction and the main results

In the present paper we are interested in the existence, multiplicity and concentration behavior of the semi-classical solutions of the singularly perturbed nonlocal elliptic equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = \varepsilon^{\mu-N}\Big(\int_{\mathbb{R}^3} \frac{G(u(y))}{|x-y|^{\mu}}dy\Big)g(u) \text{ in } \mathbb{R}^N, \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $0 < \mu < N$, $V, g = G'$ are real continuous functions on $\mathbb{R}^N$ and the fractional Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s\Psi(x) = C_{N,s}P.V.\int_{\mathbb{R}^N} \frac{\Psi(x)-\Psi(y)}{|x-y|^{N+2s}}dy, \quad \Psi \in \mathcal{S}(\mathbb{R}^N),$$

$P.V.$ stands for the Cauchy principal value, $C_{N,s}$ is a normalized constant, $\mathcal{S}(\mathbb{R}^N)$ is the Schwartz space of rapidly decaying functions, $s \in (0,1)$. As $\varepsilon$ goes to zero in (1.1), the existence and asymptotic behavior of the

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solutions of the singularly perturbed equation (1.1) is known as the semi-classical problem. It was used to describe the transition between Quantum Mechanics and Classical Mechanics.

Our motivation to study (1.1) mainly comes from the fact that solutions $u(x)$ of (1.1) correspond to standing wave solutions $\Psi(x,t) = e^{-iEt/\varepsilon}u(x)$ of the following time-dependent fractional Schrödinger equation

$$i\varepsilon\frac{\partial\Psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s\Psi + (V(x)+E)\Psi - (K(x) \star |G(\Psi)|)g(\Psi) \ \ (x,t) \in \mathbb{R}^N \times \mathbb{R} \tag{1.2}$$

where $i$ is the imaginary unit, $\varepsilon$ is related to the Planck constant. Equations of the type (1.2) was introduced by Laskin (see [25, 26]) and come from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. It also appeared in several areas such as optimization, finance, phase transitions, stratified materials, crystal dislocation, flame propagation, conservation laws, materials science and water waves (see [11]).

When $s = 1$, the equation (1.1) turns out to be the Choquard equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{\mu-N}\Big(\int_{\mathbb{R}^N}\frac{G(u(y))}{|x-y|^\mu}dy\Big)g(u) \ \text{ in } \mathbb{R}^N, \tag{1.3}$$

The existence, multiplicity and concentration of solutions for (1.3) has been widely investigated. On one hand, some people have studied the classical problem, namely $\varepsilon = 1$ in (1.3). When $V = 1$ and $G(u) = \frac{|u|^q}{q}$, (1.3) covers in particular the Choquard-Pekar equation

$$-\Delta u + u = \Big(\int_{\mathbb{R}^N}\frac{1}{|x|^\mu} \star |u|^q dy\Big)|u|^{q-2}u \ \text{ in } \mathbb{R}^N. \tag{1.4}$$

The case $N = 3$, $q = 2$ and $\mu = 1$ came from Pekar [38] in 1954 to describe the quantum mechanics of a polaron at rest. In 1976 Choquard used (1.4) to describe an electron trapped in its own hole, in a certain approximation to Hartree-Fock theory of one component plasma [27]. In this context (1.4) is also known as the nonlinear Schrödinger-Newton equation. By using critical point theory, Lions [29] obtained the existence of infinitely many radialy symmetric solutions in $H^1(\mathbb{R}^N)$ and Ackermann [1] prove the existence of infinitely many geometrically distinct weak solutions for a general case. For the properties of the ground state solutions, Ma and Zhao [30] proved that every positive solution is radially symmetric and monotone decreasing about some point for the generalized Choquard equation (1.4) with $q \geq 2$. Later, Moroz and Van Schaftingen [32, 33] eliminated this restriction and showed the regularity, positivity and radial symmetry of the ground states for the optimal range of parameters, and also derived that these solutions decay asymptotically at infinity.

On the other hand, some people have focused on the semiclassical problem, namely, $\varepsilon \to 0$ in (1.3). The question of the existence of semiclassical solutions for the non-local problem (1.3) has been posed in [5]. Note that if $v$ is a solution of (1.3) for $x_0 \in \mathbb{R}^N$, then $u = v(\varepsilon x + x_0)$ verifies

$$-\Delta u + V(\varepsilon x + x_0)u = \Big(\int_{\mathbb{R}^N}\frac{G(u(y))}{|x-y|^\mu}dy\Big)g(u) \ \text{ in } \mathbb{R}^N, \tag{1.5}$$

which means some convergence of the family of solutions to a solution $u_0$ of the limit problem

$$-\Delta u + V(x_0)u = \Big(\int_{\mathbb{R}^N}\frac{G(u(y))}{|x-y|^\mu}dy\Big)g(u) \ \text{ in } \mathbb{R}^N. \tag{1.6}$$

For this case when $N = 3, \mu = 1$ and $G(u) = |u|^2$, Wei and Winter [49] constructed families of solutions by a Lyapunov-Schmidt-type reduction when $\inf_{x\in\mathbb{R}^N} V > 0$. This method of construction depends on the existence, uniqueness and non-degeneracy up to translations of the positive solution of the limiting equation (1.6), which is a difficult problem that has only been fully solved in the case when $N = 3$, $\mu = 1$ and $G(u) = |u|^2$. Moroz and Van Schaftingen [34] used variational methods to develop a novel non-local penalization technique to show that equation (1.3) with $G(u) = |u|^q$ has a family of solutions concentrated at the local minimum of $V$, with $V$ satisfying some additional assumptions at infinity. In addition, Alves and Yang [4] investigated the multiplicity and concentration behaviour of solutions for a quasi-linear Choquard equation via the penalization method. Very recently, in an interesting paper, Alves et al. [2] study (1.3) with a critical growth, they consider the critical problem with both linear potential and nonlinear potential, and showed the existence, multiplicity and concentration behavior of solutions when the linear potential has a global minimum or maximum.

On the contrary, the results about fractional Choquard equation (1.1) are relatively few. Recently, d'Avenia, Siciliano and Squassina [17] studied the existence, regularity and asymptotic of the solutions for the following fractional Choquard equation

$$(-\Delta)^s u + \omega u = \Big(\int\limits_{\mathbb{R}^N} \frac{|u(y)|^q}{|x-y|^\mu} dy\Big)|u|^{q-2}u \ \text{ in } \mathbb{R}^N, \tag{1.7}$$

where $\omega > 0$, $\frac{2N-\mu}{N} < q < \frac{2N-\mu}{N-2s}$. Shen, Gao and Yang [42] obtain the existence of ground states for (1.7) with general nonlinearities by using variational methods. Chen and Liu [14] studied (1.7) with nonconstant linear potential and proved the existence of ground states without any symmetry property. For critical problem, Wang and Xiang [47] obtain the existence of infinitely many nontrivial solutions and the Brezis-Nirenberg type results can be founded in [36]. For the critical Choquard equations in the sense of Hardy-Littlewood-Sobolev, Cassani and Zhang [12] developed a robust method to get the existence of ground states and qualitative properties of solutions, where they do not require the nonlinearity to enjoy monotonicity nor Ambrosetti-Rabinowitz-type conditions. For other existence results we refer to [6, 8, 23, 24, 31, 48, 52] and the references therein.

It seems that the only works concerning the concentration behavior of solutions are due to [13, 51]. Assuming the global condition on $V$:

$$0 < \inf_{x\in\mathbb{R}^N} V(x) < \liminf_{|x|\to\infty} V(x) = V_\infty,$$

which was firstly introduced by Rabinowitz [39] in the study of the nonlinear Schrödinger equations. By using the method of Nehari manifold developed by Szulkin and Weth [46], authors in [13, 51] obtained the multiplicity and concentration of positive solutions for the following fractional Choquard equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = \varepsilon^{\mu-3}\Big(\int\limits_{\mathbb{R}^3} \frac{|u(y)|^{2^\star_{\mu,s}} + F(u(y))}{|x-y|^\mu} dy\Big)\Big(|u|^{2^\star_{\mu,s}-2}u + \frac{1}{2^\star_{\mu,s}} f(u)\Big) \ \text{ in } \mathbb{R}^3, \tag{1.8}$$

where $\varepsilon > 0$, $0 < \mu < 3$, $F$ is the primitive function of $f$.

Different to [13, 51], in this paper, we are devote to establishing the existence and concentration of positive solutions for the fractional Choquard equation (1.8) when the potential function satisfies the following local conditions [18]:

$(V_1)$ $V \in C(\mathbb{R}^3, \mathbb{R})$ and $0 < \inf_{x\in\mathbb{R}^3} V(x)$.

$(V_2)$ There is a bounded open domain $\Omega$ such that

$$V_0 := \inf_{\Omega} V(x) < \min_{\partial\Omega} V(x).$$

Without loss of generality, we may assume that $\mathcal{M} = \{x \in \Omega : V(x) = V_0\} \neq \emptyset$ and $V(0) = \min_{x\in\mathbb{R}^3} V(x) = V_0$.

To go on studying the problem (1.8), the following Hardy-Littlewood-Sobolev inequality [28] is the starting point.

**Lemma 1.1.** *Let $t, r > 1$ and $0 < \mu < 3$ with*

$$\frac{1}{t} + \frac{\mu}{3} + \frac{1}{r} = 2,$$

*$f \in L^t(\mathbb{R}^3)$ and $h \in L^r(\mathbb{R}^3)$. There exists a sharp constant $C(t, \mu, r)$, independent of $f$, $h$ such that*

$$\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{f(x)h(y)}{|x-y|^\mu} dy dx \le C(t,\mu,r)|f|_t|h|_r. \tag{1.9}$$

*In particular, if* $t = r = \frac{6}{6-\mu}$, *then*

$$C(t,\mu,r) = C(\mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma(\frac{3}{2}-\frac{\mu}{2})}{\Gamma(3-\frac{\mu}{2})}\left(\frac{\Gamma()\frac{3}{2}}{\Gamma(3)}\right)^{\frac{\mu}{3}-1}.$$

*In this case there is equality in* (1.9) *if and only if* $f \equiv Ch$ *and*

$$h(x) = \frac{A}{(a^2 + |x-b|^2)^{\frac{6-\mu}{2}}}$$

*for some* $A \in \mathbb{C}$, $a \in \mathbb{R}\backslash\{0\}$ *and* $b \in \mathbb{R}^3$.

Notice that, by the Hardy-Littlewood-Sobolev inequality, the integral

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu}dydx$$

ie well defined if $u^q \in L^t(\mathbb{R}^3)$ satisfies $\frac{2}{t} + \frac{\mu}{3} = 2$. Therefore, for $u \in H^s(\mathbb{R}^3)$ we will require that $t \cdot q \in [2, 2_s^\star]$, where $2_s^\star = \frac{6}{3-2s}$ is fractional critical Sobolev exponent for dimension 3. Then we have

$$\frac{6-\mu}{3} \le q \le \frac{6-\mu}{3-2s}.$$

Thus, $\frac{6-\mu}{3}$ is called the lower critical exponent and $2_{\mu,s}^\star := \frac{6-\mu}{3-2s}$ is the upper critical exponent in the sense of Hardy-Littlewood-Sobolev inequality and the fractional Laplace operator.

For the nonlinearity term, we assume that the continuous function $f$ vanishes in $(-\infty, 0)$ and satisfies:
$(f_1)$ $|f(u)| \le c(|u|^{q_1-1} + |u|^{q_2-1})$ for some $c > 0$ and $\frac{6-\mu}{3} < q_1 \le q_2 < 2_{\mu,s}^\star$.
$(f_2)$ The function $u \mapsto f(u)$ is increasing in $(0, \infty)$.
$(f_3)$ $(i)$

$$\lim_{|u|\to+\infty} \frac{F(u)}{|u|^{2_{\mu,s}^\star-1}} = +\infty \text{ for } s \in (\frac{3}{4}, 1);$$

$(ii)$

$$\lim_{|u|\to+\infty} \frac{F(u)}{|u|^{2_{\mu,s}^\star-\frac{2s}{3-2s}}(log|u|)^{\frac{1}{2}}} = +\infty \text{ for } s = \frac{3}{4};$$

$(iii)$

$$\lim_{|u|\to+\infty} \frac{F(u)}{|u|^{2_{\mu,s}^\star-\frac{2s}{3-2s}}} = +\infty \text{ for } s \in (0, \frac{3}{4}).$$

Note that there is no $(AR)$ type assumption on $f$. Then it is difficult to show that the functional satisfies the $(PS)$ condition even for the autonomous case, which is necessary to use Ljusternik-Schnirelmann category theory. We shall investigate the $(PS)$ sequence carefully and restore the compactness for $(PS)$ sequence via some compactness Lemmas.

In order to describe the multiplicity, we first recall that, if $Y$ is a closed subset of a topological space $X$, the Ljusternik-Schnirelmann category $cat_X Y$ is the least number of closed and contractible sets in $X$ which cover $Y$. Then we state our main result as follows.

**Theorem 1.1.** *If* $0 < \mu < 2s$, *assume that* $V$ *satisfies* $(V_1)$ *and* $(V_2)$ *and the function* $f$ *satisfies* $(f_1) - (f_3)$. *Then for any* $\delta > 0$ *such that*

$$\mathcal{M}_\delta = \{x \in \mathbb{R}^3 : dist(x, \mathcal{M}) \le \delta\} \subset \Omega,$$

*there exists* $\varepsilon_\delta > 0$ *such that the problem* (1.8) *has at least* $cat_{\mathcal{M}_\delta}(\mathcal{M})$ *positive solutions for any* $\varepsilon \in (0, \varepsilon_\delta)$. *Moreover, if* $u_\varepsilon$ *denotes one of these positive solutions and* $\eta_\varepsilon \in \mathbb{R}^3$ *its global maximum, then*

$$\lim_{\varepsilon\to0} V(\eta_\varepsilon) = V_0.$$

**Remark 1.1.** *Here, we make a few observations about the restriction on the parameter* $0 < \mu < 2s$. *In order to adapt the penalization method introduced by del Pino and Felmer in [18], we will proposed some control conditions on the non-local term* $(\frac{1}{|x|^{\mu}}(|u|^{2^*_{\mu,s}} + F(u)))$ *, which need some regularity (see Lemma 2.7 and 2.8), where we introduce the assumption* $0 < \mu < 2s$.

We shall use the method of Nehari manifold, concentration compactness principle and category theory to prove the main results. There are some difficulties in proving our theorems. The first difficulty is that the nonlinearity $f$ is only continuous, we can not use standard arguments on the Nehari manifold. To overcome the nondifferentiability of the Nehari manifold, we shall use some variants of critical point theorems from Szulkin and Weth [46]. The second one is the lack of compactness of the embedding of $H^s(\mathbb{R}^3)$ into the space $L^{2^*_s}(\mathbb{R}^3)$. We shall borrow the idea in [2, 12] to deal with the difficulties brought by the critical exponent. However, we require some new estimates, which are complicated because of the appearance of fractional Laplacian and the convolution-type nonlinearity. Moreover, the potential $V$ satisfies $(V_1)$ and $(V_2)$ instead of the global condition. Since we have no information on the potential $V$ at infinity, we adapt the truncation trick explored in [18]. It consists in making a suitable modification on the nonlinearity, solving a modified problem and then check that, for $\varepsilon$ small enough, the solutions of the modified problem are indeed solutions of the original one. It is worthwhile to remark that in the arguments developed in [18], one of the key points is the existence of estimates involving the $L^\infty$-norm of the modified problem. But for the critical nonlocal problem (1.8), this kind of estimates are more delicate.

This paper is organized as follows. In section 2, besides describing the functional setting to study problem (1.8), we give some preliminary Lemmas which will be used later. In section 3, influenced by the work [18] and [45], we introduce a modified functional and show it satisfies the Palais-Smale condition. In section 4, we study the autonomous problem associated. This study allows us to show that the modified problem has multiple solutions. Finally, we show the critical point of the modified functional which satisfies the original problem, and investigate its concentration behavior, which completes the proof Theorem 1.1.

# 2 Variational settings and preliminary results

Throughout this paper, we denote $|\cdot|_r$ the usual norm of the space $L^r(\mathbb{R}^3)$, $1 \le r < \infty$, $B_r(x)$ denotes the open ball with center at $x$ and radius $r$, $C$ or $C_i (i = 1, 2, \cdots)$ denote some positive constants may change from line to line. $\rightharpoonup$ and $\to$ mean the weak and strong convergence. Let $E$ be a Hilbert space, the Fréchet derivative of a functional $\Phi$ at $u$, $\Phi'(u)$, is an element of the dual space $E^*$ and we shall denote $\Phi'(u)$ evaluated at $v \in E$ by $\langle \Phi'(u), v\rangle$.

## 2.1 The functional space setting

Firstly, fractional Sobolev spaces are the convenient setting for our problem, so we will give some skrtchs of the fractional order Sobolev spaces and the complete introduction can be found in [19]. We recall that, for any $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^3) = W^{s,2}(\mathbb{R}^3)$ is defined as follows:

$$H^s(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \int\limits_{\mathbb{R}^3} \left(|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2\right) d\xi < \infty\},$$

whose norm is defined as

$$\|u\|^2_{H^s(\mathbb{R}^3)} = \int\limits_{\mathbb{R}^3} \left(|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2\right) d\xi,$$

where $\mathcal{F}$ denotes the Fourier transform. We also define the homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^3)$ as the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} := \Big(\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dxdy\Big)^{\frac{1}{2}} = [u]_{H^s(\mathbb{R}^3)}.$$

The embedding $\mathcal{D}^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^\star}(\mathbb{R}^3)$ is continuous and for any $s \in (0,1)$, there exists a best constant $S_s > 0$ such that

$$S_s := \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^3)} \frac{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)}}{|u|^2_{2_s^\star}}$$

According to [16], $S_s$ is attained by

$$u_0(x) = C\Big(\frac{b}{b^2+|x-a|^2}\Big)^{\frac{3-2s}{2}},\quad x\in\mathbb{R}^3, \tag{2.1}$$

where $C \in \mathbb{R}$, $b > 0$ and $a \in \mathbb{R}^3$ are fixed parameters. We use $S_{H,L}$ to denote the best constant defined by

$$S_{H,L} := \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2dx}{\big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(y)|^{2^\star_{\mu,s}}|u(x)|^{2^\star_{\mu,s}}}{|x-y|^\mu}dydx\big)^{\frac{1}{2^\star_{\mu,s}}}}. \tag{2.2}$$

The fractional Laplacian, $(-\Delta)^s u$, of a smooth function $u : \mathbb{R}^3 \to \mathbb{R}$, is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi),\quad \xi\in\mathbb{R}^3.$$

Also $(-\Delta)^s u$ can be equivalently represented [19] as

$$(-\Delta)^s u(x) = -\frac{1}{2}C(s)\int\limits_{\mathbb{R}^3} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{3+2s}}\,dy,\ \forall x\in\mathbb{R}^3$$

where

$$C(s) = \Big(\int\limits_{\mathbb{R}^3}\frac{(1-cos\xi_1)}{|\xi|^{3+2s}}d\xi\Big)^{-1},\ \xi=(\xi_1,\xi_2,\xi_3).$$

Also, by the Plancherel formular in Fourier analysis, we have

$$[u]^2_{H^s(\mathbb{R}^3)} = \frac{2}{C(s)}|(-\Delta)^{\frac{s}{2}}u|_2^2.$$

For convenience, we will omit the normalization constant in the following. As a consequence, the norms on $H^s(\mathbb{R}^3)$ defined below

$$u \longmapsto \Big(\int\limits_{\mathbb{R}^3}|u|^2dx + \iint\limits_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}dxdy\Big)^{\frac{1}{2}};$$

$$u \longmapsto \Big(\int\limits_{\mathbb{R}^3}(|\xi|^{2s}|\mathcal{F}(u)|^2+|\mathcal{F}(u)|^2)d\xi\Big)^{\frac{1}{2}};$$

$$u \longmapsto \Big(\int\limits_{\mathbb{R}^3}|u|^2dx + |(-\Delta)^{\frac{s}{2}}u|_2^2\Big)^{\frac{1}{2}}.$$

are equivalent.

Making the change of variable $x \mapsto \varepsilon x$, we can rewrite the equation (1.8) as the following equivalent form

$$(-\Delta)^s u + V(\varepsilon x)u = \Big(\int\limits_{\mathbb{R}^3}\frac{|u(y)|^{2^\star_{\mu,s}}+F(u(y))}{|x-y|^\mu}dy\Big)\Big(|u|^{2^\star_{\mu,s}-2}u + \frac{1}{2^\star_{\mu,s}}f(u)\Big)\ \text{in}\ \mathbb{R}^3, \tag{2.3}$$

If $u$ is a solution of the equation (2.3), then $v(x) := u(\frac{x}{\varepsilon})$ is a solution of the equation (1.8). Thus, to study the equation (1.8), it suffices to study the equation (2.3). In view of the presence of potential $V(x)$, we introduce the subspace

$$H_\varepsilon = \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x) u^2 dx < +\infty \right\},$$

which is a Hilbert space equipped with the inner product

$$(u, v)_{H_\varepsilon} = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} V(\varepsilon x) u v dx,$$

and the norm

$$\|u\|^2_{H_\varepsilon} = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x) u^2 dx.$$

We denote $\|\cdot\|_{H_\varepsilon}$ by $\|\cdot\|_\varepsilon$ in the sequel for convenience.

For the reader's convenience, we review some useful result for this class of fractional Sobolev spaces:

**Lemma 2.1.** *[19] Let $0 < s < 1$, then there exists a constant $C = C(s) > 0$, such that*

$$|u|^2_{2^\star_s} \le C[u]^2_{H^s(\mathbb{R}^3)}$$

*for every $u \in H^s(\mathbb{R}^3)$. Moreover, the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ is continuous for any $r \in [2, 2^\star_s]$ and is locally compact whenever $r \in [2, 2^\star_s)$.*

**Lemma 2.2.** *[40] If $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$ and for some $R > 0$ we have*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^2 dx = 0,$$

*then $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for any $2 < r < 2^\star_s$.*

**Lemma 2.3.** *[37] Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^3)$, $\varphi \in C_0^\infty(\mathbb{R}^3)$ and for each $r > 0$, $\varphi_r(x) = \varphi(\frac{x}{r})$. Then*

$$u\varphi_r \to 0 \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^3) \text{ as } r \to 0.$$

*If, in addition, $\varphi \equiv 1$ in a neighbourhood of the origin, then*

$$u\varphi_r \to u \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^3) \text{ as } r \to +\infty.$$

## 2.2 Preliminary lemmas

Set $G(u) = |u|^{2^\star_{\mu,s}} + F(u)$, $g(u) = \frac{dG(u)}{du}$. In the sequel, set $r_0 = \frac{6}{6-\mu}$, then $2 < r_0 q_1 \le r_0 q_2 < 2^\star_s$. So for any $u \in H^s(\mathbb{R}^3)$, we have

$$|F(u)|_{r_0} \le C(|u|^{q_1}_{q_1 r_0} + |u|^{q_2}_{q_2 r_0}) \tag{2.4}$$

and

$$|G(u)|_{r_0} \le C(|u|^{q_1}_{q_1 r_0} + |u|^{q_2}_{q_2 r_0} + |u|^{2^\star_{\mu,s}}_{2^\star_s}). \tag{2.5}$$

Therefore, the Hardy-Littlewood-Sobolev inequality implies that

$$\Big| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{G(u(y))G(u(x))}{|x-y|^\mu} dy dx \Big| \le C|G(u)|^2_{r_0} \le C(|u|^{2q_1}_{q_1 r_0} + |u|^{2q_2}_{q_2 r_0} + |u|^{22^\star_{\mu,s}}_{2^\star_s}) \tag{2.6}$$

and

$$\Big| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{G(u(y))g(u(x))u(x)}{|x-y|^\mu} dy dx \Big| \le C(|u|^{2q_1}_{q_1 r_0} + |u|^{2q_2}_{q_2 r_0} + |u|^{22^\star_{\mu,s}}_{2^\star_s}). \tag{2.7}$$

It is clear that problem (2.3) is the Euler-Lagrange equations of the functional $I : H_\varepsilon \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\|u\|_\varepsilon^2 - \frac{1}{22^\star_{\mu,s}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{G(u(y))G(u(x))}{|x-y|^\mu}\,dydx. \tag{2.8}$$

From (2.6) we know that $I(u)$ is well defined on $H_\varepsilon$ and belongs to $C^1$, with its derivative given by

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} \big((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v + V(\varepsilon x)uv\big)dx - \frac{1}{2^\star_{\mu,s}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{G(u(y))g(u(x))v(x)}{|x-y|^\mu}\,dydx \tag{2.9}$$

for all $u, v \in H_\varepsilon$. Hence the critical points of $I$ in $H_\varepsilon$ are weak solutions of problem (2.3). In the following, we will consider critical points of $I$ using variational methods.

Firstly, we give the following Lemma, whose simple proof is omit.

**Lemma 2.4.** *If $(f_1)$ and $(f_2)$ are satisfied, then*

$$0 < F(u) < f(u)u, \quad 0 < G(u) < g(u)u, \ \ \forall\, u \neq 0. \tag{2.10}$$

*In addition, $(f_2)$ and (2.10) imply*

$$\frac{F(u)}{u} \ \text{and} \ \frac{G(u)}{u} \ \text{are increasing on} \ (0, +\infty). \tag{2.11}$$

For the derivative of the functional $I$ we have the following Lemma.

**Lemma 2.5.** *Let $(V_1)$ and $(f_1)$ hold, then*
*(i) $I'$ maps bounded sets in $H^s(\mathbb{R}^3)$ into bounded sets in $(H^s(\mathbb{R}^3))^\star$.*
*(ii) $I'$ is weakly sequentially continuous. Namely, if $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, then $I'(u_n) \rightharpoonup I'(u)$ in $(H^s(\mathbb{R}^3))^\star$.*

*Proof.* $(i)$. Let $\{u_n\}$ be a bounded sequence in $H^s(\mathbb{R}^3)$. For any $v \in H^s(\mathbb{R}^3)$, from the Hardy-Littlewood-Sobolev inequality and (2.5) it follows that

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{G(u_n(y))g(u_n(x))v(x)}{|x-y|^\mu}\,dydx\Big| &\le C|G(u_n)|_{r_0}\big(|u_n|_{q_1r_0}^{q_1-1}|v|_{q_1r_0} + |u_n|_{q_2r_0}^{q_2-1}|v|_{q_2r_0} + |u_n|_{2^\star_s}^{2^\star_{\mu,s}-1}|v|_{2^\star_s}\big)\\ &\le C\|v\|_\varepsilon.\end{aligned} \tag{2.12}$$

Then $|\langle I'(u_n), v\rangle| \le C\|v\|_\varepsilon$. Hence, $\{I'(u_n)\}$ is bounded in $(H^s(\mathbb{R}^3))^\star$.

$(ii)$. Assume that $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$. For any $v \in C_0^\infty(\mathbb{R}^3)$ with support $\Omega$, by Lemma 2.1 we may assume that $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$ and $u_n \to u$ in $L^p(\Omega)$, $p < 2^\star_s$. We first check that if $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, then

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(y)|^{2^\star_{\mu,s}}|u_n(x)|^{2^\star_{\mu,s}-2-\mu}u_n(x)v(x)}{|x-y|^\mu}\,dydx \to \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(y)|^{2^\star_{\mu,s}}|u(x)|^{2^\star_{\mu,s}-2-\mu}u(x)v(x)}{|x-y|^\mu}\,dydx, \tag{2.13}$$

as $n \to \infty$. By the Hardy-Littlewood-Sobolev inequality, we have

$$|h \star |x-y|^{-\mu}|_{\frac{6}{\mu}} \le C|h|_{r_0}, \ \ \text{for all } h \in L^{r_0}(\mathbb{R}^3),$$

and it is a linear bounded operator from $L^{r_0}(\mathbb{R}^3)$ to $L^{\frac{6}{\mu}}(\mathbb{R}^3)$. Choosing $h_n(y) := |u_n(y)|^{2^\star_{\mu,s}} \in L^{r_0}(\mathbb{R}^3)$, we have

$$\big||u_n(y)|^{2^\star_{\mu,s}} \star |x-y|^{-\mu}\big|_{\frac{6}{\mu}} \le C|u_n|_{2^\star_s} \le C.$$

Therefore, by Hölder inequality we can prove the sequence

$$\big(|u_n(y)|^{2^\star_{\mu,s}} \star |x-y|^{-\mu}\big)|u_n(x)|^{2^\star_{\mu,s}-2-\mu}u_n(x)$$

is bounded in $L^{\frac{2_s^\star}{2_s^\star-1}}(\mathbb{R}^3)$. Then, by duality we have

$$\int_{\mathbb{R}^3} \frac{|u_n(y)|^{2^\star_{\mu,s}}|u_n(x)|^{2^\star_{\mu,s}-2-\mu}u_n(x)}{|x-y|^\mu}dy \rightharpoonup \int_{\mathbb{R}^3} \frac{|u(y)|^{2^\star_{\mu,s}}|u(x)|^{2^\star_{\mu,s}-2-\mu}u(x)}{|x-y|^\mu}dy, \text{ in } L^{\frac{2_s^\star}{2_s^\star-1}}(\mathbb{R}^3)$$

as $n \to \infty$. Then (2.13) follows for every $v \in H^s(\mathbb{R}^3) \subset L^{2_s^\star}(\mathbb{R}^3)$.

By $(f_1)$ we have $|f(u)| \le C(1+|u|^{q_2-1})$ for all $u \in \mathbb{R}^+$, and then $f(u_n) \to f(u)$ in $L^{\frac{q_2 r}{q_2-1}}(\Omega)$. Then Hardy-Littlewood-Sobolev inequality implies that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{G(u_n(y))(f(u_n(x))-f(u(x)))v(x)}{|x-y|^\mu}dydx \le C|G(u_n)|_r|f(u_n)-f(u)|_{\frac{q_2 r}{q_2-1},\Omega}|v|_{q_2 r} \to 0 \tag{2.14}$$

Since $F(u_n)$ is bounded in $L^{r_0}(\mathbb{R}^3)$, we may assume that $F(u_n) \rightharpoonup F(u)$ in $L^{r_0}(\mathbb{R}^3)$. Consequently,

$$\int_{\mathbb{R}^3} \frac{F(u_n(y))}{|x-y|^\mu}dy \rightharpoonup \int_{\mathbb{R}^3} \frac{F(u(y))}{|x-y|^\mu}dy \ \text{ in } L^{\frac{6}{\mu}}(\mathbb{R}^3).$$

Moreover, as (2.12) we get $\int_{\mathbb{R}^3}|f(u(x))v(x)|^r dx < \infty$, we infer

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{F(u_n(y))}{|x-y|^\mu}f(u(x))v(x)dydx \to \int_{\mathbb{R}^3} \frac{F(u(y))}{|x-y|^\mu}f(u(x))v(x)dydx \ \text{ in } L^{\frac{6}{\mu}}(\mathbb{R}^3),$$

which combining with (2.14) we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{F(u_n(y))}{|x-y|^\mu}f(u_n(x))v(x)dydx \to \int_{\mathbb{R}^3} \frac{F(u(y))}{|x-y|^\mu}f(u(x))v(x)dydx \ \text{ in } L^{\frac{6}{\mu}}(\mathbb{R}^3).$$

Notice that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{G(u(y))g(u(x))v(x)}{|x-y|^\mu}dydx = \int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3} \frac{|u(y)|^{2^\star_{\mu,s}+F(u(y))}}{|x-y|^\mu}dy\Big)\Big(|u(x)|^{2^\star_{\mu,s}-2-\mu}u(x)+\frac{1}{2^\star_{\mu,s}}f(u(x))\Big)v(x)dx,$$

from our argument above it is then easy to prove

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{G(u_n(y))g(u_n(x))v(x)}{|x-y|^\mu}dydx \to \int_{\mathbb{R}^3} \frac{G(u(y))G(u(x))v(x)}{|x-y|^\mu}dydx \ \text{ in } L^{\frac{6}{\mu}}(\mathbb{R}^3).$$

Hence, for any $v \in C_0^\infty(\mathbb{R}^3)$,

$$\langle I'(u_n), v\rangle \to \langle I'(u), v\rangle. \tag{2.15}$$

Since $\{I'(u_n)\}$ is bounded in $(H^s(\mathbb{R}^3))^\star$ and $C_0^\infty(\mathbb{R}^3)$ is dense in $H^s(\mathbb{R}^3)$, we conclude that (2.15) holds for any $v \in H^s(\mathbb{R}^3)$, and so $I'(u_n) \rightharpoonup I'(u)$ in $(H^s(\mathbb{R}^3))^\star$. □

## 2.3 Regularity of solutions and Pohožaev identity

The assumption $(f_1)$ is too weak for the standard bootstrap method as in [4, 15, 32]. Therefore, in order to prove regularity of solutions of (2.3) we shall rely on a nonlocal version of the Brezis-Kato estimate. Note that a special case of the regularity result of Brezis and Kato [10, Theorem 2.3] states that if $u \in H^1(\mathbb{R}^N)$ is a solution of the linear elliptic equation

$$-\Delta u + u = Vu \ \text{ in } \mathbb{R}^N,$$

and $V \in L^\infty(\mathbb{R}^N) + L^{\frac{N}{2}}(\mathbb{R}^N)$, then $u \in L^p(\mathbb{R}^N)$ for every $p \ge 1$. Similar to [33, 42], we extent this result to the fractional Choquard equation with critical growth. For the convenience of readers, we give here a short proof. We first have the following useful inequality.

**Lemma 2.6.** *[33] Let $p, q, r, t \in [1, +\infty)$ and $\lambda \in [0, 2]$ such that*

$$1 + \frac{N-\mu}{N} - \frac{1}{p} - \frac{1}{t} = \frac{\lambda}{q} + \frac{2-\lambda}{r}.$$

*If $\theta \in (0, 2)$ satisfies*

$$\min(q, r)(\frac{N-\mu}{N} - \frac{1}{p}) < \theta < \max(q, r)(1 - \frac{1}{p})$$

*and*

$$\min(q, r)(\frac{N-\mu}{N} - \frac{1}{t}) < 2 - \theta < \max(q, r)(1 - \frac{1}{t}),$$

*then for every $H \in L^p(\mathbb{R}^N)$, $K \in L^t(\mathbb{R}^N)$ and $u \in L^q(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{H|u(y)|^\theta K|u(x)|^{2-\theta}}{|x-y|^\mu}\,dy\,dx \le C(\int_{\mathbb{R}^N} |H|^p)^{\frac{1}{p}}(\int_{\mathbb{R}^N} |K|^t)^{\frac{1}{t}}(\int_{\mathbb{R}^N} |u|^q)^{\frac{\lambda}{q}}(\int_{\mathbb{R}^N} |u|^r)^{\frac{2-\lambda}{r}}.$$

Applying Lemma 2.6, we have the following result, which is a nonlocal counterpart of the estimate [10, Lemma 2.1]: If $V \in L^\infty(\mathbb{R}^N) + L^{\frac{N}{2}}(\mathbb{R}^N)$, then for every $\varepsilon > 0$, there exists $C_\varepsilon$ such that

$$\int_{\mathbb{R}^N} V|u|^2\,dx \le \varepsilon^2 \int_{\mathbb{R}^N} |\nabla u|^2\,dx + C_\varepsilon \int_{\mathbb{R}^N} |u|^2\,dx.$$

**Lemma 2.7.** *Let $N \ge 2s$, $\mu \in (0, 2s)$ and $\theta \in (0, 2)$. If $H, K \in L^{\frac{2N}{N-\mu}}(\mathbb{R}^N) + L^{\frac{2N}{N+2s-\mu}}(\mathbb{R}^N)$ and $\frac{N-\mu}{N} < \theta < 2 - \frac{N-\mu}{N}$, then for every $\varepsilon > 0$, there exists $C_{\varepsilon,\theta} \in \mathbb{R}$ such that for every $u \in H^s(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{H|u(y)|^\theta K|u(x)|^{2-\theta}}{|x-y|^\mu}\,dy\,dx \le \varepsilon^2 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2\,dx + C_{\varepsilon,\theta} \int_{\mathbb{R}^N} |u|^2\,dx.$$

*Proof.* Since $0 < \mu < 2s$, we may assume that $H = H^\star + H_\star$ and $K = K^\star + K_\star$ with $H^\star, K^\star \in L^{\frac{2N}{N-\mu}}(\mathbb{R}^3)$ and $H_\star, K_\star \in L^{\frac{2N}{N+2s-\mu}}(\mathbb{R}^N)$. Applying Lemma 2.6 with $q = r = 2_s^\star$, $p = t = \frac{2N}{N+2s-\mu}$ and $\lambda = 0$, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{H_\star|u(y)|^\theta K_\star|u(x)|^{2-\theta}}{|x-y|^\mu}\,dy\,dx \le C|H_\star|_{\frac{2N}{N+2s-\mu}} |K_\star|_{\frac{2N}{N+2s-\mu}} |u|_{2_s^\star}^2,$$

where we use $|\theta - 1| < \frac{\mu}{N-2s}$. Taking $p = t = \frac{2N}{N-\mu}$, $q = r = 2$ and $\lambda = 2$, we have $|\theta - 1| < \frac{\mu}{N}$ and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{H^\star|u(y)|^\theta K^\star|u(x)|^{2-\theta}}{|x-y|^\mu}\,dy\,dx \le C|H^\star|_{\frac{2N}{N-\mu}} |K^\star|_{\frac{2N}{N-\mu}} |u|_2^2.$$

Similarly, with $p = \frac{2N}{N+2s-\mu}$, $t = \frac{2N}{N-\mu}$, $q = 2$, $r = 2_s^\star$ and $\lambda = 1$, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{H_\star|u(y)|^\theta K^\star|u(x)|^{2-\theta}}{|x-y|^\mu}\,dy\,dx \le C|H_\star|_{\frac{2N}{N-+2s-\mu}} |K^\star|_{\frac{2N}{N-\mu}} |u|_2 |u|_{2_s^\star}$$

and with $p = \frac{2N}{N-\mu}$, $t = \frac{2N}{N+2s-\mu}$, $q = 2$, $r = 2_s^\star$ and $\lambda = 1$,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{H^\star|u(y)|^\theta K_\star|u(x)|^{2-\theta}}{|x-y|^\mu}\,dy\,dx \le C|K_\star|_{\frac{2N}{N-+2s-\mu}} |H^\star|_{\frac{2N}{N-\mu}} |u|_2 |u|_{2_s^\star}.$$

By the Sobolev inequality, we have thus prove that for every $u \in H^s(\mathbb{R}^N)$,

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{H|u(y)|^\theta K|u(x)|^{2-\theta}}{|x-y|^\mu}\,dy\,dx \le C\Big(&(\int_{\mathbb{R}^N} |H_\star|^{\frac{2N}{N+2s-\mu}} \int_{\mathbb{R}^N} |K_\star|^{\frac{2N}{N+2s-\mu}})^{\frac{N+2s-\mu}{2N}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2\,dx \\ &+ (\int_{\mathbb{R}^N} |H^\star|^{\frac{2N}{N-\mu}} \int_{\mathbb{R}^N} |K^\star|^{\frac{2N}{N-\mu}})^{\frac{N-\mu}{2N}} \int_{\mathbb{R}^N} |u|^2\,dx\Big).\end{aligned}$$

The conclusion follows by choosing $H^\star$ and $K^\star$ such that

$$C\Big(\int_{\mathbb{R}^N} |H_\star|^{\frac{2N}{N+2s-\mu}} \int_{\mathbb{R}^N} |K_\star|^{\frac{2N}{N+2s-\mu}}\Big)^{\frac{N+2s-\mu}{2N}} \le \varepsilon^2.$$

□

Now, we have the following result, which is a nonlocal Brezis-Kato type regularity estimate.

**Lemma 2.8.** *Let $N \ge 2s$ and $0 < \mu < 2s$. If $H, K \in L^{\frac{2N}{N-\mu}}(\mathbb{R}^N) + L^{\frac{2N}{N+2s-\mu}}(\mathbb{R}^N)$ and $u \in H^s(\mathbb{R}^N)$ solves*

$$(-\Delta)^s u + u = \Big(\int_{\mathbb{R}^N} \frac{H(u(y))u(y)}{|x-y|^\mu} dy\Big) K(u(x)), \tag{2.16}$$

*then $u \in L^p(\mathbb{R}^N)$ for every $p \in [2, \frac{N}{N-\mu}\frac{2N}{N-2s})$. Moreover, there exists a constant $C_p$ independent of $u$ such that*

$$\Big(\int_{\mathbb{R}^N} |u|^p dx\Big)^{\frac{1}{p}} \le C_p \Big(\int_{\mathbb{R}^N} |u|^2 dx\Big)^{\frac{1}{2}}.$$

*Proof.* By Lemma 2.7 with $\theta = 1$, there exists $\lambda > 0$ such that for every $\varphi \in H^s(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|H\varphi||K\varphi|}{|x-y|^\mu} dydx \le \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\varphi|^2 dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} |\varphi|^2 dx.$$

Choose sequences $\{H_k\}_{k\in\mathbb{N}}$ and $\{K_k\}_{k\in\mathbb{N}}$ in $L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)$ such that $|H_k| \le |H|$, $|K_k| \le |K|$, and $H_k \to H$ and $K_k \to K$ almost everywhere in $\mathbb{R}^N$. For each $k \in \mathbb{N}$, for $\varphi, \psi \in H^s(\mathbb{R}^N)$, the form $a_k : H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$a_k(\varphi, \psi) = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}\varphi(-\Delta)^{\frac{s}{2}}\psi + \lambda\varphi\psi - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{H_k\varphi K_k\psi}{|x-y|^\mu} dydx$$

is bilinear and coercive. Therefore, applying the Lax-Milgram theorem [9, Corollary 5.8], there exists a unique solution $u_k \in H^s(\mathbb{R}^N)$ satisfies

$$(-\Delta)^s u_k + \lambda u_k = \int_{\mathbb{R}^N} \Big(\frac{H_k u_k}{|x-y|^\mu} dy\Big) K_k u_k + (\lambda - 1)u, \tag{2.17}$$

where $u \in H^s(\mathbb{R}^N)$ is the given solution of (2.16). Moreover, we can prove that the sequences $\{u_k\}_{k\in\mathbb{N}}$ converges weakly to $u$ in $H^s(\mathbb{R}^N)$ as $k \to \infty$.

For $\mu > 0$, we define the truncation $u_{k,\mu}$ by

$$u_{k,\mu}(x) = \begin{cases} -\mu & \text{if } u_k(x) \le -\mu, \\ u_k(x) & \text{if } -\mu < u_k(x) < \mu \\ \mu & \text{if } u_k(x) \ge \mu. \end{cases}$$

For $p \ge 2$, we have $|u_{k,\mu}|^{p-2}u_{k,\mu} \in H^s(\mathbb{R}^N)$, so we can take it as a test function in (2.17), we have

$$\frac{4(p-1)}{p^2}\int_{\mathbb{R}^N} \big(|(-\Delta)^{\frac{s}{2}}(u_{k,\mu})^{\frac{p}{2}}|^2 + \lambda||u_{k,\mu}|^{\frac{p}{2}}|^2\big) \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(H_k u_k)(K_k|u_{k,\mu}|^{p-2}u_{k,\mu})}{|x-y|^\mu} dydx + (\lambda-1)u|u_{k.\mu}|^{p-2}u_{k,\mu}.$$

By Lemma 2.7 with $\theta = 1$, there exists $C > 0$ such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|H_k||u_{k,\mu}||K_k||u_{k,\mu}|^{p-2}|u_{k,\mu}|}{|x-y|^\mu} dydx \le \frac{2(p-1)}{p^2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(u_{k,\mu})^{\frac{p}{2}}|^2 dx + C\int_{\mathbb{R}^N} ||u_{k,\mu}|^{\frac{p}{2}}|^2 dx.$$

We have thus

$$\frac{2(p-1)}{p^2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}(u_{k,\mu})^{\frac{p}{2}}|^2dx \le C_1\int_{\mathbb{R}^N}(|u_k|^2+|u|^2)dx+\int_{A_{k,\mu}}\Big(\frac{|H_ku_k|}{|x-y|^{\mu}}dy\Big)|K_ku_k|,$$

where

$$A_{k,\mu}=\{x\in\mathbb{R}^3:\ |u_k(x)|>\mu\}.$$

Since $p<\frac{2N}{N-\mu}$, by the Hardy-Littlewood-Sobolev inequality with $\frac{1}{r}=\frac{2N-\mu}{2N}+1-\frac{1}{p}$ and $\frac{1}{t}=\frac{2N-\mu}{2N}+\frac{1}{p}$,

$$\int_{A_{k,\mu}}\Big(\frac{1}{|x|^{\mu}}\star|K_k||u_k|^{p-1}dy\Big)|H_ku_k|\le C\Big(\int_{\mathbb{R}^N}||K_k||u_k|^{p-1}|^rdx\Big)^{\frac{1}{r}}\Big(\int_{A_{k,\mu}}|H_ku_k|^tdx\Big)^{\frac{1}{t}}.$$

By Hölder inequality, if $u_k\in L^p(\mathbb{R}^N)$, then $|K_k||u_k|^{p-1}\in L^r(\mathbb{R}^N)$, $|H_k||u_k|\in L^t(\mathbb{R}^N)$, thus by Lebesgue's dominated convergence theorem we have

$$\lim_{\mu\to\infty}\int_{A_{k,\mu}}\Big(\frac{1}{|x|^{\mu}}\star|K_k||u_k|^{p-1}dy\Big)|H_ku_k|=0.$$

In view of the Sobolev estimate, we have proved the inequality

$$\limsup_{k\to\infty}\Big(\int_{\mathbb{R}^N}|u_k|^{2_s^\star}dx\Big)^{\frac{2}{2_s^\star}}\le C\limsup_{k\to\infty}\int_{\mathbb{R}^N}|u_k|^2dx.$$

By iterating over $p$ a finite number of times we cover the range $p\in[2,\frac{N}{N-\mu}\frac{2N}{N-2s})$. □

# 3 The penalized problem

In this section, we will adapt for our case an argument explored by the penalization method introduction by del Pino and Felmer [18] to overcome the lack of compactness. Let $K>2$ to be determined later, and take $a>0$ to be the unique number such that $\frac{G(a)}{a}=\frac{V_0}{K}$ where $V_0$ is given by $(V_1)$. We define

$$\tilde{G}(u)=\begin{cases}G(u) & \text{if } u\le a,\\ \frac{V_0}{K}u & \text{if } u>a,\end{cases}$$

and

$$H(x,u)=\chi_{\Omega}(x)G(u)+\big(1-\chi_{\Omega}(x)\big)\tilde{G}(u),$$

where $\chi$ is characteristic function of set $\Omega$. From hypotheses $(f_1)$–$(f_3)$ we get that $H$ is a Carathéodory function and satisfies the following properties:

$(g_1)$ $H(x,u)\le G(u)\le C(|u|^{q_1}+|u|^{q_2}+|u|^{2^\star_{\mu,s}})$.

$(g_2)$ The function $\frac{H(x,u)}{u}$ is increasing for $u>0$.

$(g_3)$ $(i)$

$$\lim_{|u|\to+\infty}\frac{H(x,u)}{|u|^{2^\star_{\mu,s}-1}}=+\infty \text{ for } s\in(\frac{3}{4},1);$$

$(ii)$

$$\lim_{|u|\to+\infty}\frac{H(x,u)}{|u|^{2^\star_{\mu,s}-\frac{2s}{3-2s}}(log|u|)^{\frac{1}{2}}}=+\infty \text{ for } s=\frac{3}{4};$$

$(iii)$

$$\lim_{|u|\to+\infty}\frac{H(x,u)}{|u|^{2^\star_{\mu,s}-\frac{2s}{3-2s}}}=+\infty \text{ for } s\in(0,\frac{3}{4}).$$

Moreover, in order to find positive solutions, we shall henceforth consider $H(x,u) = 0$ for all $u \le 0$. It is easy to check that if $u$ is a positive solution of the equation

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u = \varepsilon^{\mu-3}\frac{1}{2^*_{\mu,s}}(\int_{\mathbb{R}^3}\frac{H(\varepsilon x,u)}{|x-y|^\mu}dy)h(\varepsilon x,u) \quad \text{in } \mathbb{R}^3 \\ u \in C^{0,\alpha}_{loc}(\mathbb{R}^3)\cap H^s(\mathbb{R}^3), \end{cases}$$

such that $u(x) \le a$ for all $x \in \mathbb{R}^3\backslash\Omega$, then $H(x,u) = G(u)$ and therefore $u$ is also a solution of problem (1.8).

In view of this argument above, we shall deal with in the following with the penalized problem

$$(-\Delta)^s u + V(\varepsilon x)u = \frac{1}{2^*_{\mu,s}}(\int_{\mathbb{R}^3}\frac{H(\varepsilon x,u)}{|x-y|^\mu}dy)h(\varepsilon x,u) \text{ in } \mathbb{R}^3, \tag{3.1}$$

and we will look for solutions $u_\varepsilon$ of problem (3.1) verifying

$$u_\varepsilon(x) \le a \text{ for all } x \in \mathbb{R}^3\backslash\Omega_\varepsilon,$$

where $\Omega_\varepsilon = \{x \in \mathbb{R}^3 : \varepsilon x \in \Omega\}$.

The energy functional associated with (3.1) is

$$J_\varepsilon(u) = \frac{1}{2}\|u\|^2_\varepsilon - \frac{1}{22^*_{\mu,s}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{H(\varepsilon x,u(y))H(\varepsilon x,u(x))}{|x-y|^\mu}dydx.$$

which is of $C^1$ class and whose derivative is given by

$$\langle J'_\varepsilon(u),v\rangle = \int_{\mathbb{R}^3}\big((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v + V(\varepsilon x)uv\big)dx - \frac{1}{2^*_{\mu,s}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{H(\varepsilon x,u(y))h(\varepsilon x,u(x))v(x)}{|x-y|^\mu}dydx$$

for all $u, v \in H_\varepsilon$. Hence the critical points of $J_\varepsilon$ in $H_\varepsilon$ are weak solutions of problem (3.1).

Now, we denote the Nehari manifold associated to $J_\varepsilon$ by

$$\mathcal{N}_\varepsilon = \{u \in H_\varepsilon\backslash\{0\} : \langle J'_\varepsilon(u),u\rangle = 0\}.$$

Obviously, $\mathcal{N}_\varepsilon$ contains all nontrivial critical points of $I_\varepsilon$. But we do not know whether $\mathcal{N}_\varepsilon$ is of class $C^1$ under our assumptions and therefore we cannot use minimax theorems directly on $\mathcal{N}_\varepsilon$. To overcome this difficulty, we will adopt a technique developed in [45, 46] to show that $\mathcal{N}_\varepsilon$ is still a topological manifold, naturally homeomorphic to the unit sphere of $H_\varepsilon$, and then we can consider a new minimax characterization of the corresponding critical value for $I_\varepsilon$.

For this we denote by $H^+_\varepsilon$ the subset of $H_\varepsilon$ given by

$$H^+_\varepsilon = \{u \in H_\varepsilon : |supp(u^+)\cap\Omega_\varepsilon| > 0\}$$

and $S^+_\varepsilon = S_\varepsilon \cap H^+_\varepsilon$, where $S_\varepsilon$ is the unit sphere of $H_\varepsilon$.

**Lemma 3.1.** *The set $H^+_\varepsilon$ is open in $H_\varepsilon$.*

*Proof.* Suppose by contradiction there are a sequence $\{u_n\} \subset H_\varepsilon\backslash H^+_\varepsilon$ and $u \in H^+_\varepsilon$ such that $u_n \to u$ in $H_\varepsilon$. Hence $|supp(u_n^+)\cap\Omega_\varepsilon| = 0$ for all $n \in \mathbb{N}$ and $u_n^+(x) \to u^+(x)$ a.e. in $x \in \Omega_\varepsilon$. So,

$$u^+(x) = \lim_{n\to\infty} u_n^+(x) = 0, \text{ a.e. in } x \in \Omega_\varepsilon.$$

But, this contradicts the fact that $u \in H^+_\varepsilon$. Therefore $H^+_\varepsilon$ is open. □

From definition of $S^+_\varepsilon$ and Lemma 3.1 it follows that $S^+_\varepsilon$ is a incomplete $C^{1,1}$-manifold of codimension 1, modeled on $H_\varepsilon$ and contained in the open $H^+_\varepsilon$. Hence, $H_\varepsilon = T_u S^+_\varepsilon \oplus \mathbb{R}u$ for each $u \in S^+_\varepsilon$, where $T_u S^+_\varepsilon = \{v \in H_\varepsilon : (u,v)_\varepsilon = 0\}$.

In the rest of this section, we show some Lemmas related to the function $J_\varepsilon$ and the set $H^+_\varepsilon$. First, we show the functional $J_\varepsilon$ satisfying the Mountain Pass geometry.

**Lemma 3.2.** *The functional $J_\varepsilon$ satisfies the following conditions:*
(*i*) *There exist $\alpha, \rho > 0$ such that $J_\varepsilon(u) \geq \alpha$ with $\|u\|_\varepsilon = \rho$;*
(*ii*) *There exists $e \in H_\varepsilon$ satisfying $\|e\|_\varepsilon > \rho$ such that $J_\varepsilon(e) < 0$.*

*Proof.* (*i*). For any $u \in H_\varepsilon \backslash \{0\}$, it follows from $(g_1)$ and the Hardy-Littlewood-Sobolev inequality that

$$|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{H(\varepsilon x, u(y))H(\varepsilon x, u(x))}{|x-y|^\mu} dydx| \leq C(|u|_{q_1 r_0}^{2q_1} + |u|_{q_2 r_0}^{2q_2} + |u|_{2_s^\star}^{22^\star_{\mu,s}}) \tag{3.2}$$

Hence,

$$\begin{aligned} J_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{s}{2}} u|^2 + V(\varepsilon x)u^2\right) dx - \frac{1}{22^\star_{\mu,s}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{H(\varepsilon x, u(y))H(\varepsilon x, u(x))}{|x-y|^\mu} dydx. \\ &\geq \frac{1}{2}\|u\|_\varepsilon^2 - C_1\|u\|^{2q_1} - C_2\|u\|^{2q_2} - C_3\|u\|^{22^\star_{\mu,s}}. \end{aligned}$$

Therefore, we can choose positive constants $\alpha, \rho$ such that

$$J_\varepsilon(u) \geq \alpha \text{ with } \|u\|_\varepsilon = \rho.$$

(*ii*). Fix a positive function $u_0 \in H_\varepsilon^+$ with $supp(u_0) \subset \Omega_\varepsilon$, and we set

$$\psi(t) := \Sigma_\varepsilon(\frac{tu_0}{\|u_0\|_\varepsilon}) > 0,$$

where

$$\Sigma_\varepsilon(u) = \frac{1}{22^\star_{\mu,s}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{H(\varepsilon x, u(y))H(\varepsilon x, u(x))}{|x-y|^\mu} dydx.$$

Since $H(\varepsilon x, u_0) = F(u_0)$ and by using Lemma 2.4, we deduce that

$$\begin{aligned} \psi'(t) &= \Sigma'_\varepsilon(\frac{tu_0}{\|u_0\|_\varepsilon})\frac{u_0}{\|u_0\|_\varepsilon} \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (\frac{F(\frac{tu_0}{\|u_0\|_\varepsilon})f(\frac{tu_0}{\|u_0\|_\varepsilon})\frac{u_0}{\|u_0\|_\varepsilon}}{|x-y|^\mu}) dxdy \\ &= \frac{22^\star_{\mu,s}}{t} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{22^\star_{\mu,s}} (\frac{F(\frac{tu_0}{\|u_0\|_\varepsilon})f(\frac{tu_0}{\|u_0\|_\varepsilon})\frac{tu_0}{\|u_0\|_\varepsilon}}{|x-y|^\mu}) dxdy \\ &\geq \frac{22^\star_{\mu,s}}{t}\psi(t). \end{aligned} \tag{3.3}$$

Integrating (3.3) on $[1, t\|u_0\|_\varepsilon]$ with $t > \frac{1}{\|u_0\|_\varepsilon}$, we have

$$\Sigma_\varepsilon(tu_0) \geq \Sigma_\varepsilon(\frac{u_0}{\|u_0\|_\varepsilon})\|u_0\|_\varepsilon^{22^\star_{\mu,s}} t^{22^\star_{\mu,s}}$$

Therefore, we have

$$J_\varepsilon(tu_0) = \frac{t^2}{2}\|u\|_\varepsilon^2 - \Sigma_\varepsilon(tu_0) \leq C_1 t^2 - C_2 t^{22^\star_{\mu,s}} \text{ for } t > \frac{1}{\|u_0\|_\varepsilon}.$$

Taking $e = tu_0$ with $t$ sufficiently large, we can see (*ii*) holds. □

Since $f$ is only continuous, the next two results are very important because they allow us to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of $S_\varepsilon^+$.

**Lemma 3.3.** *Assume that the potential $V$ satisfies $(V_1) - (V_2)$ and the functional $f$ satisfies $(f_1) - (f_3)$. Then the following properties hold:*

$(A_1)$*For each* $u \in H_\varepsilon^+$, *let* $\varphi_u : \mathbb{R}^+ \to \mathbb{R}$ *be given by* $\varphi_u(\tau) = J_\varepsilon(\tau u)$. *Then there exists a unique* $\tau_u > 0$ *such that* $\varphi_u'(\tau) > 0$ *in* $(0, \tau_u)$ *and* $\varphi_u'(\tau) < 0$ *in* $(\tau_u, \infty)$.

$(A_2)$*There is a* $\sigma > 0$ *independent on* $u$ *such that* $\tau_u > \sigma$ *for all* $u \in S_\varepsilon^+$. *Moreover, for each compact set* $\mathcal{W} \subset S_\varepsilon^+$ *there is* $C_{\mathcal{W}} > 0$ *such that* $\tau_u \le C_{\mathcal{W}}$ *for all* $u \in \mathcal{W}$.

$(A_3)$*The map* $\hat{m}_\varepsilon : H_\varepsilon^+ \to \mathcal{N}_\varepsilon$ *given by* $\hat{m}_\varepsilon(u) = \tau_u u$ *is continuous and* $m_\varepsilon := \hat{m}_\varepsilon|_{S_\varepsilon^+}$ *is a homeomorphism between* $S_\varepsilon^+$ *and* $\mathcal{N}_\varepsilon$. *Moreover,* $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon}$.

*Proof.* $(A_1)$ From Lemma 3.2, it is sufficient to note that, $\varphi_u(0) = 0$, $\varphi_u(\tau) > 0$ when $\tau > 0$ is small and $\varphi_u(\tau) < 0$ when $\tau > 0$ is large. Since $\varphi_u \in C^1(\mathbb{R}^+, \mathbb{R})$, there is $\tau_u > 0$ global maximum point of $\varphi_u$ and $\varphi_u'(\tau_u) = 0$. Thus, $J_\varepsilon'(\tau_u u)(\tau_u u) = 0$ and $\tau_u u \in \mathcal{N}_\varepsilon$. We see that $\tau_u > 0$ is the unique positive number such that $\varphi_u'(\tau_u) = 0$. Indeed, suppose by contradiction that there are $\tau_1 > \tau_2 > 0$ with $\varphi_u'(\tau_1) = \varphi_u'(\tau_2) = 0$. Then, for $i = 1, 2$ we have that

$$0 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{|x-y|^\mu}\left(\frac{H(\varepsilon x, \tau_1 u(u))}{\tau_1} h(\varepsilon x, \tau_1 u(x))u(x) - \frac{H(\varepsilon x, \tau_2 u(u))}{\tau_2} h(\varepsilon x, \tau_2 u(x))u(x)\right) dydx. \tag{3.4}$$

By Lemma 2.4 and $(h_2)$ we know that $h(\varepsilon x, \tau u(x))u(x)$ and $\frac{H(\varepsilon x, \tau u(x))}{\tau}$ are positive and increasing in $\tau$. Then (3.4) in impossible and $(A_1)$ is proved.

$(A_2)$ Suppose $u \in S_\varepsilon^+$, then as (2.5) we have

$$\tau_u \le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{H(\varepsilon x, u(y))h(\varepsilon x, u(x))u(x)}{|x-y|^\mu} dydx \le C(\|\tau u\|^{2q_1} + \|\tau u\|^{2q_2} + \|\tau u\|^{22^*_{\mu,s}}).$$

From previous inequality we obtain $\sigma > 0$ independent on $u$, such that $\tau_u > \sigma$.

Finally, if $\mathcal{W} \subset S_\varepsilon^+$ is compact, and suppose by contradiction that there is $\{u_n\} \subset \mathcal{W}$ such that $\tau_n := \tau_{u_n} \to \infty$. Since $\mathcal{W}$ is compact, there is $u \in \mathcal{W}$ such that $u_n \to u$ in $H_\varepsilon$. Then $u \in \mathcal{W} \subset S_\varepsilon^+$. By $(h_2)$ we obtain

$$0 \le \frac{J_\varepsilon(\tau_n u_n)}{\tau_n^2} = \frac{1}{2} - \frac{1}{22^*_{\mu,s}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{H(\varepsilon x, \tau_n u_n(y))}{|x-y|^\mu \tau_n} \frac{H(\varepsilon x, \tau_n u_n(x))}{\tau_n} dydx \to -\infty,$$

which yields a contradiction. Therefore $(A_2)$ is true.

$(A_3)$ First of all we observe that $\hat{m}_\varepsilon$, $m_\varepsilon$ and $m_\varepsilon^{-1}$ are well defined. In fact, by $(A_1)$, for each $u \in H_\varepsilon^+$, there exists a unique $\tau_u > 0$ such that $\tau_u u \in \mathcal{N}_\varepsilon$, hence there is a unique $\hat{m}_\varepsilon(u) = \tau_u u \in \mathcal{N}_\varepsilon$. On the other hand, if $u \in \mathcal{N}_\varepsilon$ then $u \in H_\varepsilon^+$. Therefore, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon} \in S_\varepsilon^+$, is well defined and it is a continuous function. Since

$$m_\varepsilon^{-1}(m_\varepsilon(u)) = m_\varepsilon^{-1}(t_u u) = \frac{\tau_u u}{\tau_u \|u\|_\varepsilon} = u, \ \forall\, u \in S_\varepsilon^+,$$

we conclude that $m_\varepsilon$ is a bijection.

To prove $\hat{m}_\varepsilon : H_\varepsilon^+ \to \mathcal{N}_\varepsilon$ is continuous, let $\{u_n\} \subset H_\varepsilon^+$ and $u \in H_\varepsilon^+$ be such that $u_n \to u$ in $H_\varepsilon$. By $(A_2)$, there is a $\tau_0 > 0$ up to a subsequence such that $\tau_n := \tau_{u_n} \to \tau_0$. Since $\tau_n u_n \in \mathcal{N}_\varepsilon$ we obtain

$$\tau_n^2 \|u_n\|_\varepsilon^2 = \frac{1}{2^*_{\mu,s}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{H(\varepsilon x, \tau_n u_n(y))h(\varepsilon x, \tau_n u(x))\tau_n u_n(x)}{|x-y|^\mu} dydx, \ \forall\, n \in \mathbb{N}.$$

By Lemma 2.5 and passing to the limit as $n \to \infty$, it follows that

$$\tau_0^2 \|u\|_\varepsilon^2 = \frac{1}{2^*_{\mu,s}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{H(\varepsilon x, \tau_0 u_n(y))h(\varepsilon x, \tau_0 u(x))\tau_0 u_n(x)}{|x-y|^\mu} dydx,$$

which means that $\tau_0 u \in \mathcal{N}_\varepsilon$ and $\tau_u = \tau_0$. This proves $\hat{m}_\varepsilon(u_n) \to \hat{m}_\varepsilon(u)$ in $H_\varepsilon^+$. So, $\hat{m}_\varepsilon$, $m_\varepsilon$ are continuous functions and $(A_3)$ is proved. □

Now we define the functions

$$\hat{\Psi}_\varepsilon : H_\varepsilon^+ \to \mathbb{R} \text{ and } \Psi_\varepsilon : S_\varepsilon^+ \to \mathbb{R},$$

by $\hat{\Psi}_\varepsilon(u) = I_\varepsilon(\hat{m}_\varepsilon(u))$ and $\Psi_\varepsilon := \hat{\Psi}_\varepsilon|_{S_\varepsilon^+}$. The next result is a direct consequence of Lemma 3.3. The details can be seen in the relevant material from [46]. For the convenience of the reader, here we do a sketch of the proof.

**Lemma 3.4.** *Assume that $(V_1)$ – $(V_2)$ and $(f_1)$ – $(f_3)$ are satisfied. Then :*
*$(B_1)\hat{\Psi}_\varepsilon \in C^1(H_\varepsilon^+, \mathbb{R})$ and*

$$\hat{\Psi}'_\varepsilon(u)v = \frac{\|\hat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon} J'_\varepsilon(\hat{m}_\varepsilon(u))v, \quad \forall u \in H_\varepsilon^+ \text{ and } \forall\, v \in H_\varepsilon.$$

*$(B_2)\Psi_\varepsilon \in C^1(S_\varepsilon^+, \mathbb{R})$ and*

$$\Psi'_\varepsilon(u)v = \|m_\varepsilon(u)\|_\varepsilon J'_\varepsilon(m_\varepsilon(u))v, \quad \forall v \in T_u S_\varepsilon^+.$$

*$(B_3)$If $\{u_n\}$ is a $(PS)_c$ sequence of $\Psi_\varepsilon$, then $\{m_\varepsilon(u_n)\}$ is a $(PS)_c$ sequence of $J_\varepsilon$. If $\{u_n\} \subset \mathcal{N}_\varepsilon$ is a bounded $(PS)_c$ sequence for $J_\varepsilon$, then $\{m_\varepsilon^{-1}(u_n)\}$ is a $(PS)_c$ sequence of $\Psi_\varepsilon$.*
*$(B_4)u$ is a critical point of $\Psi_\varepsilon$ if and only if, $m_\varepsilon(u)$ is a critical point of $J_\varepsilon$. Moreover, corresponding critical values coincide and*

$$\inf_{S_\varepsilon^+} \Psi_\varepsilon = \inf_{\mathcal{N}_\varepsilon} I_\varepsilon.$$

*Proof.* $(B_1)$ Let $u \in H_\varepsilon^+$ and $v \in H_\varepsilon$. From definition of $\hat{\Psi}_\varepsilon$ and $t_u$ and the mean value theorem, we obtain

$$\begin{aligned}\hat{\Psi}_\varepsilon(u + hv) - \hat{\Psi}_\varepsilon(u) &= J_\varepsilon\big(\tau_{u+hv}(u + hv)\big) - J_\varepsilon(\tau_u u)\\ &\le J_\varepsilon\big(\tau_{u+hv}(u + hv)\big) - J_\varepsilon(\tau_{u+hv}u)\\ &= J'_\varepsilon\big(\tau_{u+hv}(u + \theta hv)\big)\tau_{u+hv}hv,\end{aligned}$$

where $|h|$ is small enough and $\theta \in (0, 1)$. Similarly,

$$\hat{\Psi}_\varepsilon(u + hv) - \hat{\Psi}_\varepsilon(u) \ge J_\varepsilon(\tau_u(u + hv)) - J_\varepsilon(\tau_u u) = J'_\varepsilon(\tau_u(u + \varsigma hv))\tau_u hv,$$

where $\varsigma \in (0, 1)$. Since the mapping $u \mapsto \tau_u$ is continuous according to Lemma 3.3, we see combining these two inequalities that

$$\lim_{h\to 0} \frac{\hat{\Psi}_\varepsilon(u + hv) - \hat{\Psi}_\varepsilon(u)}{h} = \tau_u J'_\varepsilon(\tau_u u)v = \frac{\|\hat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon} J'_\varepsilon(\hat{m}_\varepsilon(u))v.$$

Since $J_\varepsilon \in C^1$, it follows that the Gâteaux derivative of $\hat{\Psi}_\varepsilon$ is bounded linear in $v$ and continuous on $u$. From [50] we know that $\hat{\Psi}_\varepsilon \in C^1(H_\varepsilon^+, \mathbb{R})$ and

$$\hat{\Psi}'_\varepsilon(u)v = \frac{\|\hat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon} J'_\varepsilon(\hat{m}_\varepsilon(u))v, \quad \forall u \in H_\varepsilon^+ \text{ and } \forall\, v \in H_\varepsilon.$$

The item $(B_1)$ is proved.

$(B_2)$ The item $(B_2)$ is a direct consequence of the item $(B_1)$.

$(B_3)$ We first note that $H_\varepsilon = T_u S_\varepsilon^+ \oplus \mathbb{R}u$ for every $u \in S_\varepsilon^+$ and the linear projection $P : H_\varepsilon \to T_u S_\varepsilon^+$ defined by $P(v + \tau u) = v$ is continuous, namely, there is $C > 0$ such that

$$\|v\|_\varepsilon \le C\|v + \tau u\|_\varepsilon, \quad \forall\, u \in S_\varepsilon^+, v \in T_u S_\varepsilon^+ \text{ and } \tau \in \mathbb{R}. \tag{3.5}$$

Moreover, by $(B_1)$ we have

$$\|\Psi'_\varepsilon\| = \sup_{v\in T_u S_\varepsilon^+, \|v\|_\varepsilon=1} \Psi'_\varepsilon(u)v = \|w\|_\varepsilon \sup_{v\in T_u S_\varepsilon^+, \|v\|_\varepsilon=1} J'_\varepsilon(w)v, \tag{3.6}$$

where $w = m_\varepsilon(u)$. Since $w \in \mathcal{N}_\varepsilon$, we conclude that

$$J'_\varepsilon(w)u = J'_\varepsilon(w)\frac{w}{\|w\|_\varepsilon} = 0. \tag{3.7}$$

Hence, from (3.5) and (3.7) we have

$$\|\Psi'_\varepsilon(u)\| \le \|w\|_\varepsilon \|J'_\varepsilon(w)\| \le C\|w\|_\varepsilon \sup_{v\in T_u S_\varepsilon^+\setminus\{0\}} \frac{J'_\varepsilon(w)v}{\|v\|_\varepsilon} = C\|\Psi'_\varepsilon(u)\|,$$

which showing that

$$\|\Psi'_\varepsilon(u)\| \le \|w\|_\varepsilon \|J'_\varepsilon(w)\| \le C\|\Psi'_\varepsilon(u)\|, \ \forall\, u \in S_\varepsilon^+. \tag{3.8}$$

Since $w \in \mathcal{N}_\varepsilon$, we have $\|w\| \geq \gamma > 0$. Therefore, the inequality in (3.8) together with $J_\varepsilon(w) = \Psi_\varepsilon(u)$ imply the item $(B_3)$.

$(B_4)$ It follow from (3.8) that $\Psi'_\varepsilon(u) = 0$ if and only if $J'_\varepsilon(w) = 0$. The remainder follows from definition of $\Psi_\varepsilon$. □

As in [46], using the mountain pass theorem without the $(PS)$ condition, we get the existence of a $(PS)_{c_\varepsilon}$ sequence $\{u_n\} \subset H_\varepsilon$ with

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u) = \inf_{u\in H_\varepsilon^+} \max_{\tau>0} I_\varepsilon(\tau u) = \inf_{u\in S_\varepsilon^+} \max_{\tau>0} I_\varepsilon(\tau u) > 0.$$

**Lemma 3.5.** *Suppose that $(f_1)$ – $(f_3)$ hold. Assume that $\{u_n\} \subset \mathcal{N}_\varepsilon$ is a $(PS)_c$-sequence with*

$$0 < c_\varepsilon \leq c < \frac{2^\star_{\mu,s} - 1}{22^\star_{\mu,s}} S_{H,L}^{\frac{2^\star_{\mu,s}}{2^\star_{\mu,s}-1}}.$$

*Then $\{u_n\}$ is bounded in $H_\varepsilon$. Moreover, $\{u_n\}$ cannot be vanishing, namely there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that*

$$\liminf_{n\to\infty} \int\limits_{B_r(y_n)} |u_n|^2 dx \geq \delta.$$

*Proof.* We first prove the boundedness of $\{u_n\}$. Argue by contradiction we suppose that $\{u_n\}$ is unbounded in $H_\varepsilon$. Without loss of generality, assume that $\|u_n\|_\varepsilon \to \infty$. Let $v_n := \frac{u_n}{\|u_n\|_\varepsilon}$, up to a subsequence, then there exists $v \in H_\varepsilon$ such that

$$\begin{aligned}
&u_n \rightharpoonup u \text{ in } H_\varepsilon,\\
&u_n \to u \text{ in } L^r_{loc}(\mathbb{R}^3),\ 2 \leq r < 2^\star_s,\\
&u_n(x) \to u(x)\ a.e. \text{ in } \mathbb{R}^3.
\end{aligned}$$

If $v_n$ is vanishing, i.e.

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int\limits_{B_r(y)} v_n^2(x) dx = 0,$$

then Lemma 2.2 implies that $v_n \to 0$ in $L^{r_0 q_1}(\mathbb{R}^3)$ and $L^{r_0 q_2}(\mathbb{R}^3)$. By (2.4) and (2.5) we get

$$\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{F(\tau v_n(x))F(\tau v_n(y))}{|x-y|^\mu} dxdy \to 0, \quad \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{F(\tau v_n(y))}{|x-y|^\mu} |\tau v_n(x)|^{2^\star_{\mu,s}} dxdy \to 0. \tag{3.9}$$

Then for sufficiently large $n$ we have

$$\begin{aligned}
c_\varepsilon + o_n(1) = I_\varepsilon(u_n) &\geq \sup_{\tau\geq 0} I_\varepsilon(\tau v_n)\\
&\geq \sup_{\tau\geq 0} \Big(\frac{\tau^2}{2} - \frac{\tau^{22^\star_{\mu,s}}}{22^\star_{\mu,s}} S_{H,L}^{-2^\star_{\mu,s}}\Big) + o_n(1)\\
&= \frac{2^\star_{\mu,s} - 1}{22^\star_{\mu,s}} S_{H,L}^{\frac{2^\star_{\mu,s}}{2^\star_{\mu,s}-1}} + o_n(1),
\end{aligned}$$

which is a contradiction. Therefore, $\{v_n\}$ is non-vanishing, namely there exists $y_n \in \mathbb{R}^3$ and $\delta > 0$ such that

$$\int\limits_{B_r(y_n)} v_n^2(x) dx > \delta. \tag{3.10}$$

Denote $\tilde{v_n}(\cdot) = v_n(\cdot + y_n)$, then we can assume that

$$\begin{aligned}
&\tilde{v_n} \rightharpoonup \tilde{v} \text{ in } H_\varepsilon,\\
&\tilde{v_n} \to \tilde{v} \text{ in } L^r_{loc}(\mathbb{R}^3),\ 2 \leq r < 2^\star_s,\\
&\tilde{v_n}(x) \to \tilde{v}(x)\ a.e. \text{ in } \mathbb{R}^3.
\end{aligned}$$

With the use of (3.10), we have $\tilde{v} \not\equiv 0$. Then there exists a measure set $\Lambda$ such that $\tilde{v}(x) \neq 0$ for $x \in \Lambda$. Let $|\bar{u_n}| := |\tilde{v_n}| \|u_n\|_\varepsilon$. Then $|\bar{u_n}(x)| \to +\infty$ for $x \in \Lambda$. By Lemma 2.4, we have

$$\int_\Lambda \int_\Lambda \frac{1}{|x-y|^\mu} \frac{F(\bar{u_n}(y))}{|\bar{u_n}(y)|} |\tilde{v_n}(y)| \frac{F(\bar{u_n}(x))}{|\bar{u_n}(x)|} |\tilde{v_n}(x)| dydx \to +\infty.$$

Therefor, Lemma 2.4 and Fatou Lemma imply that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x-y|^\mu} \frac{F(\bar{u_n}(y))}{|\bar{u_n}(y)|} |\tilde{v_n}(y)| \frac{F(\bar{u_n}(x))}{|\bar{u_n}(x)|} |\tilde{v_n}(x)| dydx = +\infty.$$

Namely

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x-y|^\mu} \frac{F(u_n(y))}{\|u_n\|_\varepsilon} \frac{F(u_n(x))}{\|u_n\|_\varepsilon} dydx = +\infty.$$

Then,

$$\frac{c_\varepsilon}{\|u_n\|_\varepsilon} + o_n(1) = \frac{I_\varepsilon(u_n)}{\|u_n\|_\varepsilon} \to -\infty,$$

which is a contradiction. Therefore, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$.

Next we show the second conclusion. We argue by contradiction, if $\{u_n\}$ is vanishing, then similar to (3.9) we have

$$c_\varepsilon + o_n(1) = I_\varepsilon(u_n) = \frac{1}{2}\|u_n\|_\varepsilon^2 - \frac{1}{22^\star_{\mu,s}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(y)|^{2^\star_{\mu,s}} |u_n(x)|^{2^\star_{\mu,s}}}{|x-y|^\mu} dydx + o_n(1), \tag{3.11}$$

and

$$0 = \|u_n\|_\varepsilon^2 - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(y)|^{2^\star_{\mu,s}} |u_n(x)|^{2^\star_{\mu,s}}}{|x-y|^\mu} dydx + o_n(1). \tag{3.12}$$

If $\|u_n\|_\varepsilon \to 0$, then it follows from (3.11) and (3.12) that $c_\varepsilon = 0$, which is impossible. Then $\|u_n\|_\varepsilon \nrightarrow 0$ and by virtue of (3.12) we get

$$\|u_n\|_\varepsilon^2 \leq S_{H,L}^{2^\star_{\mu,s}} \|u_n\|_\varepsilon^{22^\star_{\mu,s}} + o_n(1).$$

Hence,

$$\liminf_{n\to\infty} \|u_n\|_\varepsilon^2 \geq S_{H,L}^{\frac{2^\star_{\mu,s}}{2^\star_{\mu,s}-1}}.$$

From (3.11) and (3.12) we deduce that

$$c_\varepsilon + o_n(1) = I_\varepsilon(u_n) \geq \frac{2^\star_{\mu,s} - 1}{22^\star_{\mu,s}} S_{H,L}^{\frac{2^\star_{\mu,s}}{2^\star_{\mu,s}-1}}, \tag{3.13}$$

which is a contradiction. Therefore, $\{u_n\}$ is non-vanishing. □

**Lemma 3.6.** *Assume $(V_1)-(V_2)$ and $(f_1)-(f_3)$ hold, let $\{u_n\}$ be a $(PS)_c$ sequence for $J_\varepsilon$ with $c \in [c_\varepsilon, \frac{2^\star_{\mu,s}-1}{22^\star_{\mu,s}} S_{H,L}^{\frac{2^\star_{\mu,s}}{2^\star_{\mu,s}-1}})$. Then, for each $\eta > 0$ there exists $R = R(\eta) > 0$ such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3 \setminus B_R} |(-\Delta)^{\frac{s}{2}} u|^2 + V(\varepsilon x) u_n^2 dx < \eta.$$

*Proof.* By Lemma 3.5, we can have $\{u_n\}$ is bounded in $H_\varepsilon$. Therefore, we may assume that $u_n \rightharpoonup u$ in $H_\varepsilon$ and $u_n \to u$ in $L^r_{loc}(\mathbb{R}^3)$ for any $r \in [2, 2^\star_s)$. Fix $R > 0$ and let $\psi_R \in C^\infty(\mathbb{R}^3)$ be such that $\psi_R = 0$ in $B_{\frac{R}{2}}(0)$, $\psi_R = 1$

in $B_R^c$, $\psi_R \in [0,1]$ and $|\nabla \psi_R| \le \frac{C}{R}$, where $C$ is a constant independent of $R$. Since $\{u_n\psi_R\}$ is bounded we can see that

$$\begin{aligned}\int_{\mathbb{R}^3} \big((-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}}(u_n\psi_R) + V(\varepsilon x)\psi_R u_n^2\big)\,dx &= \langle J_\varepsilon'(u_n), u_n\psi_R\rangle \\ &\quad + \frac{1}{2^\star_{\mu,s}} \int_{\mathbb{R}^3} \Big(\frac{1}{|x|^\mu} \star H(\varepsilon x, u_n)\Big) h(\varepsilon x, u_n) u_n \psi_R\,dx \\ &= o_n(1) + \frac{1}{2^\star_{\mu,s}} \int_{\mathbb{R}^3} \Big(\frac{1}{|x|^\mu} \star H(\varepsilon x, u_n)\Big) h(\varepsilon x, u_n) u_n \psi_R\,dx.\end{aligned}$$

By Lemma 2.8, taking

$$H(u) := \frac{|u|^{2^\star_{\mu,s}} + F(u)}{u},\ K(u) := |u|^{2^\star_{\mu,s}-2-\mu} + \frac{1}{2^\star_{\mu,s}} f(u) \in L^{\frac{2}{3-\mu}}(\mathbb{R}^3) + L^{\frac{6}{3+2s-\mu}}(\mathbb{R}^3),$$

we have

$$\int_{\mathbb{R}^3} \Big(\frac{1}{|x|^\mu} \star H(\varepsilon x, u_n)\Big) h(\varepsilon x, u_n) u_n\,dx \le \int_{\mathbb{R}^3} \Big(\frac{1}{|x|^\mu} \star H(u_n) u_n\Big) K(u_n) u_n\,dx \le \varepsilon^2 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2\,dx + C_\varepsilon \int_{\mathbb{R}^N} |u|^2\,dx.$$

For $n \ge n_0$ and $\varepsilon > 0$ fixed, take $R > 0$ big enough such that $\Omega_\varepsilon \subset B_{R/2}$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^3\setminus B_{R/2}} \big(|(-\Delta)^{\frac{s}{2}} u_n|^2 + V(\varepsilon x) u_n^2\big)\,dx &\le \frac{1}{2^\star_{\mu,s}} \int_{\mathbb{R}^3\setminus B_{R/2}} \Big(\frac{1}{|x|^\mu} \star H(\varepsilon x, u_n)\Big) h(\varepsilon x, u_n) u_n\,dx + o_n(1) \\ &\quad - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u_n(x)-u_n(y))(\psi_R(x)-\psi_R(y))}{|x-y|^{3+2s}} u_n(y)\,dx\,dy.\end{aligned}$$

which means

$$\frac{1}{2} \int_{\mathbb{R}^3\setminus B_{R/2}} \big(|(-\Delta)^{\frac{s}{2}} u_n|^2 + V(\varepsilon x) u_n^2\big)\,dx \le o_n(1) - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u_n(x)-u_n(y))(\psi_R(x)-\psi_R(y))}{|x-y|^{3+2s}} u_n(y)\,dx\,dy$$

Now, we note that the Hölder inequality and the boundedness of $\{u_n\}$ imply that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u_n(x)-u_n(y))(\psi_R(x)-\psi_R(y))}{|x-y|^{3+2s}} u_n(y)\,dx\,dy\Big| \\ &\le \Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}\,dx\,dy\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}} u_n^2(y)\,dx\,dy\Big)^{\frac{1}{2}} \\ &\le C\Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}} u_n^2(y)\,dx\,dy\Big)^{\frac{1}{2}}.\end{aligned}$$

Therefore, it is enough to prove that

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}} u_n^2(y)\,dx\,dy = 0$$

to conclude our result.

Let us note that $\mathbb{R}^3 \times \mathbb{R}^3$ can be written as

$$\mathbb{R}^3 \times \mathbb{R}^3 = \big((\mathbb{R}^3\setminus B_{2R}) \times (\mathbb{R}^3\setminus B_{2R})\big) \cup \big((\mathbb{R}^3\setminus B_{2R}) \times B_{2R}\big) \cup \big(B_{2R} \times \mathbb{R}^3\big) := X^1 \cup X^1 \cup X^3.$$

Then

$$\begin{aligned}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}}u_n^2(x)dxdy &= \int\int_{X^1} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}}u_n^2(x)dxdy \\ &+ \int\int_{X^2} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}}u_n^2(x)dxdy + \int\int_{X^3} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}}u_n^2(x)dxdy.\end{aligned} \tag{3.14}$$

Now, we estimate each integral in (3.14). Since $\psi_R = 1$ in $\mathbb{R}^3\backslash B_{2R}$, we have

$$\int\int_{X^1} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}}u_n^2(x)dxdy = 0. \tag{3.15}$$

Let $k > 4$, we have

$$X^2 = (\mathbb{R}^3\backslash B_{2R})\times B_{2R} = (\mathbb{R}^3\backslash B_{kR})\times B_{2R} \cup (B_{kR}\backslash B_{2R})\times B_{2R}.$$

Let us note that, if $(x,y)\in(\mathbb{R}^3\backslash B_{kR})\times B_{2R}$, then

$$|x-y| \ge |x|-|y| \ge |x|-2R > \frac{|x|}{2}.$$

Therefore, taking into account $0\le\psi_R\le 1$, $|\nabla\psi_R|\le\frac{C}{R}$ and applying Hölder inequality, we can see

$$\begin{aligned}&\int\int_{X^2} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}}u_n^2(x)dxdy \\ &= \int_{\mathbb{R}^3\backslash B_{kR}}\int_{B_{2R}} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}}u_n^2(x)dxdy + \int_{B_{kR}\backslash B_{2R}}\int_{B_{2R}} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}}u_n^2(x)dxdy \\ &\le 2^{5+2s}\int_{\mathbb{R}^3\backslash B_{kR}}\int_{B_{2R}} \frac{|u_n(x)|^2}{|x|^{3+2s}}dxdy + \frac{C}{R^2}\int_{B_{kR}\backslash B_{2R}}\int_{B_{2R}} \frac{|u_n(x)|^2}{|x-y|^{3+2(s-1)}}dxdy \\ &\le CR^3\int_{\mathbb{R}^3\backslash B_{kR}} \frac{|u_n(x)|^2}{|x|^{3+2s}}dxdy + \frac{C}{R^2}(kR)^{2(1-s)}\int_{B_{kR}\backslash B_{2R}} u_n^2(x)dxdy \\ &\le CR^3\Big(\int_{\mathbb{R}^3\backslash B_{kR}} |u_n(x)|^{2_s^*}dx\Big)^{\frac{2}{2_s^*}}\Big(\int_{\mathbb{R}^3\backslash B_{kR}} \frac{1}{|x|^{\frac{9}{2s}+3}}\Big)^{\frac{2s}{3}} + \frac{Ck^{2(1-s)}}{R^{2s}}\int_{B_{kR}\backslash B_{2R}} u_n^2(x)dx \\ &\le \frac{C}{k^3}\Big(\int_{\mathbb{R}^3\backslash B_{kR}} |u_n(x)|^{2_s^*}dx\Big)^{\frac{2}{2_s^*}} + \frac{Ck^{2(1-s)}}{R^{2s}}\int_{B_{kR}\backslash B_{2R}} u_n^2(x)dx \\ &\le \frac{C}{k^3} + \frac{Ck^{2(1-s)}}{R^{2s}}\int_{B_{kR}\backslash B_{2R}} u_n^2(x)dx.\end{aligned} \tag{3.16}$$

Now, fix $\varepsilon\in(0,\frac{1}{2})$, and we note that

$$\begin{aligned}&\int\int_{X^3} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}}u_n^2(x)dxdy \\ &\le \int_{B_{2R}\backslash B_{\varepsilon R}}\int_{\mathbb{R}^3} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}}u_n^2(x)dxdy + \int_{B_{\varepsilon R}}\int_{\mathbb{R}^3} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}}u_n^2(x)dxdy.\end{aligned} \tag{3.17}$$

Let us estimate the first integral in (3.17). First, we have

$$\int_{B_{2R}\backslash B_{\varepsilon R}}\int_{\mathbb{R}^3\cap\{y:|x-y|<R\}} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}}u_n^2(x)dxdy \le \frac{C}{R^{2s}}\int_{B_{2R}\backslash B_{\varepsilon R}} u_n^2(x)dx$$

and

$$\int_{B_{2R}\setminus B_{\varepsilon R}} \int_{\mathbb{R}^3\cap\{y:|x-y|\geq R\}} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}} u_n^2(x)dxdy \leq \frac{C}{R^{2s}} \int_{B_{2R}\setminus B_{\varepsilon R}} u_n^2(x)dx.$$

Then

$$\int_{B_{2R}\setminus B_{\varepsilon R}} \int_{\mathbb{R}^3} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}} u_n^2(x)dxdy \leq \frac{C}{R^{2s}} \int_{B_{2R}\setminus B_{\varepsilon R}} u_n^2(x)dx. \tag{3.18}$$

By using the definition of $\psi_R$, $\varepsilon \in (0,1)$ and $\psi_R \leq 1$, we have

$$\begin{aligned}\int_{B_{\varepsilon R}} \int_{\mathbb{R}^3} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}} u_n^2(x)dxdy &= \int_{B_{\varepsilon R}} \int_{\mathbb{R}^3\setminus B_R} \frac{|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{3+2s}} u_n^2(x)dxdy \\ &\leq 4\int_{B_{\varepsilon R}} \int_{\mathbb{R}^3\setminus B_R} \frac{u_n^2(x)}{|x-y|^{3+2s}} dxdy \\ &\leq C\int_{B_{\varepsilon R}} u_n^2(x)dx \int_{(\frac{1}{2}-\varepsilon)R}^{\infty} \frac{1}{r^{2+2s}} dr \\ &= \frac{C}{((\frac{1}{2}-\varepsilon)R)^{2s}} \int_{B_{\varepsilon R}} u_n^2(x)dx\end{aligned} \tag{3.19}$$

where we use the fact that if $(x,y) \in B_{\varepsilon R} \times (\mathbb{R}^3\setminus B_R)$, then $|x-y| > (\frac{1}{2}-\varepsilon)R$.

Taking into account (3.17)-(3.19) we deduce

$$\int_{X^3}\int \frac{u_n^2(x)|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{\mu}} dxdy \leq \frac{C}{R^{2s}} \int_{B_{2R}\setminus B_{\epsilon R}} |u_n(x)|^2 dx + \frac{C}{((1-\epsilon)R)^{2s}} \int_{B_{\epsilon R}} u_n^2(x)dx. \tag{3.20}$$

Putting together (3.14),(3.15),(3.16) and (3.20), we can infer

$$\begin{aligned}&\int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u_n^2(x)|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{\mu}} dxdy \\ &\leq \frac{C}{k^3} + \frac{Ck^{2(1-s)}}{R^{2s}} \int_{B_{kR}\setminus B_{2R}} u_n^2(x)dx + \frac{C}{R^{2s}} \int_{B_{2R}\setminus B_{\epsilon R}} u_n^2(x)dx + \frac{C}{((1-\epsilon)R)^{2s}} \int_{B_{\epsilon R}} u_n^2(x)dx.\end{aligned} \tag{3.21}$$

Since $\{u_n\}$ is bounded in $H_\varepsilon$, we may assume that $u_n \to u$ in $L^2_{loc}(\mathbb{R}^3)$ for some $u \in H_\varepsilon$. Then, taking the limit as $n \to \infty$ in (3.21), we have

$$\begin{aligned}&\limsup_{n\to\infty} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u_n^2(x)|\psi_R(x)-\psi_R(y)|^2}{|x-y|^{\mu}} dxdy \\ &\leq \frac{C}{k^3} + \frac{Ck^{2(1-s)}}{R^{2s}} \int_{B_{kR}\setminus B_{2R}} u^2(x)dx + \frac{C}{R^{2s}} \int_{B_{2R}\setminus B_{\epsilon R}} u^2(x)dx + \frac{C}{((1-\epsilon)R)^{2s}} \int_{B_{\epsilon R}} u^2(x)dx \\ &\leq \frac{C}{k^3} + Ck^2\Big(\int_{B_{kR}\setminus B_{2R}} |u|^{2_s^\star}(x)dx\Big)^{\frac{2}{2_s^\star}} + C\Big(\int_{B_{2R}\setminus B_{\varepsilon R}} |u|^{2_s^\star}(x)dx\Big)^{\frac{2}{2_s^\star}} + C\Big(\frac{\varepsilon}{1-\varepsilon}\Big)^{2s}\Big(\int_{B_{\varepsilon R}} |u(x)|^{2_s^\star}\Big)^{\frac{2}{2_s^\star}},\end{aligned}$$

where in the last passage we use Hölder inequality.

Since $u \in L^{2_s^\star}(\mathbb{R}^3)$, $k > 4$ and $\varepsilon \in (0,\frac{1}{2})$, we obtain

$$\limsup_{R\to\infty} \int_{B_{kR}\setminus B_{2R}} |u(x)|^{2_s^\star} dx = \limsup_{R\to\infty} \int_{B_{2R}\setminus B_{\varepsilon R}} |u(x)|^{2_s^\star} dx = 0.$$

Choosing $\varepsilon = \frac{1}{k}$, we get

$$\begin{aligned}
&\limsup_{R\to\infty}\limsup_{n\to\infty}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{u_n^2(x)|\psi_R(x)-\psi_R(y)|^2}{|x-y|^\mu}dxdy\\
&\le \lim_{k\to\infty}\limsup_{R\to\infty}\left(\frac{C}{k^3}+Ck^2\Big(\int_{B_{kR}\setminus B_{2R}}|u(x)|^{2_s^\star}dx\Big)^{\frac{2}{2_s^\star}}+C\Big(\int_{B_{2R}\setminus B_{\frac{R}{k}}}|u(x)|^{2_s^\star}dx\Big)^{\frac{2}{2_s^\star}}+C\Big(\frac{1}{k-1}\Big)^{2s}\Big(\int_{B_{\frac{R}{k}}}|u(x)|^{2_s^\star}dx\Big)^{\frac{2}{2_s^\star}}\right)\\
&\le \lim_{k\to\infty}\frac{C}{k^3}+C\Big(\frac{1}{k-1}\Big)^{2s}\Big(\int_{B_{\frac{R}{k}}}|u(x)|^{2_s^\star}dx\Big)^{\frac{2}{2_s^\star}}\\
&=0,
\end{aligned}$$

which complete our proof. □

**Lemma 3.7.** *Under the conditions of Lemma 3.6, the functional $J_\varepsilon$ satisfies the $(PS)_c$ condition for all $c \in [c_\varepsilon, \frac{2_{\mu,s}^\star-1}{22_{\mu,s}^\star}S_{H,L}^{\frac{2_{\mu,s}^\star}{2_{\mu,s}^\star-1}})$.*

*Proof.* Since $\{u_n\}$ is bounded in $H_\varepsilon$, we may assume

$$\begin{aligned}
&u_n \rightharpoonup u \text{ in } H_\varepsilon,\\
&u_n \to u \text{ in } L^r_{loc}(\mathbb{R}^3),\ 2\le r<2_s^\star,\\
&u_n(x)\to u(x)\ a.e. \text{ in } \mathbb{R}^3.
\end{aligned}$$

Let us prove that $u_n \to u$ in $H_\varepsilon$ as $n\to\infty$. Setting $\omega_n = \|u_n-u\|_\varepsilon^2$, we have

$$\omega_n = \langle J_\varepsilon'(u_n),u_n\rangle - \langle J_\varepsilon'(u_n),u\rangle + \frac{1}{2_{\mu,s}^\star}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|^\mu}\star H(\varepsilon x,u_n)\Big)h(\varepsilon x,u_n)(u_n-u)dx + o_n(1). \tag{3.22}$$

Note that $\langle J_\varepsilon'(u_n),u_n\rangle = \langle J_\varepsilon'(u_n),u\rangle = o_n(1)$, so we only need to show that

$$\int_{\mathbb{R}^3}\Big(\frac{1}{|x|^\mu}\star H(\varepsilon x,u_n)\Big)h(\varepsilon x,u_n)(u_n-u)dx = o_n(1). \tag{3.23}$$

Similar the proof in Lemma 2.7, we can see that

$$\frac{1}{|x|^\mu}H(\varepsilon x,u_n) \rightharpoonup \frac{1}{|x|^\mu}H(\varepsilon x,u) \ \text{ in } L^{\frac{6}{\mu}}(\mathbb{R}^3). \tag{3.24}$$

By using Lemma 2.1, we have

$$\lim_{n\to\infty}\int_{B_R}\frac{1}{|x|^\mu}H(\varepsilon x,u_n)h(\varepsilon x,u_n)(u_n-u)dx = 0. \tag{3.25}$$

By Lemma 2.1 and 3.6, for any $\eta>0$ there exists $R=R(\eta)>0$ such that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3\setminus B_R}\Big|\frac{1}{|x|^\mu}H(\varepsilon x,u_n)h(\varepsilon x,u_n)u_n\Big|dx \le C_1\eta$$

and

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3\setminus B_R}\Big|\frac{1}{|x|^\mu}H(\varepsilon x,u_n)h(\varepsilon x,u_n)u\Big|dx \le C_2\eta$$

Taking into account the above limits we can deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|^\mu}\star H(\varepsilon x,u_n)\Big)h(\varepsilon x,u_n)(u_n-u)dx = 0.$$

□

**Lemma 3.8.** *The functional $\Psi_\varepsilon$ verifies the $(PS)_c$ condition in $S_\varepsilon^+$ for all $c \in [c_\varepsilon, \frac{2^*_{\mu,s}-1}{22^*_{\mu,s}} S_{H,L}^{\frac{2^*_{\mu,s}}{2^*_{\mu,s}-1}})$.*

*Proof.* Let $\{u_n\} \subset S_\varepsilon^+$ be a $(PS)_c$ sequence for $\Psi_\varepsilon$. Thus, $\Psi_\varepsilon(u_n) \to c$ and $\|\Psi'_\varepsilon\|_\star \to 0$, where $\|\cdot\|_\star$ is the norm in the dual space $(T_{u_n} S_\varepsilon^+)'$. It follows from Lemma 3.4-($B_3$) that $\{m_\varepsilon(u_n)\}$ is a $(PS)_c$ sequence for $I_\varepsilon$ in $H_\varepsilon$. From Lemma 3.7 we see that there is a $u \in S_\varepsilon^+$ such that $m_\varepsilon(u_n) \to m_\varepsilon(u)$ in $H_\varepsilon$. From Lemma 3.3-($A_3$), it follows that $u_n \to u$ in $S_\varepsilon^+$. □

## 4 The autonomous problem

Since we are interestd in giving a multiplicity result for the modified problem, we start by considering the limit problem associated to (1.8), namely, the problem

$$(-\Delta)^s u + V_0 u = \Big(\int_{\mathbb{R}^3} \frac{|u(y)|^{2^*_{\mu,s}} + F(u(y))}{|x-y|^\mu} dy\Big)\Big(|u|^{2^*_{\mu,s}-2} u + \frac{1}{2^*_{\mu,s}} f(u)\Big) \text{ in } \mathbb{R}^3. \tag{4.1}$$

Set $G(u) = |u|^{2^*_{\mu,s}} + F(u)$, $g(u) = \frac{dG(u)}{du}$, then the equation (4.1) changes into

$$(-\Delta)^s u + V_0 u = \frac{1}{2^*_{\mu,s}}\Big(\int_{\mathbb{R}^3} \frac{G(u(y))}{|x-y|^\mu} dy\Big) g(u) \text{ in } \mathbb{R}^3. \tag{4.2}$$

which has the following associated functional

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + V_0 u^2) dx - \frac{1}{22^*_{\mu,s}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{G(u(y))G(u(x))}{|x-y|^\mu} dy dx.$$

The functional $I_0$ is well defined on the Hilbert space $H_0 = H^s(\mathbb{R}^3)$ with the inner product

$$(u, v)_0 = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} V_0 u v dx,$$

and the norm

$$\|u\|_0^2 = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \int_{\mathbb{R}^3} V_0 u^2 dx.$$

We denote the Nehari manifold associated to $I_0$ by

$$\mathcal{N}_0 = \{u \in H_0 \backslash \{0\} : \langle I'_0(u), u\rangle = 0\},$$

and by $H_0^+$ the open subset of $H_0$ given by

$$H_0^+ = \{u \in H_0 : |supp(u^+)| > 0\},$$

and $S_0^+ = S_0 \cap H_0^+$, where $S_0$ is the unit sphere of $H_0$.

As in section 3, $S_0^+$ is a incomplete $C^{1,1}$-manifold of codimension 1, modeled on $H_0$ and contained in the open $H_0^+$. Thus, $H_0 = T_u S_0^+ \oplus \mathbb{R}u$ for each $u \in S_0^+$, where $T_u S_0^+ = \{v \in H_0 : (u, v)_0 = 0\}$.

Next we have the following Lemmas and the proofs follow from a similar argument used in the proofs of Lemma 3.3 and Lemma 3.4.

**Lemma 4.1.** *Let $V_0$ be given in $(V_1)$ and the functional $f$ satisfies $(f_1)-(f_3)$. Then the following properties hold:*

$(a_1)$ *For each $u \in H_0^+$, let $\phi_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $\phi_u(\tau) = I_0(\tau u)$. Then there exists a unique $\tau_u > 0$ such that $\phi'_u(\tau) > 0$ in $(0, \tau_u)$ and $\phi'_u(\tau) < 0$ in $(\tau_u, \infty)$.*

$(a_2)$ *There is a $\sigma > 0$ independent on $u$ such that $\tau_u > \sigma$ for all $u \in S_0^+$. Moreover, for each compact set $\mathcal{W} \subset S_0^+$ there is $C_{\mathcal{W}} > 0$ such that $\tau_u \le C_{\mathcal{W}}$ for all $u \in \mathcal{W}$.*

$(a_3)$ *The map* $\hat{m} : H_0^+ \to \mathcal{N}_0$ *given by* $\hat{m}(u) = \tau_u u$ *is continuous and* $m := \hat{m}|_{S_0^+}$ *is a homeomorphism between* $S_0^+$ *and* $\mathcal{N}_0$. *Moreover,* $m^{-1}(u) = \frac{u}{\|u\|_0}$.

We define the applications

$$\hat{\Psi}_0 : H_0^+ \to \mathbb{R} \text{ and } \Psi_0 : S_0^+ \to \mathbb{R},$$

by $\hat{\Psi}_0(u) = I_0(\hat{m}(u))$ and $\Psi_0 := \hat{\Psi}_0|_{S_0^+}$.

**Lemma 4.2.** *Let* $V_0$ *be given in* $(V_1)$ *and* $(f_1)-(f_3)$ *are satisfied. Then :*
$(b_1)$ $\hat{\Psi}_0 \in C^1(H_0^+, \mathbb{R})$ *and*

$$\hat{\Psi}_0'(u)v = \frac{\|\hat{m}(u)\|_0}{\|u\|_0} I_0'(\hat{m}(u))v, \quad \forall u \in H_0^+ \text{ and } \forall\, v \in H_0.$$

$(b_2)$ $\Psi_0 \in C^1(S_0^+, \mathbb{R})$ *and*

$$\Psi_0'(u)v = \|m(u)\|_0 I_0'(m(u))v, \quad \forall v \in T_u S_0^+.$$

$(b_3)$ *If* $\{u_n\}$ *is a* $(PS)_c$ *sequence of* $\Psi_0$, *then* $\{m(u_n)\}$ *is a* $(PS)_c$ *sequence of* $I_0$. *If* $\{u_n\} \subset \mathcal{N}_0$ *is a bounded* $(PS)_c$ *sequence for* $I_0$, *then* $\{m^{-1}(u_n)\}$ *is a* $(PS)_c$ *sequence of* $\Psi_0$.
$(b_4)$ $u$ *is a critical point of* $\Psi_0$ *if and only if,* $m(u)$ *is a critical point of* $I_0$. *Moreover, corresponding critical values coincide and*

$$\inf_{S_0^+} \Psi_0 = \inf_{\mathcal{N}_0} I_0.$$

As in the previous section, we have the following variational characterization of the infimum of $I_0$ over $\mathcal{N}_0$:

$$c_{V_0} = \inf_{u \in \mathcal{N}_0} I_0(u) = \inf_{u \in H_0^+} \max_{\tau > 0} I_0(\tau u) = \inf_{u \in S_0^+} \max_{\tau > 0} I_0(\tau u) > 0.$$

The next Lemma allows us to assume that the weak limit of a $(PS)_c$ sequence is non-trivial.

**Lemma 4.3.** *Let* $\{u_n\} \subset H_0$ *be a* $(PS)_c$ *sequence with* $c \in [c_{V_0}, \frac{2^\star_{\mu,s}-1}{22^\star_{\mu,s}} S_{H,L}^{\frac{2^\star_{\mu,s}}{2^\star_{\mu,s}-1}})$ *for* $I_0$. *Then, only one of the following conclusions holds.*
$(i)$ $u_n \to 0$ *in* $H_0$, *or*
$(ii)$ *There exist a sequence* $\{y_n\} \subset \mathbb{R}^3$ *and constants* $R, \beta > 0$ *such that*

$$\liminf_{n \to +\infty} \int\limits_{B_R(y_n)} u_n^2 dx \geq \beta > 0.$$

*Proof.* Suppose $(ii)$ does not occur. Then, for any $R > 0$, we have

$$\lim_{n \to +\infty} \sup_{y \in \mathbb{R}^3} \int\limits_{B_R(y)} u_n^2 dx = 0.$$

Similarly to Lemma 3.5, we have $\{u_n\}$ is bounded in $H_0$, then by Lemma 2.2, we have

$$u_n \to 0 \text{ in } L^r(\mathbb{R}^3) \text{ for } r \in (2, 2_s^\star).$$

Thus, by $(f_1)$ we have

$$\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{G(u(y))g(u(x))u(x)}{|x-y|^\mu} dydx = o_n(1).$$

Recalling that $I_0'(u_n)u_n \to 0$, we get

$$\|u_n\|_0^2 = o_n(1).$$

Therefore the conclusion follows. □

From Lemma 4.3 we can see that, if $u$ is the weak limit of a $(PS)_{c_{V_0}}$ sequence $\{u_n\}$ for the functional $I_0$, then we can assume $u \neq 0$. Otherwise we would have $u_n \rightharpoonup 0$ and once it doesn't occur $u_n \to 0$, we conclude from Lemma 4.3 that there exist $\{y_n\} \subset \mathbb{R}^3$ and $R, \beta > 0$ such that

$$\liminf_{n\to+\infty} \int_{B_R(y_n)} u_n^2 dx \geq \beta > 0.$$

Then set $v_n(x) = u_n(x + y_n)$, making a change of variable, we can prove that $\{v_n\}$ ia also a $(PS)_{c_{V_0}}$ sequence for the functional $I_0$, it is bounded in $H_0$ and there is $v \in H_0$ such that $v_n \to v$ in $H_0$ with $v \neq 0$.

Next we devote to estimating the least energy $c_{V_0}$. Recall that the best Sobolev constant $S_s$ of the embedding $\mathcal{D}^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^\star}(\mathbb{R}^3)$ is defined by

$$S_s := \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^3)} \frac{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)}}{|u|^2_{2_s^\star}}.$$

In particular, we consider the following family of functions $U_\varepsilon$ defined as

$$U_\varepsilon(x) = \varepsilon^{\frac{2s-3}{2}} \bar{u}(\frac{x}{\varepsilon} S_s^{\frac{1}{2s}}), \quad \bar{u} = \frac{u_n(x)}{|u_0|_{2_s^\star}},$$

for $\varepsilon > 0$ and $x \in \mathbb{R}^3$, the minimizer of $S_s$ (see,[41]), which satisfies

$$(-\Delta)^s u = |u|^{2_s^\star - 2} u, \quad x \in \mathbb{R}^3.$$

Then, by a simple calculation, we know

$$\tilde{U}(x) = S_s^{\frac{(3-\mu)(2s-3)}{4(2+2s-\mu)}} C(\mu)^{\frac{2s-3}{2(3+2s-\mu)}} U_\varepsilon(x)$$

is the unique minimizer for $S_{H,L}$ that satisfies

$$(-\Delta)^s u = \Big(\int_{\mathbb{R}^3} \frac{|u(y)|^{2^\star_{\mu,s}}}{|x-y|^\mu}\Big)|u|^{2^\star_{\mu,s}} u, \quad \text{in } \mathbb{R}^3.$$

Moreover,

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \tilde{U}|^2 dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\tilde{U}(x)|^{2^\star_{\mu,s}} |\tilde{U}(y)|^{2^\star_{\mu,s}}}{|x-y|^\mu} dxdy = S_{H,L}^{\frac{6-\mu}{3-\mu+2s}}.$$

Let $\varphi \in C_0^\infty(\mathbb{R}^3, [0,1])$ and small $\delta > 0$ be such that $\varphi \equiv 1$ in $B_\delta(0)$ and $\phi \equiv 0$ in $\mathbb{R}^3 \backslash B_{2\delta}(0)$. For any $\varepsilon > 0$, define the best function by $u_\varepsilon = \varphi U_\varepsilon$.

Similar to [22, Lemma 1.2], we can easily draw the following conclusion.

**Lemma 4.4.** *The constant $S_{H,L}$ defined in* (2.2) *is achieved if and only if*

$$u = C\Big(\frac{b}{b^2 + |x-a|^2}\Big)^{\frac{3-2s}{2}},$$

*where $C > 0$ is a fixed constant, $a \in \mathbb{R}^3$ and $b > 0$ are parameters. Furthermore,*

$$S_{H,L} = \frac{S_s}{C(\mu)^{\frac{1}{2^\star_{\mu,s}}}}.$$

*Proof.* We sketch the proof for the completeness of this paper. By the Hardy-Littlewood-Sobolev inequality, we have

$$S_{H,L} \geq \frac{1}{C(\mu)^{\frac{1}{2^\star_{\mu,s}}}} \int_{u\in\mathcal{D}^{s,2}(\mathbb{R}^3)\backslash\{0\}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{|u|^2_{2_s^\star}} = \frac{S_s}{C(\mu)^{\frac{1}{2^\star_{\mu,s}}}}.$$

On the other hand, the equality in the Hardy-Littlewood-Sobolev (1.6) holds if and only if

$$f(x) = h(x) = C\Big(\frac{b}{b^2 + |x-a|^2}\Big)^{\frac{6-\mu}{2}},$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^3$ and $b > 0$ are parameters. Thus

$$\Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\,dxdy\Big)^{\frac{1}{2^*_{\mu,s}}} = C(\mu)^{\frac{1}{2^*_{\mu,s}}}|u|^2_{2^*_s},$$

if and only if

$$u = C\Big(\frac{b}{b^2 + |x-a|^2}\Big)^{\frac{3-2s}{2}}.$$

Then, by the definition of $S_{H,L}$, we get

$$S_{H,L} \le \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|dx}{\Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(y)|^{2^*_{\mu,s}}|u(x)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\,dydx\Big)^{\frac{1}{2^*_{\mu,s}}}} = \frac{1}{C(\mu)^{\frac{1}{2^*_{\mu,s}}}}\frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2dx}{|u|^2_{2^*_s}} \tag{4.3}$$

an thus we have

$$S_{H,L} \le \frac{S_s}{C(\mu)^{\frac{1}{2^*_{\mu,s}}}}.$$

From the arguments above, we know that $S_{H,L}$ is achieved if and only if $u = C(\frac{b}{b^2+|x-a|^2})^{\frac{3-2s}{2}}$ and

$$S_{H,L} = \frac{S_s}{C(\mu)^{\frac{1}{2^*_{\mu,s}}}}.$$

□

Next. repeat the proof in [22, Lemma 1.3], we have the following important information about the best constant $S_{H,L}$.

**Lemma 4.5.** *For every open subset $\Omega \subset \mathbb{R}^3$, we have*

$$S_{H,L}(\Omega) := \inf_{u \in \mathcal{D}^{s,2}_0(\Omega)\setminus\{0\}} \frac{\int_{\Omega} |(-\Delta)^{\frac{s}{2}} u|^2dx}{\Big(\int_{\Omega}\int_{\Omega} \frac{|u(y)|^{2^*_{\mu,s}}|u(x)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\,dydx\Big)^{\frac{1}{2^*_{\mu,s}}}} = S_{H,L} \tag{4.4}$$

*where $S_{H,L}(\Omega)$ is never achieved except when $\Omega = \mathbb{R}^3$.*

*Proof.* It is clear that $S_{H,L} \le S_{H,L}(\Omega)$ by $\mathcal{D}^{s,2}_0(\Omega) \subset \mathcal{D}^{s,2}(\mathbb{R}^3)$. Let $\{u_n\} \subset C_0^\infty(\mathbb{R}^3)$ be a minimizing sequence for $S_{H,L}$. We make translations and dilations for $\{u_n\}$ by choosing $y_n \in \mathbb{R}^3$ and $\tau_n > 0$ such that

$$v_n := \tau_n^{\frac{3-2s}{2}} u_n(\tau_n x + y_n) \in C_0^\infty(\Omega),$$

which satisfies

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2dx = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2dx$$

and

$$\int_{\Omega}\int_{\Omega} \frac{|v_n(y)|^{2^*_{\mu,s}}|v_n(x)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\,dxdy = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(y)|^{2^*_{\mu,s}}|u_n(x)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\,dxdy.$$

Hence $S_{H,L}(\Omega) \le S_{H,L}$. Moreover, since $\tilde{U}(x)$ is the only class of functions such that the equality holds in the Hardy-Littlewood-Sobolev inequality, we know that $S_{H,L}(\Omega)$ is never achieved except for $\Omega = \mathbb{R}^3$. □

**Lemma 4.6.**

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{3+2s}}dxdy \le S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}), \tag{4.5}$$

$$|u_\varepsilon|_2^2 = \begin{cases} C\varepsilon^{2s} + O(\varepsilon^{3-2s}) & \text{if } 4s < 3, \\ C\varepsilon^{2s}\log(\frac{1}{\varepsilon}) + O(\varepsilon^{2s}) & \text{if } 4s = 3, \\ C\varepsilon^{3-2s} + O(\varepsilon^{2s}) & \text{if } 4s > 3, \end{cases} \tag{4.6}$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_\varepsilon(x)|^{2^\star_{\mu,s}}|u_\varepsilon(y)|^{2^\star_{\mu,s}}}{|x-y|^\mu}dxdy \ge C(\mu)^{\frac{3}{2s}} S_{H,L}^{\frac{6-\mu}{2s}} - O(\varepsilon^{\frac{6-\mu}{2}}), \tag{4.7}$$

*In addition, if $q < 2^\star_{\mu,s}$, then there holds*

$$\int_{B_\delta}\int_{B_\delta} \frac{|U_\varepsilon(x)|^q|U_\varepsilon(y)|^q}{|x-y|^\mu}dxdy = O(\varepsilon^{6-\mu-q(3-2s)}). \tag{4.8}$$

*Proof.* For the proof of (4.5) and (4.6), we can see that in [41]. So we only need to estimate (4.7) and (4.8). Concerning (4.7), similar to [2, Lemma 7.1], we have

$$\begin{aligned} &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_\varepsilon(x)|^{2^\star_{\mu,s}}|u_\varepsilon(y)|^{2^\star_{\mu,s}}}{|x-y|^\mu}dxdy \ge \int_{B_\delta}\int_{B_\delta} \frac{|u_\varepsilon(x)|^{2^\star_{\mu,s}}|u_\varepsilon(y)|^{2^\star_{\mu,s}}}{|x-y|^\mu}dxdy \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|U_\varepsilon(x)|^{2^\star_{\mu,s}}|U_\varepsilon(y)|^{2^\star_{\mu,s}}}{|x-y|^\mu}dxdy - 2\int_{\mathbb{R}^3\setminus B_\delta}\int_{B_\delta} \frac{|U_\varepsilon(x)|^{2^\star_{\mu,s}}|U_\varepsilon(y)|^{2^\star_{\mu,s}}}{|x-y|^\mu}dxdy \\ &- \int_{\mathbb{R}^3\setminus B_\delta}\int_{\mathbb{R}^3\setminus B_\delta} \frac{|U_\varepsilon(x)|^{2^\star_{\mu,s}}|U_\varepsilon(y)|^{2^\star_{\mu,s}}}{|x-y|^\mu}dxdy := C(\mu)^{\frac{3}{2s}} S_{H,L}^{\frac{6-\mu}{2s}} - 2\mathbb{A} - \mathbb{B}, \end{aligned} \tag{4.9}$$

where

$$\mathbb{A} = \int_{\mathbb{R}^3\setminus B_\delta}\int_{B_\delta} \frac{|U_\varepsilon(x)|^{2^\star_{\mu,s}}|U_\varepsilon(y)|^{2^\star_{\mu,s}}}{|x-y|^\mu}dxdy,$$

$$\mathbb{B} = \int_{\mathbb{R}^3\setminus B_\delta}\int_{\mathbb{R}^3\setminus B_\delta} \frac{|U_\varepsilon(x)|^{2^\star_{\mu,s}}|U_\varepsilon(y)|^{2^\star_{\mu,s}}}{|x-y|^\mu}dxdy.$$

By direct computation, we have

$$\begin{aligned} \mathbb{A} &= \int_{\mathbb{R}^3\setminus B_\delta}\int_{B_\delta} \frac{|U_\varepsilon(x)|^{2^\star_{\mu,s}}|U_\varepsilon(y)|^{2^\star_{\mu,s}}}{|x-y|^\mu}dxdy, \\ &= C\varepsilon^{6-\mu}\int_{\mathbb{R}^3\setminus B_\delta}\int_{B_\delta} \frac{1}{(\varepsilon^2b^2+x^2S_s^{\frac{1}{s}})^{\frac{6-\mu}{2}}}\frac{1}{|x-y|^\mu}\frac{1}{(\varepsilon^2b^2+y^2S_s^{\frac{1}{s}})^{\frac{6-\mu}{2}}}dxdy \\ &\le C\varepsilon^{6-\mu}\Big(\int_{\mathbb{R}^3\setminus B_\delta} \frac{1}{(\varepsilon^2b^2+x^2S_s^{\frac{1}{s}})^3}dx\Big)^{\frac{6-\mu}{6}}\Big(\int_{B_\delta} \frac{1}{(\varepsilon^2b^2+y^2S_s^{\frac{1}{s}})^3}dy\Big)^{\frac{6-\mu}{6}} \\ &\le C\varepsilon^{6-\mu}\Big(\int_{\mathbb{R}^3\setminus B_\delta} \frac{1}{|x|^6S_s^{\frac{3}{s}}}dx\Big)^{\frac{6-\mu}{6}}\Big(\int_0^\delta \frac{r^2}{(\varepsilon^2b^2+r^2S_s^{\frac{1}{s}})^3}dr\Big)^{\frac{6-\mu}{6}} \\ &= O(\varepsilon^{\frac{6-\mu}{2}}) \end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
\mathbb{B} &= \int\limits_{\mathbb{R}^3\setminus B_\delta}\int\limits_{\mathbb{R}^3\setminus B_\delta} \frac{|U_\varepsilon(x)|^{2^\star_{\mu,s}}|U_\varepsilon(y)|^{2^\star_{\mu,s}}}{|x-y|^\mu}dxdy \\
&\le C\varepsilon^{6-\mu}\int\limits_{\mathbb{R}^3\setminus B_\delta}\int\limits_{\mathbb{R}^3\setminus B_\delta} \frac{1}{(\varepsilon^2b^2+x^2S_s^{\frac{1}{s}})^{\frac{6-\mu}{2}}}\frac{1}{|x-y|^\mu}\frac{1}{(\varepsilon^2b^2+y^2S_s^{\frac{1}{s}})^{\frac{6-\mu}{2}}}dxdy \\
&\le C\varepsilon^{6-\mu}\Big(\int\limits_{\mathbb{R}^3\setminus B_\delta}\frac{1}{(\varepsilon^2b^2+x^2S_s^{\frac{1}{s}})^3}dx\Big)^{\frac{6-\mu}{6}}\Big(\int\limits_{\mathbb{R}^3\setminus B_\delta}\frac{1}{(\varepsilon^2b^2+y^2S_s^{\frac{1}{s}})^3}dy\Big)^{\frac{6-\mu}{6}} \\
&\le C\varepsilon^{6-\mu}\Big(\int\limits_{\mathbb{R}^3\setminus B_\delta}\frac{1}{|x|^6S_s^{\frac{3}{s}}}dx\Big)^{\frac{6-\mu}{6}}\Big(\int\limits_{\mathbb{R}^3\setminus B_\delta}\frac{1}{|y|^6S_s^{\frac{3}{s}}}dy\Big)^{\frac{6-\mu}{6}} \\
&= O(\varepsilon^{6-\mu})
\end{aligned} \tag{4.11}$$

It follows from (4.9) to (4.11) that

$$\begin{aligned}
\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{|u_\varepsilon(x)|^{2^\star_{\mu,s}}|u_\varepsilon(y)|^{2^\star_{\mu,s}}}{|x-y|^\mu}dxdy &\ge C(\mu)^{\frac{3}{2s}}S_{H,L}^{\frac{6-\mu}{2s}} - O(\varepsilon^{\frac{6-\mu}{2}}) - O(\varepsilon^{6-\mu}) \\
&= C(\mu)^{\frac{3}{2s}}S_{H,L}^{\frac{6-\mu}{2s}} - O(\varepsilon^{\frac{6-\mu}{2}}).
\end{aligned}$$

Then (4.7) follows. Now we prove (4.8). If $q < 2^\star_{\mu,s}$, we have

$$\begin{aligned}
&\int\limits_{B_\delta}\int\limits_{B_\delta}\frac{|U_\varepsilon(x)|^q|U_\varepsilon(y)|^q}{|x-y|^\mu}dxdy = O(\varepsilon^{6-\mu-q(3-2s)}) \\
&= C\varepsilon^{q(3-2s)}\int\limits_{B_\delta}\int\limits_{B_\delta}\frac{1}{(\varepsilon^2b^2+x^2S_s^{\frac{1}{s}})^{\frac{3-2s}{2}q}}\frac{1}{|x-y|^\mu}\frac{1}{(\varepsilon^2b^2+y^2S_s^{\frac{1}{s}})^{\frac{3-2s}{2}q}}dxdy \\
&\le C\varepsilon^{q(3-2s)}\Big(\int\limits_{B_\delta}\frac{1}{(\varepsilon^2b^2+x^2S_s^{\frac{1}{s}})^{\frac{3-2s}{2}q\frac{6}{6-\mu}}}dx\Big)^{\frac{6-\mu}{6}}\Big(\frac{1}{(\varepsilon^2b^2+y^2S_s^{\frac{1}{s}})^{\frac{3-2s}{2}q\frac{6}{6-\mu}}}dy\Big)^{\frac{6-\mu}{6}} \\
&\le C\varepsilon^{q(3-2s)}\Big(\int_0^\varepsilon\frac{r^2}{(\varepsilon^2b^2+r^2S_s^{\frac{3-2s}{2}q\frac{6}{6-\mu}}}dr\Big)^{\frac{6-\mu}{3}} \\
&= O(\varepsilon^{6-\mu-q(3-2s)}
\end{aligned}$$

□

**Lemma 4.7.** *Suppose that $(f_1)-(f_3)$ hold. Then the number $c_{V_0}$ satisfies that*

$$0 < c_{V_0} < \frac{2^\star_{\mu,s}-1}{22^\star_{\mu,s}}S_{H,L}^{\frac{2^\star_{\mu,s}}{2^\star_{\mu,s}-1}}.$$

*Proof.* By the definition of $c_{V_0}$, it suffices to prove that there exists $v \in \mathcal{N}_0$ such that

$$I_0(v) < \frac{2^\star_{\mu,s}-1}{22^\star_{\mu,s}}S_{H,L}^{\frac{2^\star_{\mu,s}}{2^\star_{\mu,s}-1}}. \tag{4.12}$$

By Lemma 4.1, there exists $\tau_\varepsilon > 0$ such that $\tau_\varepsilon u_\varepsilon \in \mathcal{N}_0$. We claim that for $\varepsilon > 0$ small enough, there exist $A_1$ and $A_2$ independent of $\varepsilon$ such that

$$0 < A_1 \le \tau_\varepsilon \le A_2 < \infty. \tag{4.13}$$

Indeed, note that $\mathcal{N}_0$ is bounded away from 0, we have that $\tau_\varepsilon \geq A_1 > 0$ using (4.5) and (4.6). Moreover, since $\langle I_0'(\tau_\varepsilon u_\varepsilon), \tau_\varepsilon u_\varepsilon\rangle = 0$, from (4.7) we have that

$$\begin{aligned}\tau_\varepsilon^{22^*_{\mu,s}} &\leq C(\tau_\varepsilon^2\|u_\varepsilon\|_0^2 + \tau_\varepsilon^{2q_1}\|u_\varepsilon\|_0^{2q_1} + \tau_\varepsilon^{2q_2}\|u_\varepsilon\|_0^{2q_2} \\ &\quad + \tau_\varepsilon^{q_1+2^*_{\mu,s}}\|u_\varepsilon\|_0^{q_1+2^*_{\mu,s}} + \tau_\varepsilon^{q_2+2^*_{\mu,s}}\|u_\varepsilon\|_0^{q_2+2^*_{\mu,s}}).\end{aligned}$$

Using (4.5) and (4.6) again, there exists $A_2 > 0$ such that $\tau_\varepsilon \leq A_2$. Then (4.13) holds true.

Now we estimate $I_0(\tau_\varepsilon u_\varepsilon)$. Note that

$$\begin{aligned}I_0(\tau_\varepsilon u_\varepsilon) &\leq \left(\frac{\tau_\varepsilon^2}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\varepsilon(x)|^2dx - \frac{\tau_\varepsilon^{22^*_{\mu,s}}}{22^*_{\mu,s}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_\varepsilon(y)|^{2^*_{\mu,s}}|u_\varepsilon(x)|^{2^*_{\mu,s}}}{|x-y|^\mu}dydx\right) \\ &\quad + \left(\frac{\tau_\varepsilon^2}{2}\int_{\mathbb{R}^3}V_0u_\varepsilon^2dx - \frac{1}{22^*_{\mu,s}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{F(\tau_\varepsilon u_\varepsilon(y))F(\tau_\varepsilon u_\varepsilon(x))}{|x-y|^\mu}dydx\right) \\ &:= I_1 + I_2.\end{aligned} \tag{4.14}$$

For $I_1$, we set

$$\begin{aligned}A &:= \int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\varepsilon(x)|^2dx, \\ B &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_\varepsilon(y)|^{2^*_{\mu,s}}|u_\varepsilon(x)|^{2^*_{\mu,s}}}{|x-y|^\mu}dydx,\end{aligned}$$

and consider the function $\theta : [0,\infty) \to \mathbb{R}$ defined by

$$\theta(\tau) = \frac{1}{2}A\tau^2 - \frac{\tau_\varepsilon^{22^*_{\mu,s}}}{22^*_{\mu,s}}B,$$

we have that $\tau_0 = (\frac{A}{B})^{\frac{1}{22^*_{\mu,s}-2}}$ is a maximum point of $\theta$ and

$$\theta(\tau_0) = \frac{2^*_{\mu,s}-1}{22^*_{\mu,s}}A^{\frac{2^*_{\mu,s}}{2^*_{\mu,s}-1}}B^{\frac{1}{1-2^*_{\mu,s}}}.$$

Combining with Lemma 4.3, (4.5) and (4.7) we have

$$I_1 \leq \frac{2^*_{\mu,s}-1}{22^*_{\mu,s}}S_{H,L}^{\frac{2^*_{\mu,s}-1}{2^*_{\mu,s}}} + O(\varepsilon^{3-2s}) + O(\varepsilon^{\frac{6-\mu}{2}}). \tag{4.15}$$

For $I_2$, given $A_0 > 0$, we invoke $(f_3)$ to obtain $R = R(A_0) > 0$ such that, for $x \in \mathbb{R}^3$, $t \geq R$,

$$F(x,t) \geq \begin{cases} A_0t^{2^*_{\mu,s}-1} & \text{if } 3 < 4s, \\ A_0t^{2^*_{\mu,s}-\frac{2s}{3-2s}}(logt)^{\frac{1}{2}} & \text{if } 3 = 4s, \\ A_0t^{2^*_{\mu,s}-\frac{2s}{3-2s}} & \text{if } 3 > 4s. \end{cases} \tag{4.16}$$

By (4.6) and (4.16), we need to estimate $I_2$ in three cases. Since the argument is similar, we only consider the case that $3 < 4s$. For $|x| < \varepsilon < \delta$, noting that $\varphi \equiv 1$ in $B_\delta(0)$, by the definition of $u_\varepsilon$ and (4.13), we get a constant $\beta > 0$ such that

$$\tau_\varepsilon u_\varepsilon(x) \geq A_1U_\varepsilon(x) \geq \beta\varepsilon^{\frac{2s-3}{2}}.$$

Then we can choose $\varepsilon_1 > 0$ such that $\tau_\varepsilon u_\varepsilon \geq R$, for $|x| < \varepsilon$, $0 < \varepsilon < \varepsilon_1$. It follows from (4.16) that

$$F(x,\tau_\varepsilon u_\varepsilon(x)) \geq A_0\tau_\varepsilon^{2^*_{\mu,s}-1}u_\varepsilon^{2^*_{\mu,s}-1},$$

for $|x| < \varepsilon$, $0 < \varepsilon < \varepsilon_1$. Then for any $0 < \varepsilon < \varepsilon_1$, by (4.8) we get

$$\begin{aligned}\int_{B_\varepsilon}\int_{B_\varepsilon} \frac{F(\tau_\varepsilon u_\varepsilon(x))F(\tau_\varepsilon u_\varepsilon(y))}{|x-y|^\mu}dydx &\geq A_0^2 \int_{B_\varepsilon}\int_{B_\varepsilon} \frac{|\tau_\varepsilon u_\varepsilon(y)|^{2^\star_{\mu,s}-1}|\tau_\varepsilon u_\varepsilon(x)|^{2^\star_{\mu,s}-1}}{|x-y|^\mu}dydx \\ &\geq CA_0^2 \int_{B_\varepsilon}\int_{B_\varepsilon} \frac{|U_\varepsilon(y)|^{2^\star_{\mu,s}-1}|U_\varepsilon(x)|^{2^\star_{\mu,s}-1}}{|x-y|^\mu}dydx \\ &= A_0^2 O(\varepsilon^{3-2s}).\end{aligned}$$

Note that $F(u) \geq 0$, (4.6) and (4.13), we have

$$I_2 \leq C|u_\varepsilon|_2^2 - \frac{1}{22^\star_{\mu,s}} \int_{B_\varepsilon}\int_{B_\varepsilon} \frac{F(\tau_\varepsilon u_\varepsilon(x))F(\tau_\varepsilon u_\varepsilon(y))}{|x-y|^\mu}dydx \leq (C - C_1A_0^2)\varepsilon^{3-2s}. \tag{4.17}$$

Inserting (4.15) and (4.17) into (4.14), we get

$$I_0(\tau_\varepsilon u_\varepsilon) \leq \frac{2^\star_{\mu,s}-1}{22^\star_{\mu,s}} S_{H,L}^{\frac{2^\star_{\mu,s}-1}{2^\star_{\mu,s}}} + (C_2 + C - C_1A_0^2)\varepsilon^{3-2s} + O(\varepsilon^{\frac{6-\mu}{2}}). \tag{4.18}$$

Observe that $\frac{6-\mu}{2} > 3 - 2s$ for $3 < 4s$, and $A_0 > 0$ is arbitrary, we choose large enough $A_0$ such that $C + C_2 - C_1A_0^2 < 0$. Then for small $\varepsilon > 0$ we have $v := \tau_\varepsilon u_\varepsilon$ satisfies (4.12). □

**Theorem 4.1.** *Assume that* $(f_1)$–$(f_3)$ *hold. Then autonomous problem* (4.1) *has a positive ground state solution* $u$ *with* $I_0(u) = c_{V_0}$.

*Proof.* By Lemma 3.2 with $V(x) = V_0$ and the Mountain Pass Theorem without $(PS)$ condition (cf. [50]), there exists a $(PS)_{c_{V_0}}$-sequence $\{u_n\} \subset H_0$ of $I_0$ with

$$c_{V_0} < \frac{2^\star_{\mu,s}-1}{22^\star_{\mu,s}} S_{H,L}^{\frac{2^\star_{\mu,s}-1}{2^\star_{\mu,s}}}.$$

By Lemma 3.5 and 4.3, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$ and non-vanishing, namely there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 dx \geq \delta.$$

Up to s subsequence, there exists $u \in H^s(\mathbb{R}^3)$ such that

$$\begin{aligned}&u_n \rightharpoonup u \text{ in } H^s(\mathbb{R}^3),\\ &u_n \to u \text{ in } L^r_{loc}(\mathbb{R}^3),\ 2 \leq r < 2^\star_s,\\ &u_n(x) \to u(x)\ a.e. \text{ in } \mathbb{R}^3.\end{aligned}$$

As Lemma 2.5, we have $I_0'(u) = 0$. Since $I_0$ and $I_0'$ are both invariant by translation, without lost of generality, we can assume that $\{y_n\}$ is bounded. Note that $u_n \to u$ in $L^2_{loc}(\mathbb{R}^3)$. Then $u \not\equiv 0$. So $u \in \mathcal{N}_0$. Then

$$\begin{aligned}c_{V_0} \leq I_0(u) &= I_0(u) - \frac{1}{2}\langle I_0'(u), u\rangle \\ &= \frac{1}{22^\star_{\mu,s}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{G(u(y))}{|x-y|^\mu}\big(g(u(x))u(x) - G(u(x))\big)dydx \\ &\leq \frac{1}{22^\star_{\mu,s}} \liminf_{n\to\infty} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{G(u_n(y))}{|x-y|^\mu}\big(g(u_n(x))u_n(x) - G(u_n(x))\big)dydx \\ &= \liminf_{n\to\infty}(I_0(u_n) - \frac{1}{2}\langle I_0'(u_n), u_n\rangle) \\ &= c_{V_0},\end{aligned}$$

where we used Fatou Lemma and Lemma 2.4. Therefore, $I_0(u) = c_{V_0}$, which means that $u$ is a ground state solution for (4.1).

Next we prove that the solution $u$ is positive, using $u^- = \max\{-u, 0\}$ as a test function in (4.1) we obtain

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} u^- dx + \int_{\mathbb{R}^3} V_0 |u^-|^2 dx = 0. \tag{4.19}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} u^- dx &= \frac{1}{2} C(s) \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{3+2s}} dxdy \\ &\geq \frac{1}{2} C(s) \int_{\{u>0\}\times\{u<0\}} \frac{(u(x)-u(y))(-u^-(y))}{|x-y|^{3+2s}} dxdy \\ &\quad + \frac{1}{2} C(s) \int_{\{u<0\}\times\{u<0\}} \frac{(u^-(x)-u^-(y))^2}{|x-y|^{3+2s}} dxdy \\ &\quad + \frac{1}{2} C(s) \int_{\{u<0\}\times\{u>0\}} \frac{(u(x)-u(y))u^-(x)}{|x-y|^{3+2s}} dxdy \\ &\geq 0.\end{aligned}$$

Thus, it follows from (4.19) that $u^- = 0$ and $u \geq 0$. Rewriting the equation (4.1) in the form of

$$(-\Delta)^s u + V_0 u = \Big( \int_{\mathbb{R}^3} \frac{H(u(y))u(y)}{|x-y|^\mu} dy \Big) K(u) \ \text{ in } \mathbb{R}^3,$$

where

$$H(u) := \frac{|u|^{2^\star_{\mu,s}} + F(u)}{u}, \quad K(u) := |u|^{2^\star_{\mu,s}-2-\mu} + \frac{1}{2^\star_{\mu,s}} f(u) \in L^{\frac{2}{3-\mu}}(\mathbb{R}^3) + L^{\frac{6}{3+2s-\mu}}(\mathbb{R}^3).$$

By Lemma 2.8, we know $u \in L^p(\mathbb{R}^3)$ for all $p \in [2, \frac{18}{(3-\mu)(3-2s)})$. Using the growth assumption $(f_1)$ and the higher integrability of $u$, for some $C > 0$ we have

$$\Big| \int_{\mathbb{R}^3} \frac{G(u(y))}{|x-y|^\mu} \Big|_\infty \leq C \big| |u|^{2^\star_{\mu,s}} + |u|^{q_1} + |u|^{q_2} \big|_{\frac{3}{3-\mu}} \leq C \big( |u|^{2^\star_{\mu,s}}_{\frac{3(6-\mu)}{3-\mu}} + |u|^{q_1}_{\frac{3q_1}{3-\mu}} + |u|^{q_2}_{\frac{32_1}{3-\mu}} \big), \tag{4.20}$$

which is finite since the various exponents live within the range $[2, \frac{18}{(3-\mu)(3-2s)})$. Thus,

$$(-\Delta)^s u + V_0 u \leq C(|u|^{2^\star_{\mu,s}-2-\mu} u + \frac{1}{2^\star_{\mu,s}} f(u)) \ \text{ in } \mathbb{R}^3.$$

By the Moser iteration, similar arguments developed in Lemma 6.1 below, we can get $u \in L^\infty(\mathbb{R}^3)$ and $\lim_{|x|\to+\infty} u(x) = 0$ uniformly in $n$. Then, by regularity theory [43], there exists $\alpha \in (0, 1)$ such that $u \in C^{0,\alpha}_{loc}(\mathbb{R}^3)$. Therefore, if $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^3$, we have that $(-\Delta)^s u(x_0) = 0$ and by [19, Lemma 3.2], we have

$$(-\Delta)^s u(x_0) = -\frac{C(s)}{2} \int_{\mathbb{R}^3} \frac{u(x_0+y) + u(x_0-y) - 2u(x_0)}{|y|^{3+2s}} dy,$$

therefore,

$$\int_{\mathbb{R}^3} \frac{u(x_0+y) + u(x_0-y)}{|y|^{3+2s}} dy = 0,$$

yielding $u \equiv 0$, a contradiction. Therefore, $u$ is a positive solution of the equation (4.1) and the proof is completed. □

The next result is a compactness result on autonomous problem which we will use later.

**Lemma 4.8.** *Let $\{u_n\} \subset \mathcal{N}_0$ be a sequence such that $I_0(u_n) \to c_{V_0}$. Then $\{u_n\}$ has a convergent subsequence in $H_0$.*

*Proof.* Since $\{u_n\} \subset \mathcal{N}_0$, it follows from Lemma 4.1-$(a_3)$, Lemma 4.2-$(b_4)$ and the definition of $c_{V_0}$ that

$$v_n = m^{-1}(u_n) = \frac{u_n}{\|u_n\|_0} \in S_0^+, \quad \forall\, n \in \mathbb{N},$$

and

$$\Psi_0(v_n) = I_0(u_n) \to c_{V_0} = \inf_{S_0^+} \Psi_0.$$

Although $S_0^+$ is incomplete, due to Lemma 4.1, we can still apply the Ekeland's variational principle [21] to the functional $\Theta_0 : H \to \mathbb{R} \cup \{\infty\}$, defined by $\Theta_0(u) = \Psi_0(u)$ if $u \in S_0^+$ and $\Theta_0(u) = \infty$ if $u \in \partial S_0^+$, where $H = \overline{S_0^+}$ is the complete metric space equipped with the metric $d(u, v) := \|u - v\|_0$. In fact, take $\varepsilon = \frac{1}{k^2}$ in Theorem 1.1 of [21], we have a subsequence $\{v_{n_k}\} \subset \{v_n\}$ such that

$$c_{V_0} \le \Psi(v_{n_k}) \le c_{V_0} + \frac{1}{k^2}.$$

From Theorem 1.1 in [21], for $\lambda = \frac{1}{k}$, there exist a sequence $\{\tilde{v}_k\} \subset S_0^+$ such that

$$\Theta_0(\tilde{v}_k) \le \Theta_0(v_{n_k}) < c_{V_0} + \frac{1}{k^2}$$

and

$$\|v_{n_k} - \tilde{v}_k\|_0 \le \frac{1}{k}.$$

In particular, for any $u \in S_0^+$ we have

$$\Psi_0(u) > \Psi_0(\tilde{v}_k) - \frac{1}{k}\|u - \tilde{v}_k\|_0.$$

Hence, similar the proof for Theorem 3.1 in [21], we have that there exists $\lambda_k \in \mathbb{R}$ such that

$$\|\hat{\Psi}_0'(\tilde{v}_k) - \lambda_k g_0'(\tilde{v}_k)\|_0 \le \frac{1}{k},$$

where $g_0(u) = \|u\|_0^2 - 1$. Which means that

$$\lambda_k = \frac{1}{\|g_0'(\tilde{v}_k)\|_0^2}\langle \hat{\Psi}_0'(\tilde{v}_k), g_0'(\tilde{v}_k)\rangle + o_k(1), \quad g_0'(\tilde{v}_k) = \tilde{v}_k.$$

From Lemma 4.2-$(b_1)$,

$$\lambda_k = \langle \hat{\Psi}_0'(\tilde{v}_k), \tilde{v}_k\rangle + o_k(1) = \tau_{\tilde{v}_k}\langle I_0'(t_{\tilde{v}_k}\tilde{v}_k), \tilde{v}_k\rangle + o_k(1) = o_k(1).$$

Therefore, we can conclude there is a sequence $\{\tilde{v}_n\} \subset S_0^+$ such that $\{\tilde{v}_n\}$ is a $(PS)_{c_{V_0}}$ sequence for $\Psi_0$ on $S_0^+$ and

$$\|u_n - \tilde{v}_n\|_0 = o_n(1).$$

Now the remainder of the proof follows from Lemma 4.2, Theorem 4.1 and arguing as in the proof of Lemma 3.8. □

# 5 Solutions for the penalized problem

In this section, we shall prove the existence and multiplicity of solutions. We begin showing the existence of the positive ground-state solution for the penalized problem (3.1).

**Theorem 5.1.** *Suppose that the nonlinearity $f$ satisfies $(f_1)$ – $(f_3)$ and that the potential function $V$ satisfies assumptions $(V_1)$ – $(V_2)$. Then, for any $\varepsilon > 0$, problem (3.1) has a positive ground-state solution $u_\varepsilon$.*

*Proof.* Similar to Lemma 3.2, we can prove that $J_\varepsilon$ also satisfies the Mountain Pass geometry. Let

$$c_\varepsilon := \inf_{u\in H_\varepsilon\setminus\{0\}} \max_{\tau\geq 0} J_\varepsilon(\tau u) = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u).$$

Then, we know that there exists a $(PS)$ sequence at $c_\varepsilon$, i.e.

$$J'_\varepsilon(u_n) \to 0 \text{ and } J_\varepsilon(u_n) \to c_\varepsilon.$$

Therefore, by Lemma 3.7, the existence of ground state solution $u_\varepsilon$ is guaranteed. Moreover, similarly to the proof in Theorem 4.1, we know that $u_\varepsilon(x) > 0$ in $\mathbb{R}^3$. □

Next, we will relate the number of positive solutions of (3.1) to the topology of the set $\mathcal{M}$. For this, we consider $\delta > 0$ such that $\mathcal{M}_\delta \subset \Omega$ and by Theorem 4.1, we can choose $w \in \mathcal{N}_0$ with $I_0(w) = c_{V_0}$. Let $\eta$ be a smooth nonincreasing cut-off function defined in $[0, +\infty)$ such that $\eta(t) = 1$ if $0 \leq t \leq \frac{\delta}{2}$ and $\eta(t) = 0$ if $t \geq \delta$. For each $y \in \mathcal{M}$, let

$$\Psi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|)w(\frac{\varepsilon x - y}{\varepsilon}).$$

Then for small $\varepsilon > 0$, one has $\Psi_{\varepsilon,y} \in H_\varepsilon\setminus\{0\}$ for all $y \in \mathcal{M}$. In fact, using the change of variable $z = x - \frac{y}{\varepsilon}$, one has

$$\begin{aligned}\int_{\mathbb{R}^3} V(\varepsilon x)\Psi^2_{\varepsilon,y}(x)dx &= \int_{\mathbb{R}^3} V(\varepsilon x)\eta^2(|\varepsilon x - y|)w^2(\frac{\varepsilon x - y}{\varepsilon})dx = \int_{\mathbb{R}^3} V(\varepsilon z + y)\eta^2(|\varepsilon z|)w^2(z)dz \\ &\leq C\int_{\mathbb{R}^3} w^2(z)dz < +\infty.\end{aligned}$$

Moreover, using the change of variable $x' = x - \frac{y}{\varepsilon}$, $z' = z - \frac{y}{\varepsilon}$, we have

$$\begin{aligned}|(-\Delta)^{\frac{s}{2}}\Psi_{\varepsilon,y}|_2^2 &= \frac{1}{2}C(s)\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\left|\eta(|\varepsilon x - y|)w(\frac{\varepsilon x-y}{\varepsilon}) - \eta(|\varepsilon z - y|)w(\frac{\varepsilon z-y}{\varepsilon})\right|^2}{|x - z|^{3+2s}}dxdz \\ &= \frac{1}{2}C(s)\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\left|\eta(|\varepsilon x'|)w(x') - \eta(|\varepsilon z'|)w(z')\right|^2}{|x' - z'|^{3+2s}}dx'dz' \\ &= |(-\Delta)^{\frac{s}{2}}\eta(|\varepsilon x|)w(x)|_2^2 = |(-\Delta)^{\frac{s}{2}}\eta_\varepsilon w|_2^2,\end{aligned}$$

where $\eta_\varepsilon(x) = \eta(|\varepsilon x|)$. By Lemma 2.3, we see that $\eta_\varepsilon w \in \mathcal{D}^{s,2}(\mathbb{R}^3)$ as $\varepsilon \to 0$, and hence $\Psi_{\varepsilon,y} \in \mathcal{D}^{s,2}(\mathbb{R}^3)$ for $\varepsilon > 0$ small. Hence $\Psi_{\varepsilon,y} \in H_\varepsilon$. Now we proof $\Psi_{\varepsilon,y} \neq 0$. In fact,

$$\begin{aligned}\int_{\mathbb{R}^3} \Psi^2_{\varepsilon,y}(x)dx &= \int_{\mathbb{R}^3} \eta^2(|\varepsilon x - y|)w^2(\frac{\varepsilon x - y}{\varepsilon})dx = \int_{|\varepsilon x-y|<\delta} \eta^2(|\varepsilon x - y|)w^2(\frac{\varepsilon x - y}{\varepsilon})dx \\ &\geq \int_{|z|\leq\frac{\delta}{2\varepsilon}} \eta^2(|\varepsilon z|)w^2(z)dz \geq \int_{B_0(\frac{\delta}{2\varepsilon})} w^2(z)dz \to \int_{\mathbb{R}^3} w^2(z)dz > 0\end{aligned}$$

as $\varepsilon \to 0$. Then $\Psi_{\varepsilon,y} \neq 0$ for small $\varepsilon > 0$. Therefore, there exists unique $\tau_\varepsilon > 0$ such that

$$\max_{\tau\geq 0} I_\varepsilon(\tau\Psi_{\varepsilon,y}) = I_\varepsilon(\tau_\varepsilon\Psi_{\varepsilon,y}) \text{ and } \tau_\varepsilon\Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon.$$

We introduce the map $\Phi_\varepsilon : \mathcal{M} \to \mathcal{N}_\varepsilon$ by setting

$$\Phi_\varepsilon(y) = \tau_\varepsilon\Psi_{\varepsilon,y}.$$

By construction, $\Phi_\varepsilon(y)$ has a compact support for any $y \in \mathcal{M}$ and $\Phi_\varepsilon$ is a continuous map.

**Lemma 5.1.** *The functional $\Phi_\varepsilon(y)$ has the following property:*

$$\lim_{\varepsilon\to 0} J_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0} \text{ uniformly in } y \in \mathcal{M}.$$

*Proof.* Suppose that the result is false. Then, there exist some $\delta_0 > 0$, $\{y_n\} \subset \mathcal{M}$ and $\varepsilon_n \to 0$ such that

$$|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_{V_0}| \ge \delta_0. \tag{5.1}$$

By the definition of $\tau_{\varepsilon_n}$ we have

$$\begin{aligned} 0 < \tau_{\varepsilon_n}^2 \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon_n,y_n}|^2 dx + \tau_{\varepsilon_n}^2 \int_{\mathbb{R}^3} V(\varepsilon_n x)\Psi_{\varepsilon_n,y_n}^2 dx \\ = \frac{1}{2_{\mu,s}^\star} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{H(\varepsilon_n x, \tau_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}) h(\varepsilon_n x, \tau_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})\tau_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}}{|x-y|^\mu} dydx \end{aligned} \tag{5.2}$$

It follows from (5.2) that $\tau_{\varepsilon_n} \not\to 0$, then $\tau_{\varepsilon_n} \ge \tau_0 > 0$ for some $\tau_0 > 0$. If $\tau_{\varepsilon_n} \to +\infty$, by $(f_2)$ and the boundedness of $\Psi_{\varepsilon_n,y_n}$, we get

$$\begin{aligned} \Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon_n,y_n}|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon_n x)\Psi_{\varepsilon_n,y_n}^2 dx\Big) \\ = \frac{1}{2_{\mu,s}^\star} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{H(\varepsilon_n x, \tau_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}) h(\varepsilon_n x, \tau_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})\Psi_{\varepsilon_n,y_n}}{|x-y|^\mu \tau_{\varepsilon_n}} dydx \to +\infty \end{aligned} \tag{5.3}$$

as $n \to +\infty$. But the left side of the above inequality is boundedness, which is impossible. Hence, $0 < \tau_0 \le \tau_{\varepsilon_n} \le C$. Without loss of generality, we may assume that $\tau_{\varepsilon_n} \to T > 0$.

Next we claim that $T = 1$. By Lemma 2.3 and Lebesgue's theorem we have

$$\begin{aligned} \lim_{n\to+\infty} \|\Psi_{\varepsilon_n,y_n}\|_{\varepsilon_n}^2 = \|w\|_0^2, \\ \lim_{n\to+\infty} \Sigma_\varepsilon(\Psi_{\varepsilon_n,y_n}) = \Sigma_0(w) \end{aligned} \tag{5.4}$$

Moreover, from

$$\tau_{\varepsilon_n}^2 \|\Psi_{\varepsilon_n,y_n}\|_{\varepsilon_n}^2 = \frac{1}{2_{\mu,s}^\star} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{H(\varepsilon_n x, \tau_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}) h(\varepsilon_n x, \tau_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})\tau_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}}{|x-y|^\mu} dydx \tag{5.5}$$

we can deduce that

$$\|w\|_0^2 = \lim_{n\to\infty} \frac{1}{2_{\mu,s}^\star} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{H(\varepsilon_n x, \tau_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}) h(\varepsilon_n x, \tau_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})\tau_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}}{|x-y|^\mu} dydx \tag{5.6}$$

Taking into account that $w$ is a ground state solution to (4.1) and using $(f_2)$, we deduce that $T = 1$. It follows from (5.4), we have

$$\lim_{n\to+\infty} J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = J_0(w) = c_{V_0}, \tag{5.7}$$

which is a contradiction with (5.1). This completes the proof. □

Let $\rho = \rho(\delta) > 0$ be such that $\mathcal{M}_\delta \subset B_\rho(0)$. Consider $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $\chi(x) = x$ for $|x| \le \rho$ and $\chi(x) = \frac{\rho x}{|x|}$ for $|x| \ge \rho$. Finally, let us consider the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^3$ given by

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^3} \chi(\varepsilon x)u^2(x)dx}{\int_{\mathbb{R}^3} u^2(x)dx} \in \mathbb{R}^3.$$

**Lemma 5.2.** *The functional $\beta_\varepsilon$ satisfies*

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \text{ uniformly in } y \in \mathcal{M}.$$

*Proof.* Suppose by contradiction that there exist $\delta_0 > 0$, $\{y_n\} \subset \mathcal{M}$ and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0. \tag{5.8}$$

Using the change of variables $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$ and the definition of $\beta_\varepsilon$, we have

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^3}(\chi(\varepsilon_n z + y) - y_n)|\eta(|\varepsilon_n z|)w(z)|^2 dx}{\int_{\mathbb{R}^3}|\eta(|\varepsilon_n z|)w(z)|^2 dx}.$$

Since $\{y_n\} \subset \mathcal{M} \subset B_\rho(0)$ and $\chi\big|_{B_\rho} \equiv id$, we conclude that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = o_n(1),$$

which contradicts (5.8) and the desired conclusion holds. □

**Lemma 5.3.** *Let $\varepsilon_n \to 0$ and $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(u_n) \to c_{V_0}$. Then, there exists a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^3$ such that $v_n(x) = u_n(x + \tilde{y}_n)$ has a convergent subsequence in $H_0$. Moreover, passing to a subsequence, $y_n := \varepsilon_n \tilde{y}_n \to y_0 \in \mathcal{M}$.*

*Proof.* By Lemma 3.5, $\{u_n\}$ is bounded in $H_0$. Note that $c_{V_0} > 0$, and since $\|u_n\|_{\varepsilon_n} \to 0$ would imply $J_{\varepsilon_n}(u_n) \to 0$, we can argue as in the proof of Lemma 4.3 to obtain a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that

$$\liminf_{n \to +\infty} \int_{B_R(\tilde{y}_n)} u_n^2 dx \geq \beta > 0.$$

Define $v_n(x) := u_n(x + \tilde{y}_n)$, then $\{v_n\}$ is also bounded in $H_0$ and up to a subsequence, we have

$$v_n \rightharpoonup v \neq 0 \text{ in } H_0.$$

Let $\tau_n > 0$ be such that $\tilde{v}_n := \tau_n v_n \in \mathcal{N}_0$ and set $y_n = \varepsilon_n \tilde{y}_n$. By $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$, we have

$$c_{V_0} \leq J_0(\tilde{v}_n) = J_0(\tau_n u_n) \leq J_{\varepsilon_n}(\tau_n u_n) \leq J_{\varepsilon_n}(u_n) = c_{V_0} + o_n(1).$$

Which implies that $\lim\limits_{n \to +\infty} J_0(\tilde{v}_n) = c_{V_0}$. In virtue of $\tilde{v}_n \in \mathcal{N}_0$, we obtain $\{\tilde{v}_n\}$ is bounded in $H_0$. It follows from the boundedness of $\{v_n\}$ in $H_0$ that $\{\tau_n\}$ is bounded, without loss of generality, we may assume that $\tau_n \to \tau_0 \geq 0$. If $\tau_0 = 0$, in view of the boundedness of $\{v_n\}$ in $H_0$, we have $\tilde{v}_n = \tau_n v_n \to 0$ in $H_0$. Hence $J_0(\tilde{v}_n) \to 0$, which contradicts $c_{V_0} > 0$. Thus, $\tau_0 > 0$ and the weak limit of $\{\tilde{v}_n\}$ is different from zero. Hence, up to a subsequence, we have $\tilde{v}_n \rightharpoonup \tau_0 v := \tilde{v} \neq 0$ in $H_0$ by the uniqueness of the weak limit. From Lemma 4.8, we know that $\tilde{v}_n \to \tilde{v}$ in $H_0$. Moreover, $\tilde{v} \in \mathcal{N}_0$.

Now, we will show that $\{y_n\}$ is bounded in $\mathbb{R}^3$. Suppose that after passing to a subsequence, $|y_n| \to +\infty$. Choosing $R > 0$ such that $\Omega \subset B_R(0)$. Without loss of generality we may assume that $|y_n| > 2R$. Then, for all $z \in B_{R/\varepsilon_n}(o)$,

$$|\varepsilon_n z + y_n| \geq |y_n| - |\varepsilon_n z| > R. \tag{5.9}$$

By the change of variable $x \mapsto z + tildey_n$, using the fact that $V_0 \leq V(\varepsilon x)$ and (5.9), we have

$$\|v_n\|_0^2 \leq C \int_{\mathbb{R}^3} H(\varepsilon z + y_n, v_n) v_n dx \leq C \int_{B_{R/\varepsilon_n}} \tilde{G}(v_n) v_n dz + C \int_{\mathbb{R}^3 \setminus B_{R/\varepsilon_n}} G(v_n) v_n dz. \tag{5.10}$$

Since $\tilde{G}(u) \leq \frac{V_0}{K} u$ and $v_n \to v$ in $H_0$, we can see that (5.10) implies that

$$\|v_n\|_0^2 = o_n(1),$$

that is $v_n \to 0$ in $H_0$, which is a contradiction. Therefore, up to s subsequence, we may assume that $y_n \to y_0 \in \mathbb{R}^3$. It remains to check that $y_0 \in \mathcal{M}$. Clearly, if $y_0 \notin= \Omega$, then we can argue as before and we deduce that

$u_n \to 0$ in $H_0$, which is impossible. Hence we only need to show that $V(y_0) = V_0$. Arguing by contradiction again, we suppose that $V(y_0) > V_0$. Then, by using $\tilde{v}_n \to \tilde{v}$ in $H_0$ and Fatou's Lemma, we have

$$\begin{aligned} c_{V_0} &= J_0(\tilde{v}) \\ &< \liminf_{n\to\infty} \Big(\frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\tilde{v}|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(y_0)\tilde{v}^2 dx - \Sigma_0(\tilde{v})\Big) \\ &\le \liminf_{n\to\infty} \Big(\frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\tilde{v}_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon_n x + y_n)\tilde{v}_n^2 dx - \Sigma_0(\tilde{v}_n)\Big) \\ &\le \liminf_{n\to\infty} J_{\varepsilon_n}(\tau_n u_n) \\ &\le \liminf_{n\to\infty} J_{\varepsilon_n}(u_n) \\ &= c_{V_0}, \end{aligned}$$

which yields a contradiction. So, $y_0 \in \mathcal{M}$ and the proof is completed. □

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be any positive function satisfying $h(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$. Define the set

$$\tilde{\mathcal{N}}_\varepsilon = \{u \in \mathcal{N}_\varepsilon : J_\varepsilon(u) \le c_{V_0} + h(\varepsilon)\}.$$

Given $y \in \mathcal{M}$, we conclude from Lemma 5.1 that $h(\varepsilon) = \sup_{y\in\mathcal{M}} |I_\varepsilon(\Phi_\varepsilon(y)) - c_{V_0}| \to 0$ as $\varepsilon \to 0^+$. Thus, $\Phi_\varepsilon(y) \in \tilde{\mathcal{N}}_\varepsilon$ and $\tilde{\mathcal{N}}_\varepsilon \neq \emptyset$ for $\varepsilon > 0$. Moreover, we have the following Lemma.

**Lemma 5.4.** *For any $\delta > 0$, there holds that*

$$\lim_{\varepsilon\to 0} \sup_{u\in\tilde{\mathcal{N}}_\varepsilon} \inf_{y\in\mathcal{M}_\delta} |\beta_\varepsilon(u) - y| = 0.$$

*Proof.* Let $\{\varepsilon_n\} \subset \mathbb{R}^+$ be such that $\varepsilon_n \to 0$. By definition, there exists $\{u_n\} \subset \tilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\inf_{y\in\mathcal{M}_\delta} |\beta_{\varepsilon_n}(u_n) - y| = \sup_{u\in\tilde{\mathcal{N}}_{\varepsilon_n}} \inf_{y\in\mathcal{M}_\delta} |\beta_{\varepsilon_n}(u) - y| + o_n(1).$$

So, it suffices to find a sequence $\{y_n\} \subset \mathcal{M}_\delta$ satisfying

$$\lim_{n\to+\infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0. \tag{5.11}$$

Since $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we get

$$c_{V_0} \le c_{\varepsilon_n} \le J_{\varepsilon_n}(u_n) \le c_{V_0} + h(\varepsilon_n).$$

It follows that $J_{\varepsilon_n}(u_n) \to c_{V_0}$. Thus, we can invoke Lemma 5.3 to obtain a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^3$ such that $y_n = \varepsilon_n \tilde{y}_n \in \mathcal{M}_\delta$ for $n$ large enough. Then

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^3} (\chi(\varepsilon_n x + y_n) - y_n)|u_n(x + \tilde{y}_n)|^2 dx}{\int_{\mathbb{R}^3} |u_n(x + \tilde{y}_n)|^2 dx}.$$

For $\forall x \in \mathbb{R}^3$ fixed, since $\varepsilon_n x + y_n \to y \in \mathcal{M}_\delta$, we have that the sequence $\{y_n\}$ satisfies (5.11). This completes the proof. □

Next we prove our multiplicity result by presenting a relation between the topology of $\mathcal{M}$ the number of solutions of the modified problem (3.1), we will apply the Ljusternik-Schnirelmann abstract result in [44, 46].

**Theorem 5.2.** *Assume that conditions $(V_1) - (V_2)$ and $(f_1) - (f_3)$ hold. Then, given $\delta > 0$ there is $\hat{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \hat{\varepsilon}_\delta)$, problem (3.1) has at least $cat_{\mathcal{M}_\delta}(\mathcal{M})$ positive solutions.*

*Proof.* For $y \in \mathcal{M}$, set $y_\varepsilon(y) = m_\varepsilon^{-1}(\Phi_\varepsilon(y))$. It follows from Lemma 3.4 and Lemma 5.1 that

$$\lim_{\varepsilon \to 0} \Psi_\varepsilon(y_\varepsilon(y)) = \lim_{\varepsilon \to 0} I_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0}, \tag{5.12}$$

uniformly in $y \in \mathcal{M}$. Let

$$\tilde{S}_\varepsilon^+ = \{w \in S_\varepsilon^+ : \Psi_\varepsilon(w) \le c_{V_0} + h(\varepsilon)\},$$

where $h$ is given in the definition of $\tilde{\mathcal{N}}_\varepsilon$. From (5.12), we know that there is a number $\hat{\varepsilon}$ such that $\tilde{S}_\varepsilon^+ \neq \emptyset$ for $\varepsilon \in (0, \hat{\varepsilon})$.

For a fixed $\delta > 0$, by Lemmas 3.3, 5.1-5.2 and 5.4, we know that there exists a $\hat{\varepsilon} = \hat{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \hat{\varepsilon}_\delta)$, the diagram

$$\mathcal{M} \xrightarrow{\Phi_\varepsilon} \tilde{\mathcal{N}}_\varepsilon \xrightarrow{m_\varepsilon^{-1}} \tilde{S}_\varepsilon^+ \xrightarrow{m_\varepsilon} \tilde{\mathcal{N}}_\varepsilon \xrightarrow{\beta_\varepsilon} \mathcal{M}_\delta$$

is well defined. From Lemma 5.2, there is a function $\lambda(\varepsilon, y)$ with $|\lambda(\varepsilon, y)| < \frac{\delta}{2}$ uniformly in $y \in \mathcal{M}$, for all $\varepsilon \in (0, \hat{\varepsilon})$,such that $\beta_\varepsilon(\Phi_\varepsilon(y)) := y + \lambda(\varepsilon, y)$ for all $y \in \mathcal{M}$. Define $H(t, y) = y + (1 - t)\lambda(\varepsilon, y)$. Then, $H : [0, 1] \times \mathcal{M} \to \mathcal{M}_\delta$ is continuous. Obviously, $H(0, y) = \beta_\varepsilon(\Phi_\varepsilon(y)), H(1, y) = y$ for all $y \in \mathcal{M}$. That is, $H(t, y)$ is homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon$ and the inclusion map $id : \mathcal{M} \to \mathcal{M}_\delta$. This fact and Lemma 4.3 in [7] implies that

$$cat_{\tilde{S}_\varepsilon^+} y_\varepsilon(\mathcal{M}) \ge cat_{\mathcal{M}_\delta}(\mathcal{M}).$$

On the other hand, using the definition of $\tilde{\mathcal{N}}_\varepsilon$ and choosing $\hat{\varepsilon}_\delta$ small if necessary, we see that $I_\varepsilon$ satisfies the $(PS)$ condition in $\tilde{\mathcal{N}}_\varepsilon$ recalling Lemma 3.7. By Lemma 3.4 and 3.8, we obtain that $\Psi_\varepsilon$ satisfies the $(PS)$ condition in $\tilde{S}_\varepsilon^+$. Therefore, the standard Ljusternik-Schnirelmann theory provides at least $cat_{\tilde{S}_\varepsilon^+} y_\varepsilon(\mathcal{M})$ critical points of $\Psi_\varepsilon$ restricted to $\tilde{S}_\varepsilon^+$. Using Lemma 3.7 again, we infer that $I_\varepsilon$ has at least $cat_{\mathcal{M}_\delta}(\mathcal{M})$ critical points. Using the same arguments contained in the proof Theorem 4.1, we see that the equation (3.1) has at least $cat_{\mathcal{M}_\delta}(\mathcal{M})$ positive solutions. □

# 6 Proof of Theorem 1.1

In this section we will prove our main result. The idea is to show that the solutions obtained in Theorem 5.2 verify the following estimate $u_\varepsilon(x) \le a, \forall x \in \Omega_\varepsilon^c$ for $\varepsilon$ small enough. This fact implies that these solutions are in fact solutions of the original problem (2.3). The key ingredient is the following result, whose proof uses an adaptation of the arguments found in [20], which are related to the Moser iteration method [35].

**Lemma 6.1.** *Let $\varepsilon_n \to 0^+$ and $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ be a solution of (3.1). Then up to a subsequence, $v_n = u_n(x + \tilde{y}_n)$ satisfies that $v_n \in L^\infty(\mathbb{R}^3)$ and there exists $C > 0$ such that*

$$\|v_n\|_{L^\infty(\mathbb{R}^3)} \le C, \ \forall\, n \in \mathbb{N},$$

*where $\{\tilde{y}_n\}$ is given in Lemma 5.3. Furthermore,*

$$\lim_{|x| \to \infty} v_n(x) = 0 \ \textit{uniformly in } n \in \mathbb{N}.$$

*Proof.* Rewriting the equation (3.1) in the form of

$$(-\Delta)^s u + V(\varepsilon x)u = \Big(\int_{\mathbb{R}^3} \frac{H(u(y))u(y)}{|x - y|^\mu} dy\Big)K(u) \ \text{ in } \mathbb{R}^3,$$

where

$$H(u) := \frac{|u|^{2^\star_{\mu,s}} + F(u)}{u}, \ K(u) := |u|^{2^\star_{\mu,s} - 2 - \mu} + \frac{1}{2^\star_{\mu,s}} f(u) \in L^{\frac{2}{3-\mu}}(\mathbb{R}^3) + L^{\frac{6}{3+2s-\mu}}(\mathbb{R}^3).$$

By Lemma 2.8, we know $u \in L^p(\mathbb{R}^3)$ for all $p \in [2, \frac{18}{(3-\mu)(3-2s)})$. Using the growth assumption $(f_1)$ and the higher integrability of $u$, for some $C > 0$ we have

$$\Big|\int_{\mathbb{R}^3} \frac{G(u(y))}{|x-y|^\mu}\Big|_\infty \le C\big||u|^{2^*_{\mu,s}} + |u|^{q_1} + |u|^{q_2}\big|_{\frac{3}{3-\mu}} \le C\big(|u|^{2^*_{\mu,s}}_{\frac{3(6-\mu)}{3-\mu}} + |u|^{q_1}_{\frac{3q_1}{3-\mu}} + |u|^{q_2}_{\frac{3 2_1}{3-\mu}}\big),$$

which is finite since the various exponents live within the range $[2, \frac{18}{(3-\mu)(3-2s)})$. Therefore, we have

$$|h(\varepsilon x, v_n)| := \Big(\int_{\mathbb{R}^3} \frac{H(\varepsilon_n x + \varepsilon_n \tilde{y}_n, v_n)}{|x-y|^\mu} dy\Big) h(\varepsilon_n x + \varepsilon_n \tilde{y}_n, v_n) - V(\varepsilon_n x + \varepsilon_n \tilde{y}_n) v_n \le C(1 + |v_n|^{2^*_s - 1}) \tag{6.1}$$

for $n$ large enough.

Let $T > 0$, we define

$$H(t) = \begin{cases} 0, & \text{if } t \le 0, \\ t^\beta, & \text{if } 0 < t < T, \\ \beta T^{\beta-1}(t-T) + T^\beta, & \text{if } t \ge T, \end{cases}$$

with $\beta > 1$ to be determined later. Since $H$ is Lipschitz with constant $L_0 = \beta T^{\beta-1}$, we have

$$\begin{aligned}
[H(v_n)]_{\mathcal{D}^{s,2}} &= \Big(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|H(v_n(x)) - H(v_n(y))|^2}{|x-y|^{3+2s}} dxdy\Big)^{\frac{1}{2}} \\
&\le \Big(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{L_0^2 |v_n(x) - v_n(y)|^2}{|x-y|^{3+2s}} dxdy\Big)^{\frac{1}{2}} \\
&= L_0 [v_n]_{\mathcal{D}^{s,2}}.
\end{aligned}$$

Therefore, $H(v_n) \in \mathcal{D}^{s,2}(\mathbb{R}^3)$. Moreover, by the definition of $H$, we know that $H$ is a convex function, then we have

$$(-\Delta)^s H(v_n) \le H'(v_n)(-\Delta)^s v_n \tag{6.2}$$

in the weak sense. Thus, from $H(v_n) \in \mathcal{D}^{s,2}(\mathbb{R}^3)$ and (6.1)-(6.2), we have

$$\begin{aligned}
\|H(v_n)\|^2_{2^*_s} &\le C\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} H(v_n)|^2 dx = C\int_{\mathbb{R}^3} H(v_n)(-\Delta)^s H(v_n) dx \\
&\le C\int_{\mathbb{R}^3} H(v_n)H'(v_n)(-\Delta)^s v_n dx = C\int_{\mathbb{R}^3} H(v_n)H'(v_n) h(\varepsilon_n x, v_n) dx \\
&\le C\int_{\mathbb{R}^3} H(v_n)H'(v_n) dx + C\int_{\mathbb{R}^3} H(v_n)H'(v_n) v_n^{2^*_s - 1} dx.
\end{aligned}$$

Using the fact that $H(v_n)H'(v_n) \le \beta v_n^{2\beta-1}$ and $v_n H'(v_n) \le \beta H(v_n)$, we have

$$\Big(\int_{\mathbb{R}^3} (H(v_n))^{2^*_s} dx\Big)^{\frac{2}{2^*_s}} = C\beta\Big(\int_{\mathbb{R}^3} v_n^{2\beta-1} dx + \int_{\mathbb{R}^3} (H(v_n))^2 v_n^{2^*_s-2} dx\Big), \tag{6.3}$$

where $C$ is a positive constant that does not depend on $\beta$. Notice that the last integral is well defined for $T$ in the definition of $H$. Indeed

$$\begin{aligned}
\int_{\mathbb{R}^3} (H(v_n))^2 v_n^{2^*_s-2} dx &= \int_{v_n \le T} (H(v_n))^2 v_n^{2^*_s-2} dx + \int_{v_n > T} (H(v_n))^2 v_n^{2^*_s-2} dx \\
&\le T^{2\beta-2} \int_{\mathbb{R}^3} v_n^{2^*_s} dx + C\int_{\mathbb{R}^3} v_n^{2^*_s} dx < \infty.
\end{aligned}$$

We choose now $\beta$ in (6.3) such that $2\beta - 1 = 2_s^\star$, and we name it $\beta_1$, that is

$$\beta_1 := \frac{2_s^\star + 1}{2}. \tag{6.4}$$

Let $\hat{R} > 0$ to be fixed later. Attending to the last integral in (6.3) and applying the Holder's inequality with exponents $y := \frac{2_s^\star}{2}$ and $y' := \frac{2_s^\star}{2_s^\star - 2}$,

$$\begin{aligned}\int_{\mathbb{R}^3} (H(v_n))^2 v_n^{2_s^\star-2} dx &= \int_{v_n \le \hat{R}} (H(v_n))^2 v_n^{2_s^\star-2} dx + \int_{v_n > \hat{R}} (H(v_n))^2 v_n^{2_s^\star-2} dx \\ &\le \int_{v_n \le \hat{R}} \frac{(H(v_n))^2}{v_n} \hat{R}^{2_s^\star-1} dx + \Big(\int_{\mathbb{R}^3} (H(v_n))^{2_s^\star} dx\Big)^{\frac{2}{2_s^\star}} \Big(\int_{v_n > \hat{R}} v_n^{2_s^\star} dx\Big)^{\frac{2_s^\star-2}{2_s^\star}}.\end{aligned} \tag{6.5}$$

By Lemma 4.8, we know that $\{v_n\}$ has a convergent subsequence in $H_0$, therefore we can choose $\hat{R}$ large enough so that

$$\Big(\int_{v_n > \hat{R}} v_n^{2_s^\star} dx\Big)^{\frac{2_s^\star-2}{2_s^\star}} \le \frac{1}{2C\beta_1},$$

where $C$ is the constant appearing in (6.3). Therefore, we can absorb the last term in (6.5) by the left hand side of (6.3) to get

$$\Big(\int_{\mathbb{R}^3} (H(v_n))^{2_s^\star} dx\Big)^{\frac{2}{2_s^\star}} \le 2C\beta_1 \Big(\int_{\mathbb{R}^3} v_n^{2_s^\star} dx + \hat{R}^{2_s^\star-1} \int_{\mathbb{R}^3} \frac{(H(v_n))^2}{v_n} dx\Big),$$

Now we use the fact that $H(v_n) \le v_n^{\beta_1}$ and we take $T \to \infty$, we obtain

$$\Big(\int_{\mathbb{R}^3} v_n^{2_s^\star \beta_1} dx\Big)^{\frac{2}{2_s^\star}} \le 2C\beta_1 \Big(\int_{\mathbb{R}^3} v_n^{2_s^\star} dx + \hat{R}^{2_s^\star-1} \int_{\mathbb{R}^3} v_n^{2_s^\star} dx\Big).$$

and therefore

$$v_n \in L^{2_s^\star \beta_1}(\mathbb{R}^3). \tag{6.6}$$

Let us suppose now $\beta > \beta_1$. Thus, using that $H(v_n) \le v_n^\beta$ in the right hand side of (6.3) and letting $T \to \infty$ we get

$$\Big(\int_{\mathbb{R}^3} v_n^{2_s^\star \beta} dx\Big)^{\frac{2}{2_s^\star}} \le C\beta \Big(\int_{\mathbb{R}^3} v_n^{2\beta-1} dx + \hat{R}^{2_s^\star-1} \int_{\mathbb{R}^3} v_n^{2\beta+2_s^\star-2} dx\Big). \tag{6.7}$$

Set $c_0 := \frac{2_s^\star(2_s^\star-1)}{2(\beta-1)}$ and $c_1 := 2\beta - 1 - c_0$. Notice that, since $\beta > \beta_1$, then $0 < c_0 < 2_s^\star$, $c_1 > 0$. Hence, applying Young's inequality with exponents $y := 2_s^\star/c_0$ and $y' := 2_s^\star/2_s^\star - c_0$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} v_n^{2\beta-1} dx &\le \frac{c_0}{2_s^\star} \int_{\mathbb{R}^3} v_n^{2_s^\star} dx + \frac{2_s^\star}{2_s^\star - c_0} \int_{\mathbb{R}^3} v_n^{\frac{2_s^\star c_1}{2_s^\star - c_0}} dx \\ &\le \int_{\mathbb{R}^3} v_n^{2_s^\star} dx + \int_{\mathbb{R}^3} v_n^{2\beta+2_s^\star-2} dx \\ &\le C\big(1 + \int_{\mathbb{R}^3} v_n^{2\beta+2_s^\star-2} dx\big),\end{aligned}$$

with $C > 0$ independent of $\beta$. Plugging into (6.7),

$$\Big(\int_{\mathbb{R}^3} v_n^{2_s^\star \beta} dx\Big)^{\frac{2}{2_s^\star}} \le C\beta\big(1 + \int_{\mathbb{R}^3} v_n^{2\beta+2_s^\star-2} dx\big),$$

with $C$ changing from line to line, but remaining independent of $\beta$. Therefore

$$\Big(1+\int_{\mathbb{R}^3} v_n^{2_s^\star\beta}\,dx\Big)^{\frac{1}{2_s^\star(\beta-1)}} \le (C\beta)^{\frac{1}{2(\beta-1)}}\Big(1+\int_{\mathbb{R}^3} v_n^{2\beta+2_s^\star-2}\,dx\Big)^{\frac{1}{2(\beta-1)}}. \tag{6.8}$$

Repeating this argument we will define a sequence $\beta_m$, $m \ge 1$ such that

$$2\beta_{m+1}+2_s^\star-2 = 2_s^\star\beta_m.$$

Thus,

$$\beta_{m+1}-1 = \Big(\frac{2_s^\star}{2}\Big)^m(\beta_1-1).$$

Replacing it in (6.8) one has

$$\Big(1+\int_{\mathbb{R}^3} v_n^{2_s^\star\beta_{m+1}}\,dx\Big)^{\frac{1}{2_s^\star(\beta_{m+1}-1)}} \le (C\beta_{m+1})^{\frac{1}{2(\beta_{m+1}-1)}}\Big(1+\int_{\mathbb{R}^3} v_n^{2_s^\star\beta_m}\,dx\Big)^{\frac{1}{2_s^\star(\beta_m-1)}}.$$

Defining $C_{m+1} := C\beta_{m+1}$ and

$$A_m := \Big(1+\int_{\mathbb{R}^3} v_n^{2_s^\star\beta_m}\,dx\Big)^{\frac{1}{2_s^\star(\beta_m-1)}},$$

we conclude that there exists a constant $C_0 > 0$ independent of $m$, such that

$$A_m \le \prod_{k=1}^{m} C_k^{\frac{1}{2(\beta_k-1)}} A_1 \le C_0A_1.$$

Thus,

$$\|v_n\|_\infty \le C_0A_1 < \infty, \tag{6.9}$$

uniformly in $n \in \mathbb{N}$, thanks to (6.6). Now argue as in the proof of [3, Lemma 2.6], we conclude that

$$u_n(x) \to 0 \text{ as } |x| \to \infty,$$

uniformly in $n \in \mathbb{N}$. This finishes the proof of Lemma 6.1. □

We are now ready to prove the main result of the paper.

**Proof of Theorem 1.1.** We fix a small $\delta > 0$ such that $\mathcal{M}_\delta \subset \Omega$. We first claim that there exists some $\tilde{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$ and any solution $u_\varepsilon \in \tilde{\mathcal{N}}_\varepsilon$ of the problem (3.1), there holds

$$\|u_\varepsilon\|_{L^\infty(\mathbb{R}^3\setminus\Omega_\varepsilon)} < a. \tag{6.10}$$

In order to prove the claim we argue by contradiction. So, suppose that for some sequence $\varepsilon_n \to 0^+$ we can obtain $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ such that $I'_{\varepsilon_n}(u_n) = 0$ and

$$\|u_n\|_{L^\infty(\mathbb{R}^3\setminus\Omega_\varepsilon)} \ge a. \tag{6.11}$$

As in the proof of Lemma 5.3, we have that $J_{\varepsilon_n}(u_n) \to c_{V_0}$ and we can obtain a sequence $\{\tilde{y}_n\} \in \mathbb{R}^3$ such that $\varepsilon_n\tilde{y}_n \to y_0 \in \mathcal{M}$.

If we take $r > 0$ such that $B_r(y_0) \subset B_{2r}(y_0) \subset \Omega$ we have that

$$B_{\frac{r}{\varepsilon_n}}\Big(\frac{y_0}{\varepsilon_n}\Big) = \frac{1}{\varepsilon_n}B_r(y_0) \subset \Omega_{\varepsilon_n}.$$

Moreover, for any $z \in B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$, there holds

$$\Big|z-\frac{y_0}{\varepsilon_n}\Big| \le ||z-\tilde{y}_n| + \Big|\tilde{y}_n-\frac{y_0}{\varepsilon_n}\Big| < \frac{1}{\varepsilon_n}(r+o_n(1)) < \frac{2r}{\varepsilon_n},$$

for $n$ large. For these values of $n$ we have that $B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset \Omega_{\varepsilon_n}$, that is, $\mathbb{R}^3 \backslash \Omega_{\varepsilon_n} \subset \mathbb{R}^3 \backslash B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$. On the other hand, it follows from Lemma 6.1 that there is $R > 0$ such that

$$u_n(x) < a \text{ for } |x| \geq R \text{ and } \forall\, n \in \mathbb{N},$$

from where it follows that

$$v_n(x - \tilde{y}_n) < a \text{ for } x \in B_R^c(\tilde{y}_n) \text{ and } n \in \mathbb{N}.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and $\frac{r}{\varepsilon_n} > R$, there holds

$$\mathbb{R}^3 \backslash \Omega_{\varepsilon_n} \subset \mathbb{R}^3 \backslash B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset \mathbb{R}^3 \backslash B_R(\tilde{y}_n).$$

Then, there holds

$$u_n(x) < a \;\; \forall\, x \in \mathbb{R}^3 \backslash \Omega_{\varepsilon_n},$$

which contradicts to (6.11) and the claim holds true.

Let $\hat{\varepsilon}_\delta$ given by Theorem 5.2 and let $\varepsilon_\delta := \min\{\hat{\varepsilon}_\delta, \tilde{\varepsilon}_\delta\}$. We will prove the theorem for this choice of $\varepsilon_\delta$. Let $\varepsilon \in (0, \varepsilon_\delta)$ be fixed. By using Theorem 5.2 we get $cat_{\mathcal{M}_\delta}(\mathcal{M})$ nontrivial solutions of problem (3.1). If $u \in H_\varepsilon$ is one of these solutions, we have that $u \in \tilde{\mathcal{N}}_\varepsilon$, and we can use (6.10) and the definition of $g$ to conclude that $H(\cdot, u) = G(u)$. Hence, $u$ is also a solution of the problem (2.1). An easy calculation shows that $\omega(x) = u(\frac{x}{\varepsilon})$ is a solution of the original problem (1.8). Then, (1.8) has at least $cat_{\mathcal{M}_\delta}(\mathcal{M})$ positive solutions.

Now we consider $\varepsilon_n \to 0^+$ and take a sequence $u_n \in H_{\varepsilon_n}$ of positive solutions of the problem (3.1) as above. In order to study the behavior of the maximum points of $u_n$, we first notice that, by the definition of $H$ and $(h_1)$, $(h_2)$, there exists $0 < \gamma < a$ such that

$$H(\varepsilon_n x, u)u \leq \frac{V_0}{K} u^2, \text{ for all } x \in \mathbb{R}^3,\ u \leq \gamma. \tag{6.12}$$

Using a similar discussion above, we obtain $R > 0$ and $\{\tilde{y}_n\} \subset \mathbb{R}^3$ such that

$$\|u_n\|_{L^\infty(B_R^c(\tilde{y}_n))} < \gamma. \tag{6.13}$$

Up to a subsequence, we may assume that

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n))} \geq \gamma. \tag{6.14}$$

Indeed, if this is not the case, we have $\|u_n\|_{L^\infty} < \gamma$, and therefore it follows from $J'_{\varepsilon_n}(u_n) = 0$ and (6.12) that

$$\|u_n\|_{\varepsilon_n}^2 \leq \frac{V_0}{K} \int_{\mathbb{R}^3} u_n^2 dx. \tag{6.15}$$

The above expression implies that $\|u_n\|_{\varepsilon_n} \to 0$ as $n \to \infty$, which leads to a contradiction. Thus, (6.14) holds.

By using (6.13) and (6.14) we conclude that the maximum points $p_n \in \mathbb{R}^3$ of $u_n$ belongs to $B_R(\tilde{y}_n)$. Hence, $p_n = \tilde{y}_n + q_n$ for some $q_n \in B_R(0)$. Recalling that the associated solution of (1.8) is of the form $\omega_n(x) = u_n(\frac{x}{\varepsilon})$, we conclude that the maximum point $\eta_\varepsilon$ of $v_n$ is $\eta_\varepsilon := \varepsilon_n \tilde{y}_n + \varepsilon_n q_n$. Since $\{q_n\} \subset B_R(0)$ is bounded and $\varepsilon_n \tilde{y}_n \to y_0 \in \mathcal{M}$, we obtain

$$\lim_{n\to\infty} V(\eta_\varepsilon) = V(y_0) = V_0.$$

Thus, the proof of Theorem 1.1 is completed.

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