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Fujita exponent and nonexistence result for the Rockland heat equation

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A R T I C L E I N F O

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This article presents nonexistence results for semilinear parabolic equation with hypoelliptic operator. In particular, we show Fujita exponent for the Rockland heat equation on graded Lie group, which depends on the homogeneous dimension of group and the order of the Rockland operator.

1. Introduction and preliminaries

In 1966, Fujita [1] started to study the following Cauchy problem,

$$u_t(t,x) - \Delta u(t,x) = |u(t,x)|^p, \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N, \tag{1.1}$$

for $N > 1$ and $p > 1$, with the Cauchy data

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^N.$$

He proved that there is a critical exponent (named Fujita exponent today)

$$p^*(N) \doteq 1 + \frac{2}{N}$$

such that problem (1.1) possess a global existence result for $p > p^*(N)$ and the nonexistence of global in time solutions under certain assumptions on the initial data for $1 < p < p^*(N)$. Later, Hayakawa [2] and Sugitani [3] proved that in the critical case $p = p^*(N)$ it holds a blow-up result as well. Recently, the case of the sub-Laplacian on the Heisenberg groups and stratified Lie groups was considered in [4–6]. Then in this paper, we want to study this kind of problem on the more general graded Lie group.

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A homogeneous Lie group is a nilpotent Lie group with a dilation action of $\mathbb{R}_{>0}$ by group automorphisms. The dilation action allows to scale with different speed in different tangent directions. A slightly less general class is graded nilpotent Lie group. We firstly recall that $\mathbb{G}$ is a graded Lie group if its Lie algebra $\mathfrak{g}$ admits a gradation

$$\mathfrak{g}=\bigoplus_{l=1}^{\infty}\mathfrak{g}_l,$$

where the $\mathfrak{g}_l, l=1,2,\ldots$, are vector subspaces of $\mathfrak{g}$, all but finitely many equal to $\{0\}$, and satisfy

$$[\mathfrak{g}_l,\mathfrak{g}_{l'}]\subset\mathfrak{g}_{l+l'},\quad \forall l,l'\in\mathbb{N}.$$

We fix a basis $\{X_1,\ldots,X_n\}$ of a Lie algebra $\mathfrak{g}$ adapted to the gradation. By the exponential mapping $\exp_{\mathbb{G}}:\mathfrak{g}\to\mathbb{G}$ we get points in $\mathbb{G}$:

$$x=\exp_{\mathbb{G}}\left(x_1X_1+\cdots+x_nX_n\right).$$

A family of linear mappings of the form

$$D_r=\exp(A\ln r)=\sum_{k=0}^{\infty}\frac{1}{k!}(\ln(r)A)^k$$

is a family of dilations of $\mathfrak{g}$. Here A is a diagonalizable linear operator on $\mathfrak{g}$ with positive eigenvalues. Every D_r is a morphism of the Lie algebra $\mathfrak{g}$, i.e., D_r is a linear mapping from $\mathfrak{g}$ to itself with the property

$$\forall X,Y\in\mathfrak{g},r>0,[D_rX,D_rY]=D_r[X,Y]$$

as usual $[X,Y]:=XY-YX$ is the Lie bracket. One can extend these dilations through the exponential mapping to the group $\mathbb{G}$ by

$$D_r(x)=rx:=\left(r^{\nu_1}x_1,\ldots,r^{\nu_n}x_n\right),x=(x_1,\ldots,x_n)\in\mathbb{G},r>0,$$

where $\nu_1,\ldots,\nu_n$ are weights of the dilations. The sum of these weights

$$Q:=\operatorname{Tr}A=\nu_1+\cdots+\nu_n$$

is called the homogeneous dimension of $\mathbb{G}$.

We also recall that the standard Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}^n$ is the Haar measure for $\mathbb{G}$. Also, we can define a homogeneous quasi-norm on a homogeneous group $\mathbb{G}$ to be a continuous function $x\mapsto|x|$ from $\mathbb{G}$ to $[0,\infty)$ that satisfies for all $x\in\mathbb{G}$ and $r>0$,

(a) (symmetric) $\left|x^{-1}\right|=|x|$,
(b) (homogeneous) $|rx|=r|x|$,
(c) (non-degeneracy) $|x|=0$ if and only if $x=0$.

Here and elsewhere we denote by $rx=D_rx$ the dilation of x induced by the dilations on the Lie algebra through the exponential mapping. Recalling the homogeneous dimension of $\mathbb{G}$, we have

$$|D_r(E)|=r^Q|E|,\quad d(rx)=r^Qdx.$$

Remark 1.1. The Heisenberg group $\mathbb{H}^n$ is a graded Lie group with dilations

$$D_r\left(z_1,\ldots,z_n,t\right)=\left(rz_1,\ldots,rz_n,r^2t\right),$$

and its Lie algebra $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g}=V_1\oplus V_2\quad\text{where}\quad V_1=\oplus_{i=1}^n\mathbb{R}X_i\oplus\mathbb{R}Y_i\quad\text{and}\quad V_2=\mathbb{R}T.$$

Definition 1.1. A Rockland operator on $\mathbb{G}$ is a left-invariant differential operator $\mathcal{R}$ which is homogeneous of positive degree and satisfies the Rockland condition:

(R) for each unitary irreducible representation π on $\mathbb{G}$, except for the trivial representation, the operator $\pi(\mathcal{R}) := d\pi(\mathcal{R})$ (infinitesimal representation) is injective on $\mathcal{H}_\pi^\infty$, that is,

$$\forall v \in \mathcal{H}_\pi^\infty \quad \pi(\mathcal{R})v = 0 \Longrightarrow v = 0,$$

where $\mathcal{H}_\pi^\infty$ is space of all smooth vectors of π which means $\pi(x)v \in \mathcal{H}_\pi$ is of class C^∞.

Then for any graded Lie group $\mathbb{G}$ with dilation weights $\nu_1, \ldots, \nu_n$ as above. If ν_0 is any common multiple of $\nu_1, \ldots, \nu_n$, the operator

$$\mathcal{R}_x = \sum_{j=1}^n (-1)^{\frac{v_0}{v_j}} a_j X_j^{2\frac{v_0}{v_j}}, \quad a_j > 0$$

is a Rockland operator of homogeneous degree $2v_0 \doteq \alpha$. We will mainly consider this kind of Rockland operator in this paper. For more information about Rockland operators and graded Lie groups, the reader is referred to the book of Fischer and Ruzhansky [7].

Remark 1.2. The Rockland operator we consider here is not an elliptic operator, it is hypoellipticity in the sense: if $\mathcal{R}_x u$ is smooth on every open subset, then u is smooth in the sense of distribution, which can be seen as the weaker versions of ellipticity.

2. Main results and proofs

We start by studying the Cauchy problem for the nonlinear Rockland heat equation

$$\begin{cases} u_t(t,x) + \mathcal{R}_x u(t,x) = |u(t,x)|^p, & (t,x) \in (0,+\infty) \times \mathbb{G} := \Omega, \\ u(0,x) = u_0(x), & x \in \mathbb{G}. \end{cases} \tag{2.1}$$

We firstly give the definition of weak solutions and some useful results.

Definition 2.1. A locally integrable function $u \in L^p_{\mathrm{loc}}(\Omega_T)$ $(\Omega_T = (0,T) \times \mathbb{G})$ is called a local weak solution of (2.1) in Ω_T subject to the initial data $u_0 \in L^1_{\mathrm{loc}}(\mathbb{G})$ if the equality

$$\begin{aligned} &\int_{\Omega_T} -u(t,x)\varphi_t(t,x)dxdt + \int_{\Omega_T} u(t,x)\mathcal{R}_x\varphi(t,x)dxdt \\ &= \int_{\mathbb{G}} u_0(x)\varphi(0,x)dx + \int_{\Omega_T} |u(t,x)|^p\varphi(t,x)dxdt \end{aligned} \tag{2.2}$$

is satisfied for any regular function

$$\varphi \in C^1\left((0,T]; L^2(\mathbb{G})\right) \cap C\left([0,T]; H^\alpha(\mathbb{G})\right),$$

where $H^\alpha(\mathbb{G})$ stands for the homogeneous Sobolev space related to the Rockland operator $\mathcal{R}_x$, $\varphi(x,T) = 0, \varphi \geq 0$. The solution is called global if $T = +\infty$.

Lemma 2.1. *Let $f \in L^1(\mathbb{G})$ and $\int_{\mathbb{G}} f dx > 0$. Then there exists a test function $0 \leq \varphi \leq 1$ such that*

$$\int_{\mathbb{G}} f\varphi dx > 0.$$

Proof. Indeed, we have

$$\int_{\mathbb{G}} f\varphi dx = \int_{|x|\leq R} f\varphi dx + \int_{R\leq |x|} f\varphi dx.$$

Take a smooth function $\varphi = \varphi_R(x), 0 \leq \varphi_R \leq 1$, such that $\varphi_R(x) \equiv 1$ for $|x| \leq R$. Then

$$\int_{\mathbb{G}} f\varphi_R dx = \int_{|x|\leq R} f dx + \int_{R\leq |x|} f\varphi_R dx. \tag{2.3}$$

By virtue of the convergence of the integral $\int_{\mathbb{G}} |f(x)|dx$, we have

$$\left|\int_{R\leq |x|} f\varphi_R dx\right| \leq \int_{R\leq |x|} |f|dx \to 0 \quad \text{as} \quad R\to\infty.$$

Then, after the passage to the limit as $R\to\infty$ in relation (2.3), we obtain

$$\lim_{R\to\infty}\int_{\mathbb{G}} f\varphi_R dx = \lim_{R\to\infty}\int_{|x|\leq R} f dx = \int_{\mathbb{G}} f dx > 0$$

This implies the assertion. ■

Theorem 2.1. *Let the homogeneous dimension of the graded Lie group $\mathbb{G}$ be $Q \geq 2$. Suppose that $1 < p < p^*(Q,\alpha) \doteq 1 + \frac{\alpha}{Q}$ and $\int_{\mathbb{G}} u_0(x)\mathrm{d}x \geq 0$. Then, the Cauchy problem (2.1) does not have a nontrivial global weak solution.*

Proof. The proof is by contradiction. For that, let u be a global weak solution and φ be a smooth nonnegative test function such that

$$\mathcal{E}(\varphi) = \int_{\Omega_T}\left(\left|\frac{\partial\varphi(t,x)}{\partial t}\right|^{\frac{p}{p-1}} \varphi(t,x)^{\frac{-1}{p-1}} + |\mathcal{R}_x\varphi(t,x)|^{\frac{p}{p-1}}\varphi(t,x)^{\frac{-1}{p-1}}\right) dxdt < \infty.$$

Then, from Definition 2.1 and Lemma 2.1, we have

$$\begin{aligned}\int_{\Omega_T} |u(t,x)|^p\varphi(t,x)dxdt &\leq \int_{\Omega_T} |u(t,x)|^p\varphi(t,x)dxdt + \int_{\mathbb{G}} u_0(x)\varphi(0,x)dx \\ &= -\int_{\Omega_T} u(t,x)\frac{\partial\varphi(t,x)}{\partial t}dxdt + \int_{\Omega_T} u(t,x)\mathcal{R}_x\varphi(t,x)dxdt.\end{aligned} \tag{2.4}$$

By using ε-Young's inequality

$$ab \leq \varepsilon a^r + C(\varepsilon)b^{r'}, \quad \frac{1}{r}+\frac{1}{r'} = 1, a, b\geq 0,$$

we obtain

$$\begin{aligned}&\int_{\Omega_T} |u(t,x)|^p\varphi(t,x)dxdt \\ &\leq -\int_{\Omega_T} u(t,x)\frac{\partial\varphi(t,x)}{\partial t}dxdt + \int_{\Omega_T} u(t,x)\mathcal{R}_x\varphi(t,x)dxdt \\ &\leq \int_{\Omega_T} |u(t,x)|\varphi(t,x)^{\frac{1}{p}}\frac{\partial\varphi(t,x)}{\partial t}\varphi(t,x)^{-\frac{1}{p}}dxdt + \int_{\Omega_T} |u(t,x)|\varphi(t,x)^{\frac{1}{p}}\mathcal{R}_x\varphi(t,x)\varphi(t,x)^{-\frac{1}{p}}dxdt \\ &\leq \frac{1}{2}\int_{\Omega_T} |u(t,x)|^p\varphi(t,x)dxdt + C\left(\int_{\Omega_T}\left|\frac{\partial\varphi(t,x)}{\partial t}\right|^{\frac{p}{p-1}}\varphi(t,x)^{\frac{-1}{p-1}}dxdt\right. \\ &\quad \left. + \int_{\Omega_T}|\mathcal{R}_x\varphi(t,x)|^{\frac{p}{p-1}}\varphi(t,x)^{\frac{-1}{p-1}}dxdt\right).\end{aligned}$$

So, we obtain

$$\int_{\Omega_T} |u(t,x)|^p \varphi(t,x)dxdt \leq C\mathcal{E}(\varphi)$$

for some constant $C > 0$.

Set

$$\varphi(t,x) = \varphi_1(x)\varphi_2(t) = \Phi\left(\frac{|x|}{R}\right)\Phi\left(\frac{t}{R^\alpha}\right),$$

where $R > 0$, and $\Phi : \mathbb{R}_+ \to [0,1]$ is the smooth cut-off function such that

$$\Phi(r) = \begin{cases} 1, & 0 \leq r \leq 1 \\ \searrow, & 1 \leq r \leq 2 \\ 0, & r \geq 2 \end{cases}$$

By changing variables $x = R\tilde{x}$ and $t = R^\alpha \tilde{t}$ we obtain the estimates

$$\int_{\Omega_T} \left|\frac{\partial \varphi(t,x)}{\partial t}\right|^{\frac{p}{p-1}} \varphi(t,x)^{\frac{-1}{p-1}} dxdt \leq CR^{-\frac{\alpha p}{p-1}+Q+\alpha}$$

and

$$\int_{\Omega_T} |\mathcal{R}_x \varphi(t,x)|^{\frac{p}{p-1}} \varphi(t,x)^{\frac{-1}{p-1}} dxdt \leq CR^{-\frac{\alpha p}{p-1}+Q+\alpha}$$

The constraint $1 < p < 1 + \frac{\alpha}{Q}$ allows us to obtain the contradiction

$$\int_{\Omega_T} |u|^p dxdt = \lim_{R\to\infty} \int_{\Omega_T} |u|^p \varphi dxdt = 0 \Rightarrow u \equiv 0. \quad \blacksquare$$

Corollary 2.1. *In the Abelian case* $(\mathbb{R}^n, +)$ *with* $Q = n, \mathcal{R}_x = -\Delta$, *and by taking Euclidean distance instead of the quasi-norm, we claim the well-known results by Fujita [1].*

Corollary 2.2. *If* $\mathbb{G}$ *is a stratified Lie group with* $\mathcal{R}_x = -\Delta_{\mathbb{G}} = -\sum_1^n X_i^2$, *where* $\Delta_{\mathbb{G}}$ *is a sub-Laplacian* $(i.e., \nu_0 = \nu_1 = \cdots = \nu_n,)$ *then we obtain Fujita exponent for the heat equation on stratified Lie groups similar to [5]. Moreover, the particular case of the stratified Lie groups is the Heisenberg group which has been obtained by [4] and [6].*

Remark 2.1. We only consider the subcritical Fujita exponent in this paper by the test function method. For the critical case and blow-up phenomenon, we need more information on the heat kernel of the Rockland operator, which we will consider in the later paper [8].

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