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POSITIVE EIGENFUNCTIONS OF A CLASS OF FRACTIONAL SCHRÖDINGER OPERATOR WITH A POTENTIAL WELL

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Abstract. In this paper, we are concerned with the following eigenvalue problem

$$(-\Delta)^s u + \lambda g(x)u = \alpha u, \quad u \in H^s(\mathbb{R}^N),\ N \geq 3,$$

where $s \in (0,1)$, $\alpha, \lambda \in \mathbb{R}$ and

$$g(x) \equiv 0 \text{ on } \bar{\Omega}, \quad g(x) \in (0,1] \text{ on } \mathbb{R}^N \backslash \bar{\Omega} \text{ and } \lim_{|x|\to\infty} g(x) = 1$$

for some bounded open set $\Omega \subset \mathbb{R}^N$. We discuss the existence and some properties of the first two eigenvalues for this problem, which extend some classical results for semilinear Schrödinger equations to the nonlocal fractional setting.

1. Introduction and the main results

In this paper, we discuss the eigenvalue problem

$$(-\Delta)^s u + \lambda g(x)u = \alpha u, \quad u \in H^s(\mathbb{R}^N), \quad N \geq 3, \tag{1.1}$$

where $(-\Delta)^s$ denotes the fractional Laplacian operator of order $s \in (0,1)$, λ, α are real numbers, and the function g satisfies the condition:

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(G) $g \in L^\infty(\mathbb{R}^N, \mathbb{R})$ and there exists a nonempty bounded open set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary such that

$$g(x) \equiv 0 \text{ on } \bar{\Omega}, \quad g(x) \in (0,1] \text{ on } \mathbb{R}^N \setminus \bar{\Omega} \text{ and } \lim_{|x| \to \infty} g(x) = 1.$$

By the condition (G), the potential functional $g(x)$ represents a potential well with the bottom Ω and whose depth is controlled by the parameter λ.

A basic motivation for the study of Eq. (1.1) arises in looking for the standing wave solutions of the type

$$\Psi(x,t) = e^{-iEt/\varepsilon} u(x)$$

for the following time-dependent fractional Schrödinger equation

$$i\varepsilon \frac{\partial \Psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s \Psi + (V(x) + E)\Psi - f(x, \Psi) \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{1.2}$$

Eq. (1.2) was introduced by Laskin [22, 23], which describes how the wave function of a physical system evolves over time. In the last few years, the study of elliptic equation involving fractional Laplacian operator appears widely in both pure mathematical research and concrete applications, such as the thin obstacle problem [7, 30], minimal surfaces [6, 9], phase transitions [31], anomalous diffusion [4, 28, 34] and mathematical finance [12]. See [13, 14, 3] and references therein for an elementary introduction to the literature. Recently, problem (1.1) and problems similar as (1.1) have captured a lot of interest, especially on the existence and nonexistence of positive solutions, multiple solutions, ground states and regularity, see for example, [5, 8, 11, 16, 17, 20, 36, 37, 38, 39] and the references therein.

For a bounded domain $\Omega \subset \mathbb{R}^N$, R. Servadei and E. Valdinoci in [29] studied the eigenvalue problem

$$\begin{cases} \mathcal{L}_K u = \xi u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3}$$

where $\mathcal{L}_K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is defined as

$$\mathcal{L}_K u(x) = -\frac{1}{2} \int_{\mathbb{R}^N} \big(u(x+y) + u(x-y) - 2u(x)\big) K(y) dy \quad \text{for all } x \in \mathbb{R}^N,$$

which satisfies the following conditions:

(i) $mK \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^2, 1\}$,

(ii) There exist $\lambda > 0$ and $s \in (0,1)$ such that $K(x) \geq \lambda |x|^{-(N+2s)}$,

(iii) $K(x) = K(-x)$ for any $x \in \mathbb{R}^N \setminus \{0\}$.

In case $K(x) = |x|^{-(N+2s)}$, (1.3) transform into

$$\begin{cases} (-\Delta)^s u = \xi u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.4}$$

The authors prove that problem (1.3) or (1.4) has a sequence of eigenvalues $\{\xi_k\}$ satisfy

$$0 < \xi_1 < \xi_2 \leq \cdots \leq \xi_k \leq \cdots \text{ and } \xi_k \to \infty \text{ as } k \to \infty.$$

Moreover, for each $k \in \mathbb{N}$, there exists an eigenfunction φ_k corresponding to the eigenvalue ξ_k such that

$$\int_\Omega \varphi_k^2 dx = 1.$$

In particular, the first eigenvalue ξ_1 is simple, isolated and there is a unique eigenfunction satisfying the conditions

$$\int_\Omega \varphi_1^2 dx = 1 \quad \text{and} \quad \varphi_1 > 0 \text{ on } \Omega.$$

When the domain Ω is replaced by $\mathbb{R}^N$ and there exists a potential function, one of the main difficulty, as opposed to their study on bounded domains, is the fact that the spectrum contains points which are not eigenvalues. C.A. Stuart and H. Zhou in [33] discusses the problem

$$-\Delta u + \lambda g(x) u = \alpha u, \quad u \in H^1(\mathbb{R}^N), \quad u > 0. \tag{1.5}$$

Let the condition (G) be satisfied. They proved that the principal eigenvalue $\lambda = \Lambda(\alpha)$ always exists for any $\alpha \in (\Gamma, \mu_1)$ with

$$\Gamma = \inf \Big\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : \; u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (1-g) u^2 dx = 1 \Big\},$$

where μ_1 is the first eigenvalue of (1.4) with $s = 1$ and the $\Lambda(\alpha)$-eigenfunction φ_Λ is the only eigenfunction which does change sign. Later, X. Liu and Y. Huang in [26] discuss the existence and properties of the second eigenvalue for (1.5), and they prove the corresponding eigenfunctions change sign. All the conclusions in [26, 33] are useful result to study the asymptotically linear Schrödinger equation, see [1, 24, 25, 27, 35] for example.

Our aim in this paper is to show that the results of [26, 33] can be extended to problem (1.1). In order to do this, as in the classical case, we will use a variational technique for spectral analysis. We first set

$$\Gamma_1 = \inf \Big\{ \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx : \; u \in H^s(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (1-g) u^2 dx = 1 \Big\}.$$

In the first part of this paper, we follows [33] describing the eigenvalue λ as a function of the parameter α rather than the eigenvalue α as a function of the parameter λ in the traditional treatment. Which can help us yield some no-trivial conclusions. We begin by establishing the following result concerning the quantity Γ_1.

Theorem 1.1. *Let the condition* (G) *be satisfied.*

(*i*) *If* $\alpha \geq \xi_1$, *then there is no eigenvalue of* (1.1) *in* $[\alpha, +\infty)$ *with a non-negative eigenfunction.*

(*ii*) *If* $\Gamma_1 < \alpha < \xi_1$, *then there exists a unique eigenvalue* $\lambda = \Lambda(\alpha)$ *of* (1.1) *having a positive eigenfunction. Furthermore,* $\Lambda(\alpha) > \alpha$, *and it is simple in the sense that*

$$\ker((-\Delta)^s - \alpha + \Lambda(\alpha)g) = span\{u_{\Lambda(\alpha)}\} := V_1,$$

where $u_{\Lambda(\alpha)} > 0$ *on* $\mathbb{R}^N$. *All other eigenvalues of* (1.1) *are less than* $\Lambda(\alpha)$, *and their eigenfunctions change sign.*

(*iii*) *The function* $\Lambda \in C^\infty((\Gamma_1, \xi_1))$ *and is strictly increasing with*

$$\lim_{\alpha \to \Gamma_1^+} \Lambda(\alpha) = \Gamma_1 \text{ and } \lim_{\alpha \to \xi_1^-} \Lambda(\alpha) = +\infty.$$

(*iv*) *For* $\Gamma_1 < \alpha < \xi_1$, $\Lambda(\alpha)$ *is characterized as the unique value of* λ *for which* $\Sigma(\lambda) = 0$, *where*

$$\Sigma(\lambda) = \inf \left\{ a_\lambda(u) : u \in H^s(\mathbb{R}^N) \text{ and } |u|_2 = 1 \right\}$$

and

$$a_\lambda(u) = \int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} u|^2 - \alpha u^2 + \lambda g u^2 \right) dx.$$

(*v*) *If* $\alpha \leq \Gamma_1$, *then problem* (1.1) *has no eigenvalues* λ *in the interval* (α, ∞).

In the following, taking the alternative view, from Theorem 1.1, let λ be fixed, for the fractional Schrödinger operator

$$L_\lambda := (-\Delta)^s + \lambda g(x),$$

we can define

$$\alpha_1(\lambda) := \inf \left\{ \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda g(x) u) dx : u \in H^s(\mathbb{R}^N) \text{ and } |u|_2 = 1 \right\}.$$

Next, we define

$$\Gamma_2 = \inf_{u \in H^s(\mathbb{R}^N) \cap V_1^\perp} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{\int_{\mathbb{R}^N} (1-g) u^2 dx}.$$

Clearly, $\Gamma_2 \geq \Gamma_1$. Let X_1 denote the set of all closed subspaces of $H^s(\mathbb{R}^N)$ with codimension 1. Fixing $\lambda \in (\Gamma_2, \infty)$, we define

$$\alpha_2(\lambda) = \inf\Big\{ \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda g(x) u) dx : u \in H^s(\mathbb{R}^N) \cap V_1^{\perp} \text{ and } |u|_2 = 1 \Big\}$$

and

$$\tilde{\alpha}_2(\lambda) = \sup_{V \in X_1} \inf_{u \in V} \Big\{ \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda g(x) u) dx : |u|_2 = 1 \Big\}.$$

Let W_2 denote the set of all closed subspaces of $H^s(\mathbb{R}^N)$ with dimension 2. Define

$$\hat{\alpha}_2(\lambda) = \sup_{\hat{V} \in W_2} \inf_{u \in \hat{V}} \Big\{ \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda g(x) u) dx : |u|_2 = 1 \Big\}.$$

Now, we state the second result in this paper as following:

Theorem 1.2. *Let the condition (G) be satisfied. If $\lambda > \Gamma_2$, then the following assertions holds:*

- *(i)* $\alpha_2(\lambda)$ *is the second eigenvalue of the fractional Schrödinger operator* $L_\lambda := (-\Delta)^s + \lambda g(x)$, *and corresponding eigenfunctions change sign.*
- *(ii)* $\alpha_2(\lambda) = \tilde{\alpha}_2(\lambda) = \hat{\alpha}_2(\lambda)$.
- *(iii)* $\alpha_2(\lambda)$ *is strictly increasing with* λ.
- *(iv)* $\lim_{\lambda \to \infty} \alpha_2(\lambda) = \xi_2$.

Throughout this paper, we denote $|\cdot|_p$ the usual norm of the space $L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, $B_r(x)$ denotes the open ball with center at x and radius r, C or $C_i (i = 1, 2, \cdots)$ denote some positive constants may change from line to line. $a_n \rightharpoonup a$ and $a_n \to a$ mean the weak and strong convergence, respectively, as $n \to \infty$.

2. Variational settings and preliminary results

First, fractional Sobolev spaces are the convenient setting for our problem, so we will give some introduction of the fractional order Sobolev spaces and the complete introduction can be found in [13]. We recall that, for any $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$ is defined as follows:

$$H^s(\mathbb{R}^N) = \Big\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\xi|^{2s}| + 1) |\mathcal{F}(u)|^2 d\xi < \infty \Big\},$$

whose norm is defined as

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\xi|^{2s}| + 1) |\mathcal{F}(u)|^2 d\xi,$$

where $\mathcal{F}$ denotes the Fourier transform. We also define the homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as the completion of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$[u] := \Big(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \Big)^{\frac{1}{2}}.$$

The embedding $\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ is continuous, where $2_s^* = \frac{2N}{N-2s}$ is the fractional critical exponent, then there exists a best constant $S_s > 0$ such that

$$S_s := \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^N)} \frac{[u]^2}{|u|_{2_s^*}^2}.$$

The fractional Laplacian operator, $(-\Delta)^s u$, of a smooth function $u : \mathbb{R}^N \to \mathbb{R}$, is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^N,$$

that is

$$\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-i\xi\cdot x}\phi(x)dx,$$

for functions ϕ in the Schwartz class. Also $(-\Delta)^s u$ can be equivalently represented [13] as

$$(-\Delta)^s u(x) = -\frac{1}{2}C_{N,s}\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}dy, \ \forall x \in \mathbb{R}^N,$$

where

$$C_{N,s} = \Big(\int_{\mathbb{R}^N} \frac{1-cos\xi_1}{|\xi|^{N+2s}} d\xi \Big)^{-1}, \ \xi \in \mathbb{R}^N.$$

Also, by the Plancherel formula in Fourier analysis, we have

$$[u]^2 = \frac{2}{C_{N,s}} |(-\Delta)^{\frac{s}{2}} u|_2^2.$$

As a consequence, the norms on $H^\alpha(\mathbb{R}^3)$ defined above

$$u \longmapsto \Big(\int_{\mathbb{R}^N} |u|^2 dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} dxdy \Big)^{\frac{1}{2}};$$

$$u \longmapsto \Big(\int_{\mathbb{R}^N} (|\xi|^{2\alpha}+1)|\mathcal{F}(u)|^2 d\xi \Big)^{\frac{1}{2}};$$

$$u \longmapsto \Big(\int_{\mathbb{R}^N} |u|^2 dx + |(-\Delta)^{\frac{\alpha}{2}} u|_2^2 \Big)^{\frac{1}{2}}$$

are equivalent.

In view of the presence of potential $g(x)$, we introduce the subspace

$$E = \Big\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} \lambda g(x) u^2 dx < +\infty \Big\},$$

which is a Hilbert space equipped with the inner product

$$(u,v)_E = \int_{\mathbb{R}^N} \big((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + \lambda g(x) uv \big) dx,$$

and the norm

$$\|u\|_E = \Big(\int_{\mathbb{R}^N} \big(|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda g(x) u^2 \big) dx \Big)^{\frac{1}{2}}.$$

For the reader's convenience, we review the main embedding result for this class of fractional Sobolev spaces :

Lemma 2.1. [13] *Let $0 < s < 1$, then there exists a constant $C = C_{N,s} > 0$, such that*

$$|u|_{2_s^*}^2 \leq C[u]^2$$

for every $u \in H^s(\mathbb{R}^N)$. Moreover, the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for any $r \in [2, 2_s^*)$ and is locally compact whenever $r \in [2, 2_s^*)$.

Lemma 2.2. *Under assumption (G), for fixed $\lambda \in (0, \infty)$, the norm*

$$\|u\|_E = \Big(\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda g(x) u^2) dx \Big)^{\frac{1}{2}}$$

is equivalent to the form

$$\|u\| = \Big(\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + u^2) dx \Big)^{\frac{1}{2}}.$$

Proof. By the condition (G), there exists $R > 0$ such that $\overline{\Omega} \subset B_R(0)$ and $g(x) \geq \frac{1}{2}$ for almost all $x \in \mathbb{R}^N \backslash B_R(0)$, then

$$\int_{\mathbb{R}^N \backslash \Omega} g(x) u^2 dx \geq \int_{\mathbb{R}^N \backslash B_R(0)} g(x) u^2 dx \geq \frac{1}{2} \int_{\mathbb{R}^N \backslash B_R(0)} u^2 dx. \tag{2.1}$$

By Lemma 2.1, we have

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^N \backslash \Omega} |(-\Delta)^{\frac{s}{2}} u|^2 dx &\geq \frac{1}{2} \int_{(\mathbb{R}^N \backslash \Omega) \cap B_R(0)} |(-\Delta)^{\frac{s}{2}} u|^2 dx \\ &\geq \frac{1}{2c_1} \int_{(\mathbb{R}^N \backslash \Omega) \cap B_R(0)} u^2 dx. \end{aligned} \tag{2.2}$$

By (2.1) and (2.2), we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega}|(-\Delta)^{\frac{s}{2}}u|^2dx+\int_{\mathbb{R}^N\setminus\Omega}\lambda g(x)u^2dx \qquad (2.3)\\
&\geq \frac{1}{2c_1}\int_{(\mathbb{R}^N\setminus\Omega)\cap B_R(0)}u^2dx+\frac{1}{2}\lambda\int_{\mathbb{R}^N\setminus B_R(0)}u^2dx\\
&\geq \min\{\frac{1}{2c_1},\frac{1}{2}\lambda\}\Big(\int_{(\mathbb{R}^N\setminus\Omega)\cap B_R(0)}u^2dx+\int_{\mathbb{R}^N\setminus B_R(0)}u^2dx\Big)\\
&=\min\{\frac{1}{2c_1},\frac{1}{2}\lambda\}\int_{\mathbb{R}^N\setminus\Omega}u^2dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_{\mathbb{R}^N\setminus\Omega}\big(|(-\Delta)^{\frac{s}{2}}u|^2+\lambda g(x)u^2\big)dx \qquad (2.4)\\
&\geq \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega}|(-\Delta)^{\frac{s}{2}}u|^2dx+\min\{\frac{1}{2c_1},\frac{1}{2}\lambda\}\int_{\mathbb{R}^N\setminus\Omega}u^2dx\\
&\geq c_2\int_{\mathbb{R}^N\setminus\Omega}(|(-\Delta)^{\frac{s}{2}}u|^2+u^2)dx,
\end{aligned}
$$

where

$$
c_2=\min\{\frac{1}{2},\min\{\frac{1}{2c_1},\frac{1}{2}\lambda\}\}.
$$

Also by the condition (G), we have

$$
\begin{aligned}
\int_{\mathbb{R}^N\setminus\Omega}(|(-\Delta)^{\frac{s}{2}}u|^2+\lambda g(x)u^2)dx&\leq\int_{\mathbb{R}^N\setminus\Omega}(|(-\Delta)^{\frac{s}{2}}u|^2dx+\lambda u^2)dx \qquad (2.5)\\
&\leq\max\{1,\lambda\}\int_{\mathbb{R}^N\setminus\Omega}(|(-\Delta)^{\frac{s}{2}}u|^2dx+u^2)dx.
\end{aligned}
$$

By Lemma 2.1, we get

$$
\int_\Omega u^2dx\leq c_3\int_\Omega|(-\Delta)^{\frac{s}{2}}u|^2dx,
$$

then

$$
\begin{aligned}
\int_\Omega|(-\Delta)^{\frac{s}{2}}u|^2dx&\geq\frac{1}{2}\int_\Omega|(-\Delta)^{\frac{s}{2}}u|^2dx+\frac{1}{2c_3}\int_\Omega u^2dx\\
&\geq\min\{\frac{1}{2},\frac{1}{2c_3}\}\int_\Omega(|(-\Delta)^{\frac{s}{2}}u|^2dx+u^2)dx.
\end{aligned}
$$

Since $g(x) = 0$ for $x \in \Omega$, we have

$$\int_\Omega (|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda g(x) u^2) dx = \int_\Omega (|(-\Delta)^{\frac{s}{2}} u|^2 dx \qquad (2.6)$$
$$\geq \min\{\frac{1}{2}, \frac{1}{2c_3}\} \int_\Omega (|(-\Delta)^{\frac{s}{2}} u|^2 dx + u^2) dx.$$

and

$$\int_\Omega (|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda g(x) u^2 dx = \int_\Omega (|(-\Delta)^{\frac{s}{2}} u|^2 dx \qquad (2.7)$$
$$\leq \int_\Omega (|(-\Delta)^{\frac{s}{2}} u|^2 dx + u^2) dx.$$

Hence, by (2.4)-(2.7), we know that $\|\cdot\|_E$ is equivalent to the norm $\|\cdot\|$. □

Next, we deal with the fractional Schrödinger operators

$$H = (-\Delta)^s + V$$

acting on $L^2(\mathbb{R}^N)$ with $V = \lambda g(x)$. A suitable class of potentials V for the fractional Schrödinger operators discussed here is the following Kato class (denoted by $K_s(\mathbb{R}^N)$).

Definition 2.1. *Let $0 < s < 1$. We say that the potential $V \in K_s(\mathbb{R}^N)$ if and only if $V : \mathbb{R}^N \to \mathbb{R}$ is measurable and satisfies*

$$\lim_{E\to 0} |((-\Delta)^s + E)^{-1}|V||_{L^\infty \to L^\infty} = 0,$$

where $E > 0$ is a positive number.

Remark 2.1. If $V \in K_s$, then $H = (-\Delta)^s + V$ defines a unique self-adjoint operator on $L^2(\mathbb{R}^N)$ with form domain $H^{2s}(\mathbb{R}^N)$, and the corresponding heat kernel e^{-tH} maps $L^2(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N) \cap C^0(\mathbb{R}^N)$ for any $t > 0$. In particular, any L^2-eigenfunction of H is continuous and bounded. See also [10, 18, 19] for equivalent definitions of K_s and further background material.

Lemma 2.3. [18, 19] *Let $0 < s < 1$ and $V : \mathbb{R}^N \to \mathbb{R}$ be given. If $V \in L^p(\mathbb{R}^N)$ with some $\max\{1, \frac{N}{2s}\} < p \leq +\infty$, then $V \in K_s(\mathbb{R}^N)$.*

Definition 2.2. *Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space. Let $L : D(L) \subset H \to H$ be a self-adjoint operator whose form domain is dense subspace of H. Its resolvent set is*

$$\rho(L) = \Big\{\lambda \in \mathbb{R} : L - \lambda I : D(L) \to H \text{ is an isomorphism}\Big\}$$

and its spectrum is the set

$$\sigma(L) = \mathbb{R} \backslash \rho(L).$$

The elements of $\rho(L)$ *are called regular values for* $L : D(L) \subset H \to H$. *The following subsets of* $\sigma(L)$ *are of primary importance. The point spectrum is the set*

$$\sigma_p(L) = \Big\{\lambda \in \mathbb{R} : ker(L - \lambda I) \neq \{0\}\Big\},$$

its elements being the eigenvalues of L. *The discrete spectrum is the set*

$$\sigma_d(L) = \Big\{\lambda \in \sigma_p(L) : \dim ker(L - \lambda I) < \infty \text{ and } \lambda \text{ is an isolated point of } \sigma(L)\Big\}$$

and its complement in $\sigma(L)$ *is called the essential spectrum*

$$\sigma_e(L) = \sigma(L) \backslash \sigma_d(L).$$

From Remark 2.1, Lemma 2.3, and our condition (G), we know that the fractional Schrödinger operators $L_\lambda = (-\Delta)^s + \lambda g(x)$ is a self-adjoint operator on $L^2(\mathbb{R}^N)$ with form domain $H^{2s}(\mathbb{R}^N)$. Moreover, similar to in [2] and Theorem 3.8 in [32], for the point spectrum, we have the following result.

Lemma 2.4. *Let the condition* (G) *be satisfied. The fractional Schrödinger operators* $L_\lambda = (-\Delta)^s + \lambda g(x)$ *has no* L^2*-eigenvalues in the interval* $(0, +\infty)$. *In particularly, for* $g \equiv 0$, *the fractional operators* $(-\Delta)^s$ *has no* L^2*-eigenvalues on* $\mathbb{R}^N$.

Proof. Assume that $f \in D((-\Delta)^s) = D(L_\lambda)$ is such that $L_\lambda f = \lambda f$ with $\lambda > 0$. By [2, Proposition 10.10], we know that f must have compact support. Next, one may choose a periodic $W : \mathbb{R}^N \to \mathbb{R}$ such that $W(x) = \lambda g(x)$ for all x in the support of f. If $L'_\lambda = (-\Delta)^s + W$, then $L'_\lambda f = L_\lambda f = \lambda f$, since $\lambda g(x) f = W f$. Hence, f is an eigenvector of the periodic Schrödinger operator L'_λ. By [2, Proposition 10.9], we conclude that $f = 0$.

For the second part, let $L_0 = (-\Delta)^s$. Suppose that $u \in \ker(L_0 - \lambda I)$ for some $\lambda \in \mathbb{R}$. Then

$$(-\Delta)^s u = \lambda u,$$

which means that $u \in L^2(\mathbb{R}^N)$ and

$$\int u(-\Delta)^s z dx = \int (\lambda u) z dx \quad \text{for all } z \in L^2(\mathbb{R}^N).$$

By the Fourier transform, we have

$$|\xi|^{2s} \widehat{u}(\xi) = \lambda \widehat{u}(\xi) \quad \text{for almost all } \xi \in \mathbb{R}^N.$$

Since $\big\{\xi \in \mathbb{R}^N : |\xi|^{2s} = \lambda\big\}$ has N-dimensional zero, this implies that $\widehat{u}(\xi) = 0$ for almost all $\xi \in \mathbb{R}^N$. Thus, $u \equiv 0$ and $\ker(L_0 - \lambda I) = \{0\}$. $\square$

3. Proof of theorem 1.1

Inspired by [33], we first describe the eigenvalue λ as a function of the parameter α rather than the eigenvalue α as a function of the parameter λ in the traditional treatment. In this case, if we denote $\alpha(\lambda)$ the lowest eigenvalue of L_λ, we can see that $\alpha(\lambda)$ increases from Γ_1 to ξ_1 as λ increases from Γ_1 to ∞. Moreover, from Lemma 2.4, we can restrict eigenvalues λ for problem (1.1) in the interval $(\alpha, +\infty)$. In fact, if u satisfies (1.1), then

$$(-\Delta)^s u - \lambda(1-g)u = (\alpha - \lambda)u,$$

and so $\alpha - \lambda$ is an L^2-eigenvalue of the fractional Schrödinger operators $(-\Delta)^s - \lambda(1-g)$. Using Lemma 2.4, this implies that $\lambda > \alpha$. Henceforth, we concentrate on the existence of eigenvalues of (1.1) in the interval $(\alpha, +\infty)$.

In the following, we define the fractional Schrödinger operators

$$A_\lambda : H^{2s}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$$

by

$$A_\lambda u = (-\Delta)^s u + \lambda g u - \alpha u = (-\Delta)^s u + (\lambda g - \alpha)u.$$

Then A_λ is a self-adjoint operator in $L^2(\mathbb{R}^N)$ with spectrum $\sigma(A_\lambda)$ and essential spectrum $\sigma_e(A_\lambda) = [\lambda - \alpha, \infty)$. Furthermore, setting

$$\Sigma(\lambda) = \inf \sigma(A_\lambda),$$

we have

$$\Sigma(\lambda) = \inf \Big\{ a_\lambda(u) : u \in H^s(\mathbb{R}^N) \text{ and } |u|_2 = 1 \Big\} > -\infty, \tag{3.1}$$

where

$$a_\lambda(u) = \int_{\mathbb{R}^N} \big(|(-\Delta)^{\frac{s}{2}} u|^2 - \alpha u^2 + \lambda g u^2 \big) dx.$$

If we set

$$S_\alpha := \big\{ \lambda \geq \alpha : \Sigma(\lambda) < 0 \big\} \quad \text{and} \quad T_\alpha := \big\{ \lambda \geq \alpha : \Sigma(\lambda) > 0 \big\},$$

it is clear from (3.1) that S_α and T_α are intervals since $\Sigma(\lambda)$ is non-decreasing in λ.

In most of the discussion, the value of α is fixed and it is the variation with respect to λ that is of interest. However, when the dependence on α is relevant, we use the more explicit notation

$$A^\alpha_\lambda, \quad a^\alpha_\lambda, \quad \text{and} \quad \Sigma^\alpha(\lambda).$$

Lemma 3.1. *If (G) holds and $\lambda > \alpha$, we have $\Sigma(\lambda) = 0$ if and only if λ is an eigenvalue of (1.1) with a non-negative eigenfunction u_λ. In this case, 0 is a simple eigenvalue of A_λ, $\ker A_\lambda = span\{u_\lambda\}$ and $u_\lambda > 0$ on $\mathbb{R}^N$.*

Proof. We first suppose that $\Sigma(\lambda) = 0$. Then $0 = \inf \sigma(A_\lambda)$ by (3.1) and

$$0 < \lambda - \alpha = \inf \sigma_e(A_\lambda).$$

Hence, 0 is an eigenvalue of A_λ and there exists $u_\lambda \in C(\mathbb{R}^N) \cap H^{2s}(\mathbb{R}^N)$ such that $\ker A_\lambda = span\{u_\lambda\}$ and $u_\lambda > 0$ on $\mathbb{R}^N$ (see [19, Appendix]). Thus, λ is an eigenvalue of (1.1) with eigenfunction u_λ.

On the other hand, if λ is an eigenvalue of (1.1) with an eigenfunction $u_\lambda \geq 0$, then we have already observed that $u_\lambda \in C(\mathbb{R}^N) \cap H^{2s}(\mathbb{R}^N)$ and $A_\lambda u_\lambda = 0$. Thus, $0 \in \sigma(A_\lambda)$, and so

$$\Sigma(\lambda) \leq 0 < \inf \sigma_e(A_\lambda).$$

This implies that $\Sigma(\lambda)$ is a simple eigenvalue of A_λ with a positive eigenfunction $v \in H^{2s}(\mathbb{R}^N)$. Thus,

$$\Sigma(\lambda)\langle u_\lambda, v\rangle_2 = \langle u_\lambda, A_\lambda v\rangle_2 = \langle A_\lambda u_\lambda, v\rangle_2 = 0 \quad \text{and} \quad \langle u_\lambda, v\rangle_2 > 0,$$

showing that $\Sigma(\lambda) = 0$. □

Lemma 3.2. *If (G) holds, then $\alpha \in S_\alpha$ if and only if $\Gamma_1 < \alpha$.*

Proof. If $\Sigma(\lambda) < 0$, then there exists $u \in H^s(\mathbb{R}^N)$ such that

$$|u|_2 = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} u|^2 - \alpha(1-g)u^2\right) dx < 0.$$

It follows that

$$\int_{\mathbb{R}^N} (1-g)u^2 dx > 0$$

and that $\Gamma_1 < \alpha$.

On the other hand, if $\Gamma_1 < \alpha$, then there exists $u \in H^s(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx < \alpha \int_{\mathbb{R}^N} (1-g)u^2 dx,$$

hence, $\Sigma(\lambda) < 0$. □

Lemma 3.3. *Let the condition (G) be satisfied. Then*

$$0 \leq \Gamma_1 < \xi_1.$$

Proof. Let $\varphi_1 \in H_0^s(\Omega)$ be an eigenfunction of (1.4) corresponding to ξ_1 with $|\varphi_1|_2 = 1$. Extending φ by

$$\varphi = \varphi_1 \text{ in } \Omega, \quad \varphi \equiv 0 \text{ in } \mathbb{R}^N \backslash \Omega.$$

We now have $\varphi \in H^s(\mathbb{R}^N)$, $g\varphi \equiv 0$ on $\mathbb{R}^N$, and hence

$$\int_{\mathbb{R}^N} (1-g)\varphi^2 dx = 1.$$

Thus,

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\varphi|^2 dx = \int_\Omega |(-\Delta)^{\frac{s}{2}}\varphi_1|^2 dx = \xi_1 \int_\Omega \varphi^2 dx = \xi_1 \int_{\mathbb{R}^N} (1-g)\varphi^2 dx,$$

showing that $\Gamma_1 \le \xi_1$. However, if $\Gamma_1 = \xi_1$, it follows that $\varphi \in H^s(\mathbb{R}^N)$ minimizes

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx$$

under the constraint

$$\int_{\mathbb{R}^N} (1-g)u^2 dx = 1$$

and consequently

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}\varphi(-\Delta)^{\frac{s}{2}} v dx = \xi_1 \int_{\mathbb{R}^N} (1-g)\varphi v dx, \quad \text{for all } v \in H^s(\mathbb{R}^N).$$

Since $g\varphi \equiv 0$ on $\mathbb{R}^N$, this implies that φ is an L^2-eigenfunction of $(-\Delta)^s$ on $\mathbb{R}^N$. However, from Lemma 2.4, $(-\Delta)^s$ has no such L^2-eigenfunctions and hence $\Gamma_1 < \xi_1$. □

Lemma 3.4. *If* (G) *holds,*

(*i*) S_α *and* T_α *are open subsets of* $[\alpha, +\infty)$.
(*ii*) *If* $\alpha \ge \xi_1$, *then* $S_\alpha = [\alpha, +\infty)$.
(*iii*) *If* $\Gamma_1 < \alpha < \xi_1$, *then there exists* $\Lambda(\alpha) \in (\alpha, +\infty)$ *such that* $S_\alpha = [\alpha, \Lambda(\alpha))$, *where* $\alpha < \Lambda(\alpha) < \infty$.

Proof. (*i*) By the definition of a_λ, we see that, for all $\lambda, \mu \in \mathbb{R}$ and $u \in H^s(\mathbb{R}^N)$,

$$a_\lambda(u) - a_\mu(u) = (\lambda - \mu) \int_{\mathbb{R}^N} g(x)u^2 dx. \tag{3.2}$$

Suppose that $\lambda \in S_\alpha$. Then there exists $u \in H^s(\mathbb{R}^N)$ such that $|u|_2 = 1$ and $a_\lambda(u) < 0$. Since

$$a_\mu(u) \le a_\lambda(u) + |\lambda - \mu| \int_{\mathbb{R}^N} g(x)u^2 dx \le a_\lambda(u) + |\lambda - \mu|,$$

it follows that $\Sigma(\mu) < 0$ for all $\mu \ge \alpha$ such that $|\lambda - \mu| \le \frac{1}{2}|a_\lambda(u)|$, showing that S_α is open.

Suppose now that $\lambda \in T_\alpha$. Then for all $u \in H^s(\mathbb{R}^N)$ with $|u|_2 = 1$, we have

$$a_\mu(u) \ge a_\lambda(u) - |\lambda - \mu| \ge \Sigma(\lambda) - |\lambda - \mu| \ge \frac{1}{2}\Sigma(\lambda) > 0$$

for all μ such that $|\lambda - \mu| \le \frac{1}{2}\Sigma(\lambda)$. Thus,

$$\Sigma(\mu) \ge \frac{1}{2}\Sigma(\lambda) > 0 \text{ for all } \mu$$

such that $|\lambda - \mu| \le \frac{1}{2}\Sigma(\lambda)$, showing that T_α is open.

(*ii*) Let $\varphi_1 \in H_0^s(\Omega)$ be the eigenfunction of (1.2), and set

$$\varphi = \varphi_1 \text{ in } \Omega, \quad \varphi \equiv 0 \text{ in } \mathbb{R}^N \backslash \Omega.$$

We now have $\varphi \in H^s(\mathbb{R}^N)$ and $g\varphi \equiv 0$ on $\mathbb{R}^N$. Thus,

$$a_\lambda(\varphi) = \int_\Omega \big(|(-\Delta)^{\frac{s}{2}}\varphi_1|^2 - \alpha\varphi_1^2\big)dx = \xi_1 - \alpha \text{ and } |u|_2 = 1,$$

showing that $\Sigma(\lambda) < 0$ if $\alpha > \xi_1$. Furthermore, if $\alpha = \xi_1$ and $\Sigma(\lambda) = 0$, then

$$0 = a_\lambda(\varphi) = \min\Big\{a_\lambda(u) : u \in H^s(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u^2 dx = 1\Big\}.$$

Hence, there is a Lagrange multiplier $\zeta \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} \big((-\Delta)^{\frac{s}{2}}\varphi(-\Delta)^{\frac{s}{2}}v - (\alpha - \lambda g)\varphi v\big)dx = \zeta\int_{\mathbb{R}^N}\varphi v dx \text{ for all } v \in H^s(\mathbb{R}^N).$$

Putting $v = \varphi$, we see that $\zeta = \xi_1 - \alpha = 0$, and then

$$\int_{\mathbb{R}^N} \big((-\Delta)^{\frac{s}{2}}\varphi(-\Delta)^{\frac{s}{2}}v + \lambda g\varphi v - \xi_1\varphi v\big)dx = 0 \text{ for all } v \in H^s(\mathbb{R}^N)$$

since $g\varphi = 0$ in $\mathbb{R}^N$, which contradicts to Lemma 2.4. Hence, $\Sigma(\lambda) < 0$ if $\alpha = \xi_1$, too.

(*iii*) Suppose now that $\Gamma_1 < \alpha < \xi_1$. By Lemma 3.2, we have $\alpha \in S_\alpha$, and there exists $\Lambda(\alpha) > \alpha$ such that $S_\alpha = [\alpha, \Lambda(\alpha))$ since S_α is an open interval of $[\alpha, \infty)$. If $\Lambda(\alpha) = \infty$, then $S_\alpha = [\alpha, +\infty)$, and for any integer $n \ge \alpha$, there exists $u_n \in H^s(\mathbb{R}^N)$ with $|u_n|_2 = 1$ such that

$$a_n(u_n) = \int_{\mathbb{R}^N} \big(|(-\Delta)^{\frac{s}{2}}u_n|^2 - (\alpha - ng)u_n^2\big)dx < 0. \tag{3.3}$$

Since $g(x) \ge 0$, this implies that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_n|^2 dx \le \alpha\int_{\mathbb{R}^N} u_n^2 dx = \alpha,$$

and so $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Passing to a subsequence, still denoted by u_n, we may assume that, for some $u \in H^s(\mathbb{R}^N)$,

$$u_n \rightharpoonup u \text{ in } H^s(\mathbb{R}^N), \quad u_n \to u \text{ in } L^2_{loc}(\mathbb{R}^N).$$

By (3.3),

$$n\int_{\mathbb{R}^N} gu_n^2dx < \alpha\int_{\mathbb{R}^N} u_n^2dx = \alpha. \tag{3.4}$$

Since $\lim\limits_{|x|\to\infty} g(x) = 1$, there exists a compact set $K \subset \mathbb{R}^N$ such that $g(x) \geq \frac{1}{2}$ for almost all $x \notin K$. By (3.3), we have

$$\frac{n}{2}\int_{\mathbb{R}^N\setminus K} u_n^2dx \leq n\int_{\mathbb{R}^N\setminus K} gu_n^2dx \leq n\int_{\mathbb{R}^N} gu_n^2dx < \alpha,$$

that is,

$$\int_{\mathbb{R}^N\setminus K} u_n^2dx \leq \frac{2\alpha}{n},$$

and so

$$1 = \int_{\mathbb{R}^N} u_n^2dx = \int_{\mathbb{R}^N\setminus K} u_n^2dx + \int_K u_n^2dx < \int_K u_n^2dx + \frac{2\alpha}{n}.$$

Since K is compact, this implies that

$$1 \leq \lim_{n\to\infty}\int_K u_n^2dx = \int_K u^2dx \leq \int_{\mathbb{R}^N} u^2dx.$$

However,

$$\int_{\mathbb{R}^N} u^2dx \leq \liminf_{n\to\infty}\int_{\mathbb{R}^N} u_n^2dx = 1$$

and hence

$$\int_{\mathbb{R}^N} u^2dx = \int_K u^2dx = 1.$$

However,

$$\begin{aligned} a_n(u_n) &= \int_{\mathbb{R}^N} \big(|(-\Delta)^{\frac{s}{2}}u_n|^2 - (\alpha - ng)u_n^2\big)dx \\ &\geq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_n|^2dx - \alpha\int_{\mathbb{R}^N} u_n^2dx, \end{aligned}$$

and by (3.3),

$$0 \geq \liminf_{n\to\infty} a_n(u_n) \geq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2dx - \alpha. \tag{3.5}$$

On the other hand, by (3.4),

$$0 \leq \int_{\mathbb{R}^N} gu^2dx \leq \liminf_{n\to\infty}\int_{\mathbb{R}^N} gu_n^2dx \leq \liminf_{n\to\infty}\frac{\alpha}{n} = 0.$$

However, $g(x) \equiv 0$ in Ω and $g(x) > 0$ in $\mathbb{R}^N\setminus\Omega$. Hence, this implies that

$$u = 0 \text{ a.e. on } \mathbb{R}^N\setminus\overline{\Omega} \quad\text{and}\quad u = 0 \text{ a.e. on } \mathbb{R}^N\setminus\Omega.$$

Since Ω has a Lipschitz boundary, we have $\tilde{u} \in H_0^s(\Omega)$, where $\tilde{u}$ is the restriction of u to Ω. By (1.4),

$$\int_{\Omega}(|(-\Delta)^{\frac{s}{2}}\tilde{u}|^2 - \xi_1\tilde{u}^2)dx \geq 0.$$

Thus,

$$\begin{aligned}0 \leq \int_{\Omega}(|(-\Delta)^{\frac{s}{2}}\tilde{u}|^2 - \xi_1\tilde{u}^2)dx &= \int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}u|^2 dx - \xi_1 \\ &< \int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}u|^2 dx - \alpha,\end{aligned}$$

since $|u|_2 = 1$ and $\alpha < \xi_1$, which contradicts (3.5). Thus,

$$\Lambda(\alpha) = \sup S_\alpha < +\infty. \qquad \square$$

Lemma 3.5. *Let (G) be satisfied with $\Gamma_1 < \alpha < \xi_1$, and consider $\lambda \geq \alpha$. Then $\Sigma(\lambda) = 0$ if and only if $\lambda = \Lambda(\alpha)$, where $\Lambda(\alpha)$ is given by Lemma* 3.3(iii). *Furthermore, $\Lambda(\alpha) < \Lambda(\beta)$ for $\Gamma_1 < \alpha < \beta < \xi_1$.*

Proof. By Lemma 3.2, $\alpha \in S_\alpha$. If $\lambda \geq \alpha$ and $\Sigma(\lambda) = 0$, then $\lambda \notin S_\alpha$ and $\lambda > \alpha$. By Lemma 3.1, there exists $u_\lambda \in C(\mathbb{R}^N) \cap H^{2s}(\mathbb{R}^N)$ with

$$u_\lambda > 0, \quad A_\lambda u_\lambda = 0 \quad \text{and } |u_\lambda|_2 = 1.$$

Since $g(x) > 0$ on $\mathbb{R}^N \backslash \overline{\Omega}$,

$$\int_{\mathbb{R}^N} g u_\lambda^2 dx \neq 0.$$

For any $\varepsilon > 0$, it follows from (3.2) that

$$a_{\lambda-\varepsilon}(u_\lambda) = a_\lambda(u_\lambda) - \varepsilon\int_{\mathbb{R}^N} g u_\lambda^2 dx = -\varepsilon\int_{\mathbb{R}^N} g u_\lambda^2 dx < 0,$$

and this means that $\lambda - \varepsilon \in S_\alpha$ for any $\varepsilon > 0$. Therefore,

$$\lambda = \sup S_\alpha = \Lambda(\alpha).$$

On the other hand, if $\lambda = \Lambda(\alpha)$, it follows from Lemma 3.4 that $\lambda \notin S_\alpha \cup T_\alpha$, and since $\lambda \geq \alpha$, we must have $\Sigma(\lambda) = 0$.

Consider $\alpha, \beta \in (\Gamma_1, \xi_1)$ with $\alpha < \beta$. Since $\Sigma^\alpha(\Lambda(\alpha)) = 0$, it follows from Lemma 3.1 that there exists $z_\alpha \in H^{2s}(\mathbb{R}^N)\backslash\{0\}$ such that $\ker A^\alpha_{\Lambda(\alpha)} = \text{span}\{z_\alpha\}$ and hence $a^\alpha_{\Lambda(\alpha)}(z_\alpha) = 0$. However,

$$a^\beta_{\Lambda(\alpha)}(z_\alpha) = a^\alpha_{\Lambda(\alpha)}(z_\alpha) + (\alpha - \beta)\int_{\mathbb{R}^N} z_\alpha^2 dx = (\alpha - \beta)\int_{\mathbb{R}^N} z_\alpha^2 dx < 0,$$

showing that $\Lambda(\alpha) \in S_\beta$ and consequently $\Lambda(\beta) > \Lambda(\alpha)$. $\square$

Lemma 3.6. *Consider $\lambda > \alpha$ and any $p \in [2, \infty)$.*

(*i*) *The map $A_\lambda : X = W^{2s,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is a Fredholm operator of the index zero.*

(*ii*) *Let $\{v_n\} \subset X$, $v_n \rightharpoonup v$ in X and let $\{A_\lambda(v_n)\}$ converge strongly in $L^p(\mathbb{R}^N)$. Then $v_n \to v$ in X.*

Proof. (*i*) Since $\lim_{|x|\to\infty} (\lambda g(x) - \alpha) = \lambda - \alpha > 0$, we write

$$A_\lambda = (-\Delta)^s + (V - \lambda)^+ + (\lambda - \alpha) - (V - \lambda)^-,$$

where $V \in L^\infty(\mathbb{R}^N)$. From [21, Theorem 4.2 (i)], we have that

$$W^{2s,p}\left(\mathbb{R}^N\right) \longrightarrow L^p\left(\mathbb{R}^N\right), \quad u \longmapsto (-\Delta)^s u + (V - \lambda)^+ u + (\lambda - \alpha) u$$

is an isomorphism for all $\lambda > \alpha$ and is so a Fredholm operator with index 0. Moreover, we have $\lim_{|x|\to\infty} (V - \lambda)^-(x) = 0$. It follows from [21, Theorem 4.2 (ii)] that the multiplication operator

$$W^{2,q}\left(\mathbb{R}^N\right) \longrightarrow L^q\left(\mathbb{R}^N\right), \quad u \longmapsto (V - \lambda)^- u$$

is a compact operator. Recall that if $T : X \to Y$ is a bounded linear Fredholm operator and $K : X \to Y$ is a compact operator, then $T + K$ is Fredholm and $\operatorname{ind}(T) = \operatorname{ind}(T + K)$. Thus, the operator A_λ is a Fredholm operator with index 0 for all $\lambda > \alpha$.

(*ii*) Since $A_\lambda : X \to L^p\left(\mathbb{R}^N\right)$ is a Fredholm operator of index zero, by [15, Theorem 3.15], there exists $T \in \mathcal{B}\left(L^p\left(\mathbb{R}^N\right), X\right)$ such that

$$TA_\lambda = I + K,$$

where $K : X \to X$ is a compact linear operator. Let $A_\lambda v_n \to w$ strongly in $L^p\left(\mathbb{R}^N\right)$ for some $w \in L^p\left(\mathbb{R}^N\right)$; then

$$(I + K)v_n = TA_\lambda v_n \to Tw$$

strongly in X. Since K is compact, it follows that $Kv_n \to Kv$ strongly in X. Therefore, $v_n \to Tw - Kv$ strongly in X, and hence that $v_n \to v = Tw - Kv$ strongly in X. □

Proof of Theorem 1.1. (*i*) If $\alpha \geq \xi_1$, it follows from Lemma 3.4 that $\Sigma(\lambda) < 0$ for all $\lambda \geq \alpha$. Thus,

$$\inf \sigma(A_\lambda) = \Sigma(\lambda) < 0 \quad \text{and} \quad \inf \sigma_e(A_\lambda) = \lambda - \alpha \geq 0 \quad \text{for } \lambda \geq \alpha.$$

Hence, there exists $v_\lambda \in C(\mathbb{R}^N) \cap H^{2s}(\mathbb{R}^N)$ such that $A_\lambda v_\lambda = \Sigma(\lambda) v_\lambda$ and $v_\lambda > 0$ on $\mathbb{R}^N$. However, if $u \geq 0$ satisfies (1.1), as in the proof of Lemma 3.1, this leads to a contradiction. Hence, (1.1) has no non-negative eigenfunction with $\lambda \geq \alpha$.

(ii) We now have $0 \leq \Gamma_1 < \alpha < \xi_1$. It follows from Lemma 3.4 (iii) and Lemma 3.5 that

$$S_\alpha = [\alpha, \Lambda(\alpha)), \quad T_\alpha = (\Lambda(\alpha), +\infty)$$

and $\lambda = \Lambda(\alpha) > \alpha$ is the unique point in $[\alpha, \infty)$ such that $\Sigma(\lambda) = 0$. By Lemma 3.1, $\Lambda(\alpha)$ is an eigenvalue of (1.1) and 0 is a simple eigenvalue of $A_{\Lambda(\alpha)}$ with $\ker A_{\Lambda(\alpha)} = \text{span}\{z_\alpha\}$, where $z_\alpha = u_{\Lambda(\alpha)} > 0$ on $\mathbb{R}^N$. Suppose that $\mu \neq \Lambda(\alpha)$ is also an eigenvalue of (1.1) with eigenfunction $w \in H^s(\mathbb{R}^N)$. Since $\Sigma(\mu) = \inf \sigma(A_\lambda))$, this shows that $\Sigma(\mu) \leq 0$ and hence $\mu \leq \sup S_\alpha = \Lambda(\alpha)$. Therefore, $\Lambda(\alpha)$ is the largest eigenvalue of (1.1). Furthermore,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \big((-\Delta)^{\frac{s}{2}} z_\alpha (-\Delta)^{\frac{s}{2}} w - \alpha z_\alpha w + \Lambda(\alpha) g(x) z_\alpha w\big) dx \\ &= \int_{\mathbb{R}^N} \big((-\Delta)^{\frac{s}{2}} w (-\Delta)^{\frac{s}{2}} z_\alpha - \alpha w z_\alpha + \mu(\alpha) g(x) w z_\alpha\big) dx \end{aligned}$$

so that

$$(\Lambda(\alpha) - \mu) \int_{\mathbb{R}^N} g(x) z_\alpha w dx = 0.$$

For $\mu < \Lambda(\alpha)$, this implies that

$$\int_{\mathbb{R}^N \setminus \overline{\Omega}} g(x) z_\alpha w dx = 0.$$

Since $z_\alpha > 0$ and $g(x) > 0$ on $\mathbb{R}^N \setminus \overline{\Omega}$, it follows that either $w \equiv 0$ on $\mathbb{R}^N \setminus \overline{\Omega}$ or w must change sign. However, if $w \equiv 0$ on $\mathbb{R}^N \setminus \overline{\Omega}$, then its restriction $\tilde{w}$ to Ω belongs to $H^{2s}(\Omega) \cap H_0^s(\Omega) \setminus \{0\}$, since $\partial\Omega$ is Lipschitz and satisfies

$$(-\Delta)^s \tilde{w} - \alpha \tilde{w} = 0 \quad \text{on } \Omega.$$

However, $\alpha < \xi_1$, so this is impossible, and consequently w must change sign on $\mathbb{R}^N \setminus \overline{\Omega}$.

(iii) By part *(ii)*, we know that for any $\alpha \in (\Gamma_1, \xi_1)$, there exists $\Lambda(\alpha) \in (\alpha, +\infty)$ such that $\Sigma^\alpha(\Lambda(\alpha)) = 0$, and it is a strictly increasing function of α by Lemma 3.5.

Suppose that $\{\alpha_n\} \subset (\Gamma_1, \xi_1)$ is an increasing sequence such that $\alpha_n \to \xi_1$. Then $\Lambda(\alpha_n) \to \Lambda$, where $\Lambda \geq \xi_1$, since $\Lambda(\alpha_n) > \alpha_n$. If $\Lambda < \infty$, for any $u \in H^s(\mathbb{R}^N)$, $a^{\alpha_n}_{\Lambda(\alpha_n)} \to a^{\xi_1}_\Lambda$. However, by Lemma 3.5, for all $n \in \mathbb{N}$,

$$0 = \Sigma^{\alpha_n}(\Lambda(\alpha_n)) = \inf \Big\{ a^{\alpha_n}_{\Lambda(\alpha_n)}(u) : \ u \in H^s(\mathbb{R}^N) \text{ and } |u|_2 = 1 \Big\},$$

and so $a^{\alpha_n}_{\Lambda(\alpha_n)}(u) \geq 0$ for all $u \in H^s(\mathbb{R}^N)$. This implies that

$$a^{\xi_1}_{\Lambda(\alpha_n)}(u) \geq 0 \quad \text{for all } u \in H^s(\mathbb{R}^N)$$

and hence that

$$\Sigma^{\xi_1}(\Lambda) = \inf\left\{a_\Lambda^{\xi_1}(u):\ u \in H^s(\mathbb{R}^N) \text{ and } |u|_2 = 1\right\} \geq 0.$$

This means that $\Lambda \notin S_{\xi_1}$, contradicting the fact that $S_{\xi_1} = [\xi_1, \infty)$, which was established in Lemma 3.4. Thus, $\lim_{\alpha\to\xi_1^-} \Lambda(\alpha) = \infty$.

Let $\tau = \lim_{\alpha\to\Gamma_1^+} \Lambda(\alpha)$, and observe that since $\Lambda(\alpha) > \alpha$, we must have $\tau \geq \Gamma_1$. Let us suppose that $\tau > \Gamma_1$. Consider a decreasing sequence $\{\alpha_n\}$ such that $\alpha_n \to \Gamma_1$. As in part (ii), there exists $\{z_n\} \subset C(\mathbb{R}^N) \cap H^{2s}(\mathbb{R}^N)$ such that $|z_n|_2 = 1$ and

$$(-\Delta)^s z_n - \alpha_n z_n + \Lambda(\alpha) g z_n = 0 \text{ on } \mathbb{R}^N.$$

Hence, $\{z_n\}$ is bounded in $H^{2s}(\mathbb{R}^N)$. Passing to a subsequence, we suppose henceforth that $z_n \rightharpoonup z$ in $H^{2s}(\mathbb{R}^N)$. However,

$$(-\Delta)^s z_n - \Gamma_1 z_n + \tau g z_n = (\alpha_n - \Gamma_1) z_n + (\tau - \Lambda(\alpha_n)) g z_n \text{ on } \mathbb{R}^N,$$

where

$$(\alpha_n - \Gamma_1) z_n + (\tau - \Lambda(\alpha_n)) g z_n \to 0 \text{ in } L^2(\mathbb{R}^N)$$

as $n \to \infty$ and

$$(-\Delta)^s - \Gamma_1 + \tau g : H^{2s}(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$$

is a Fredholm operator of index zero since

$$\lim_{|x|\to\infty} \{-\Gamma_1 + \tau g(x)\} = -\Gamma_1 + \tau > 0$$

(see [21, Theorem 2.3]). Then Lemma 3.6 implies that $z_n \to z$ in $H^{2s}(\mathbb{R}^N)$, and hence

$$(-\Delta)^s z - \Gamma_1 z + \tau g z = 0 \text{ with } |z|_2 = 1.$$

Furthermore,

$$\int_{\mathbb{R}^N} g z^2 dx > 0,$$

since otherwise $z \equiv 0$ on $\mathbb{R}^N \backslash \Omega$, and we would then have $(-\Delta)^s z = \Gamma_1 z$ on $\mathbb{R}^N$, contradicting the fact that $(-\Delta)^s$ has no L^2-eigenfunctions on $\mathbb{R}^N$. However, by the definition of Γ_1, we have

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} z|^2 - \Gamma_1 (1-g) z^2\right) dx \\ &= \int_{\mathbb{R}^N} \left(\Gamma_1 z^2 - \tau g z^2 - \Gamma_1 (1-g) z^2\right) dx \end{aligned}$$

$$= (\Gamma_1 - \tau) \int_{\mathbb{R}^N} g z^2 dx < 0.$$

This contradiction means that our assumption $\tau > \Gamma_1$ must be rejected, and so $\tau = \Gamma_1$.

The smoothness of the function $\Lambda : (\Gamma_1, \xi_1) \to \mathbb{R}$ could follows by a standard application of the implicit function theorem. We omit here.

(iv) This follows from Lemma 3.5.

(v) Suppose that u satisfies (1.1) with $\lambda > \alpha$. Then

$$\int_{\mathbb{R}^n} g u^2 dx \neq 0,$$

since otherwise we have $gu \equiv 0$ on $\mathbb{R}^N$ and u would be an L^2-eigenfunction of $(-\Delta)^s$ on $\mathbb{R}^N$, and, as we have already remarked several times, this is false. However, (1.1) now yields

$$\int_{\mathbb{R}^N} \big(|(-\Delta)^{\frac{s}{2}} u|^2 - \alpha(1-g)u^2 \big) dx = (\alpha - \lambda) \int_{\mathbb{R}^N} g u^2 dx < 0,$$

from which it follows that

$$\int_{\mathbb{R}^n} (1-g) u^2 dx \neq 0$$

and that $\alpha > \Gamma_1$.

4. Proof of theorem 1.2

(i) We first show that $\alpha_2(\lambda)$ is an eigenvalue of (1.1). First, we claim that

$$\alpha_2(\lambda) < \lambda. \tag{4.1}$$

Indeed, let $\lambda > \Gamma_2$, by the definition of Γ_2, we have

$$\lambda > \inf_{u \in H^s(\mathbb{R}^N) \cap V_1^{\perp}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{\int_{\mathbb{R}^N} (1-g) u^2 dx}.$$

Thus, there exists $v \in H^s(\mathbb{R}^N) \cap V_1^{\perp}$ such that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx < \lambda \int_{\mathbb{R}^N} (1-g) u^2 dx.$$

Then

$$\frac{\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} v|^2 dx + \lambda g(x) v^2) dx}{\int_{\mathbb{R}^N} v^2 dx} < \lambda,$$

which implies that $\alpha_2(\lambda) < \lambda$, i.e., (4.1) is proved.

Next, we shall prove that there exists a function $u \in H^s(\mathbb{R}^N) \cap V_1^{\perp}$ such that

$$\Phi(u) = \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} v|^2 dx + \lambda g(x) v^2) dx$$

archives $\alpha_2(\lambda)$. We borrow a method in the proof of [26, 35]. Let $\{u_n\} \in H^s(\mathbb{R}^N) \cap V_1^{\perp}$ with $|u_n|_2 = 1$ be such that

$$\lim_{n\to\infty} \Phi(u_n) = \alpha_2(\lambda).$$

Obviously, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N) \cap V_1^{\perp}$, passing to a subsequence, we may assume that, for some $u \in H^s(\mathbb{R}^N) \cap V_1^{\perp}$,

$$u_n \rightharpoonup u \text{ in } H^s(\mathbb{R}^N) \cap V_1^{\perp}, \quad u_n \to u \text{ in } L^2_{loc}(\mathbb{R}^N).$$

By the condition (G), for every $\varepsilon > 0$ there exists $R > 0$ such that $\overline{\Omega} \subset B_R(0)$, and

$$\Big| \int_{\mathbb{R}^N \setminus B_R(0)} (g(x) - 1) u_n^2 dx \Big| \leq \varepsilon |u_n|_2^2 = \varepsilon$$

for all $n \in \mathbb{N}$. Since $u_n \to u$ in $L^2_{loc}(\mathbb{R}^N)$,

$$\Big| \int_{B_R(0)} (g(x) - 1)(u_n^2 - u^2) dx \Big| \leq \int_{B_R(0)} (u_n^2 - u^2) dx \leq \varepsilon$$

for n large enough. Therefore,

$$\int_{\mathbb{R}^N} (g(x) - 1) u_n^2 dx \to \int_{\mathbb{R}^N} (g(x) - 1) u^2 dx$$

as $n \to \infty$.

Define

$$Q(u_n) = \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_n|^2 + \lambda (g - 1) u_n^2) dx,$$

then

$$\begin{aligned} Q(u) &= \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda (g-1) u^2) dx \\ &\leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \lim_{n\to\infty} \int_{\mathbb{R}^N} \lambda (g-1) u_n^2 dx \qquad (4.2) \\ &\leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_n|^2 + \lambda (g-1) u_n^2) dx = \alpha_2(\lambda) - \lambda. \end{aligned}$$

By the definition of $\alpha_2(\lambda)$, we get

$$Q(u) \geq (\alpha_2(\lambda) - \lambda) \int_{\mathbb{R}^N} u^2 dx,$$

which together with (4.1) and (4.2), implies

$$\int_{\mathbb{R}^N} u^2 dx \geq 1.$$

Note that

$$\int_{\mathbb{R}^N} u^2 dx \leq \lim_{n\to\infty} \int_{\mathbb{R}^N} u_n^2 dx = 1.$$

Therefore,

$$\int_{\mathbb{R}^N} u^2 dx = 1,$$

then by the definition of $\alpha_2(\lambda)$, we have $\Phi(u) \geq \alpha_2(\lambda)$. However,

$$\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n) = \alpha_2(\lambda),$$

and hence $\Phi(u) = \alpha_2(\lambda)$.

By using the Lagrange-multipliers method, there exists $\eta \in \mathbb{R}$ such that

$$\begin{aligned} &\int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + \lambda g(x) uv) dx \\ &= \eta \int_{\mathbb{R}^N} uv dx, \quad \forall\ v \in H^s(\mathbb{R}^N) \cap V_1^{\perp}. \end{aligned} \tag{4.3}$$

For every $w \in H^s(\mathbb{R}^N)$, we have $w = s u_{\alpha_1(\lambda)} + v$ for some $s \in \mathbb{R}$ and $v \in H^s(\mathbb{R}^N) \cap V_1^{\perp}$. By (4.3), replacing v with $w - s u_{\alpha_1(\lambda)}$, we get

$$\begin{aligned} &\int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} (w - s u_{\alpha_1(\lambda)}) + \lambda g(x) u (w - s u_{\alpha_1(\lambda)})) dx \\ &= \eta \int_{\mathbb{R}^N} u (w - s u_{\alpha_1(\lambda)}) dx \end{aligned}$$

for all $w \in H^s(\mathbb{R}^N)$. Since $u_{\alpha_1(\lambda)}$ is an eigenfunction corresponding to $\alpha_1(\lambda)$,

$$\int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} u_{\alpha_1(\lambda)} + \lambda g(x) u u_{\alpha_1(\lambda)}) dx = \alpha_1(\lambda) \int_{\mathbb{R}^N} u u_{\alpha_1(\lambda)} dx = 0.$$

Thus,

$$\int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} w + \lambda g(x) uw) dx = \eta \int_{\mathbb{R}^N} uw dx \tag{4.4}$$

for all $w \in H^s(\mathbb{R}^N)$. Putting $w = u$ in (4.4), then

$$\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda g(x) u^2) dx = \eta \int_{\mathbb{R}^N} u^2 dx = \eta$$

i.e., $\alpha_2(\lambda) = \eta$. Hence, $\alpha_2(\lambda)$ is an eigenvalue of (1.1).

For fixed $\lambda > 0$, let $u_{\alpha_2(\lambda)}$ be an arbitrary eigenfunction corresponding to the eigenvalue $\alpha_2(\lambda)$. Since $\langle \alpha_1(\lambda), \alpha_2(\lambda)\rangle_2 = 0$, and $\alpha_1(\lambda) > 0$, we know that $\alpha_2(\lambda)$ must change sign, that is, all eigenfunctions corresponding to $\alpha_2(\lambda)$ must change sign. Note that all eigenfunctions corresponding to $\alpha_1(\lambda)$ do not change sign since $\alpha_1(\lambda)$ is simple and there exists a positive eigenfunction corresponding to $\alpha_1(\lambda)$. By the definition of $\alpha_2(\lambda)$, $\alpha_2(\lambda) \geq \alpha_1(\lambda)$, we finally get $\alpha_2(\lambda) > \alpha_1(\lambda)$. If $\alpha(\lambda)$ is an eigenvalue of (1.1) with an eigenfunction $u_{\alpha(\lambda)}$, it is clear that $u_{\alpha(\lambda)} \in V_1^{\perp}$, then $\alpha(\lambda) \geq \alpha_2(\lambda)$ by the definition of $\alpha_2(\lambda)$, that is, $\alpha_2(\lambda)$ is the second eigenvalue of L_λ.

(ii)We will prove that $\tilde{\alpha}_2(\lambda) = \alpha_2(\lambda)$ and $\hat{\alpha}_2(\lambda) = \alpha_2(\lambda)$.

Step 1. $\tilde{\alpha}_2(\lambda) = \alpha_2(\lambda)$. Obviously, $\tilde{\alpha}_2(\lambda) \geq \alpha_2(\lambda)$. On the other hand, let $v \in V_1$, $u \in V_1^{\perp}$ with $|v|_2 = 1$ and $|u|_2 = 1$. $\pi(v,u)$ denotes the plane expanded by u, v. For any given $V \in X_1$, then $V \cap \pi(v,u) \neq \{0\}$, thus there exists $w \in V \cap \pi(v,u) \neq \{0\}$ such that $|w|_2 = 1$, and $w = \beta v + \mu u$ with $\beta^2 + \mu^2 = 1$. By the definition of $\alpha_1(\lambda)$, we have $\Phi(v) \leq \Phi(u)$, and then

$$\begin{aligned}\Phi(w) &= (\beta^2 + \mu^2)\Phi(w) = \Phi(\beta v) + \Phi(\mu u)\\ &= \beta^2\Phi(v) + \mu^2\Phi(u) \leq \beta^2\Phi(u) + \mu^2\Phi(u) = \Phi(u).\end{aligned}$$

Thus,

$$\Phi(w) \leq \inf_{u \in V_1^{\perp}} \Phi(u),$$

and hence

$$\inf_{u \in V} \Phi(u) \leq \Phi(w) \leq \inf_{u \in V_1^{\perp}} \Phi(u).$$

Notice that V is arbitrary, and so

$$\sup_{V \in X_1} \inf_{u \in V} \Phi(u) \leq \inf_{u \in V_1^{\perp}} \Phi(u),$$

that is, $\tilde{\alpha}_2(\lambda) \leq \alpha_2(\lambda)$.

Step 2. $\hat{\alpha}_2(\lambda) = \alpha_2(\lambda)$. Choose

$$V_2 = span\{u_{\alpha_1(\lambda)}, u_{\alpha_2(\lambda)}\},$$

then

$$\sup_{u \in V_2} \{\Phi(u) : \ |u|_2 = 1\} = \alpha_2(\lambda).$$

Thus, for any $\hat{V} \in W_2$, we have

$$\inf_{\hat{V} \in W_2} \sup_{u \in \hat{V}} \{\Phi(u) : \ |u|_2 = 1\} \leq \alpha_2(\lambda),$$

i.e., $\hat{\alpha}_2(\lambda) \leq \alpha_2(\lambda)$

On the other hand, for any given $\hat{V} \in W_2$, we have $\dim \hat{V} = 2$ and $\hat{V} \cap V_1^{\perp} \neq \{0\}$, so that there exists $w \in \hat{V} \cap V_1^{\perp}$ such that $|w|_2 = 1$. Hence,

$$\sup_{u \in \hat{V}} \{\Phi(u) : \ |u|_2 = 1\} \geq \Phi(w) \geq \alpha_2(\lambda),$$

and then

$$\inf_{\hat{V} \in W_2} \sup_{u \in \hat{V}} \{\Phi(u) : \ |u|_2 = 1\} \geq \alpha_2(\lambda),$$

that is, $\hat{\alpha}_2(\lambda) \geq \alpha_2(\lambda)$.

Combining Step 1 and Step 2, (ii) is proved.

(iii) Suppose that $u \neq 0$ satisfies (1.1), then

$$\int_{\mathbb{R}^N} gu^2 dx \neq 0,$$

since otherwise we have $gu \equiv 0$ on $\mathbb{R}^N$, this implies that u is an L^2-eigenfunction of $(-\Delta)^s$ on $\mathbb{R}^N$. However, from Lemma 2.4, we know that $(-\Delta)^s$ has no such eigenfunctions, and hence

$$\int_{\mathbb{R}^N} gu^2 dx > 0.$$

Since $u_{\alpha_i(\lambda)} (i = 1, 2)$ is an eigenfunction corresponding to $\alpha_i(\lambda)$, $\alpha_i(\lambda)$ satisfies (1.1),

$$\int_{\mathbb{R}^N} gu^2_{\alpha_i(\lambda)} dx > 0.$$

We choose

$$V_2 = span\{u_{\alpha_1(\lambda)}, u_{\alpha_2(\lambda)}\} \in W_2,$$

then

$$\sup_{u \in V_2, |u|_2 = 1} \Phi(u) = \alpha_2(\lambda).$$

If $\lambda > \mu$, then

$$\begin{aligned} \alpha_2(\lambda) &= \sup_{u \in V_2, |u|_2=1} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + \lambda g(x) u^2) dx \\ &> \sup_{u \in V_2, |u|_2=1} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + \mu g(x) u^2) dx \\ &\geq \inf_{\hat{V} \in W_2} \sup_{u \in \hat{V}, |u|_2=1} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + \mu g(x) u^2) dx = \alpha_2(\mu). \end{aligned}$$

(iv) Let $\tilde{\varphi}_i = \varphi_i (i = 1, 2)$ for $x \in \Omega$, $\tilde{\varphi}_i = 0$ for $x \in \mathbb{R}^N \backslash \Omega$. Choose

$$V_2 = span\{\tilde{\varphi}_1, \tilde{\varphi}_2\} \in W_2,$$

then

$$\sup_{u\in V_2} \Phi(u) = \xi_2.$$

Therefore, for any $\hat{V} \in W_2$, we have

$$\inf_{\hat{V}\in W_2} \sup_{u\in\hat{V},|u|_2=1} \Phi(u) \le \xi_2.$$

By $\hat{\alpha}_2(\lambda) = \alpha_2(\lambda)$, we get $\alpha_2(\lambda) \le \xi_2$. Thus,

$$\lim_{\lambda\to\infty} \alpha_2(\lambda) = \beta_0 \le \xi_2.$$

If $\beta_0 = \xi_2$, then the conclusion is true. Now, we suppose that $\beta_0 < \xi_2$, then there exists $\{\lambda_n\} \subset (\Gamma_2, \infty)$ with $\lambda_n \to \infty$ as $n \to \infty$, and there exists $\{u_n\} \subset H^s(\mathbb{R}^N) \cap V_1^{\perp}$ with $|u_n|_2 = 1$ such that

$$\alpha_2(\lambda_n) = \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_n|^2 + \lambda_n g(x) u_n^2) dx \le \beta_0. \tag{4.5}$$

By Lemma 2.2, we know that there exists $c > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_n|^2 + u_n^2) dx &\le c \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_n|^2 + g(x) u_n^2) dx \\ &\le c \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_n|^2 + \lambda_n g(x) u_n^2) dx \le c\beta_0,\end{aligned}$$

and so $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N) \cap V_1^{\perp}$. Passing to a subsequence, we may assume that for some $u \in H^s(\mathbb{R}^N) \cap V_1^{\perp}$,

$$u_n \rightharpoonup u \text{ in } H^s(\mathbb{R}^N) \cap V_1^{\perp}, \quad u_n \to u \text{ in } L^2_{loc}(\mathbb{R}^N).$$

By the condition (G), there exists $R > 0$ such that $\overline{\Omega} \subset B_R(0)$ and $g(x) \ge \frac{1}{2}$ for almost all $x \in \mathbb{R}^N \backslash B_R(0)$. By (4.5), we have

$$\lambda_n \int_{\mathbb{R}^N} g u_n^2 dx \le \beta_0,$$

and so

$$\int_{\mathbb{R}^N\backslash B_R(0)} u_n^2 dx \le \frac{2\beta_0}{\lambda_n}.$$

Since $u_n \to u$ in $L^2_{loc}(\mathbb{R}^N)$, which implies that

$$\begin{aligned}1 = \int_{B_R(0)} u_n^2 dx + \int_{\mathbb{R}^N\backslash B_R(0)} u_n^2 dx &\le \lim_{n\to\infty} \int_{B_R(0)} u_n^2 dx \\ &= \int_{B_R(0)} u^2 dx \le \int_{\mathbb{R}^N} u^2 dx.\end{aligned}$$

Note that

$$\int_{\mathbb{R}^N} u^2 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} u_n^2 dx = 1,$$

therefore,

$$\int_{\mathbb{R}^N} u^2 dx = \int_{B_R(0)} u^2 dx = 1.$$

Moreover, by (4.5),

$$\int_{\mathbb{R}^N} gu^2 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} gu_n^2 dx \leq \lim_{n\to\infty} \frac{\beta_0}{\lambda_n} = 0.$$

Hence, by the condition (G), we see that $u = 0$ a.e. on $\mathbb{R}^N \backslash \overline{\Omega}$, so that $u \in H_0^s(\Omega)$. Since $u_n \in H^s(\mathbb{R}^N) \cap V_1^{\perp}$, we see that $\langle u, \varphi_1 \rangle_2 = 0$. By the definition of ξ_2 and $\beta_0 < \xi_2$, we have

$$\begin{aligned} 0 \leq \int_\Omega (|(-\Delta)^{\frac{s}{2}} u|^2 - \xi_2 u^2) dx &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \xi_2 \\ &< \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \beta - 0. \end{aligned} \tag{4.6}$$

On the other hand, by (4.5),

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \leq \beta_0,$$

which contradicts (4.6), thus $\lim_{\lambda\to\infty} \alpha_2(\lambda) = \xi_2$, and the proof of Theorem 1.2 is complete.

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