Research Article

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On the singularly perturbation fractional Kirchhoff equations: Critical case

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Abstract: This article deals with the following fractional Kirchhoff problem with critical exponent

$$\left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)(-\Delta)^{s}u=(1+\varepsilon K(x))u^{2^{*}_{s}-1},\quad\text{in }\mathbb{R}^N,$$

where a, b > 0 are given constants, ε is a small parameter, $2_s^* = \frac{2N}{N-2s}$ with 0 < s < 1 and $N \ge 4s$. We first prove the nondegeneracy of positive solutions when $\varepsilon = 0$. In particular, we prove that uniqueness breaks down for dimensions N > 4s, i.e., we show that there exist two nondegenerate positive solutions which seem to be completely different from the result of the fractional Schrödinger equation or the low-dimensional fractional Kirchhoff equation. Using the finite-dimensional reduction method and perturbed arguments, we also obtain the existence of positive solutions to the singular perturbation problems for ε small.

Keywords: fractional Kirchhoff equations, nondegeneracy, Lyapunov-Schmidt reduction

MSC 2020: 35A01, 35B25, 35A15

1 Introduction and main results

In this article, we are concerned with the following fractional Kirchhoff problem:

$$\left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)(-\Delta)^s u=(1+\varepsilon K(x))u^{2^*_s-1},\quad\text{in }\mathbb{R}^N,\tag{1.1}$$

where a, b > 0 are given constants, ε is a small parameter, $K(x) : \mathbb{R}^N \to \mathbb{R}$, $(-\Delta)^s$ is the pseudo-differential operator defined by

$$\mathcal{F}((-\Delta)^{s}u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^{N},$$

where \mathcal{F} denotes the Fourier transform, $2_s^* = \frac{2N}{N-2s}$ is the standard fractional Sobolev critical exponent. Recently, when $\varepsilon = 0$ in (1.1), Yang and Yu [1] established uniqueness and nondegeneracy for 1 < N < 4s. Then in this article, we will consider the high-dimensional cases, i.e., $N \ge 4s$ and the associated singularly perturbation problems.

If s = 1, equation (1.1) reduces to the well-known Kirchhoff-type problem; this problem and its variants have been studied extensively in the literature. The equation that goes under the name of Kirchhoff equation was proposed in [2] as a model for the transverse oscillation of a stretched string in the form

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$$\rho h \partial_{tt}^2 u - \left(p_0 + \frac{\mathcal{E}h}{2L} \int_0^L |\partial_x u|^2 dx \right) \partial_{xx}^2 u = 0, \qquad (1.2)$$

for $t \ge 0$ and 0 < x < L, where u = u(t, x) is the lateral displacement at time t and at position x, \mathcal{E} is the Young's modulus, ρ is the mass density, h is the cross-sectional area, L is the length of the string, and p_0 is the initial stress tension. Problem (1.2) and its variants have been studied extensively in the literature. Bernstein obtains the global stability result in [3], which has been generalized to arbitrary dimension $N \ge 1$ by Pohozaev in [4]. We point out that such problems may describe a process of some biological systems dependent on the average of itself, such as the density of population (see, e.g., [5]). Many interesting work on Kirchhoff equations can be found in [6–9] and references therein. We also refer to [10] for a recent survey of the results connected to this model.

On the other hand, the interest in generalizing the model introduced by Kirchhoff to the fractional case does not arise only for mathematical purposes. In fact, following the ideas of [11] and the concept of fractional perimeter, Fiscella and Valdinoci proposed in [12] an equation describing the behavior of a string constrained at the extrema in which appears the fractional length of the rope. Recently, problem similar to (1.1) has been extensively investigated by many authors using different techniques and producing several relevant results (see, e.g., [13–23]).

Besides, if $b, \varepsilon = 0$ in (1.1), then we are led immediately to the following fractional Schrödinger equation:

$$(-\Delta)^{s}u = u^{2^{*}_{s}-1}, \quad \text{in } \mathbb{R}^{N},$$
 (1.3)

which is also of practical interest and importance. For instance, it arises as the Euler-Lagrange equation of the functional

$$I(u) = \frac{\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2s}} dx \right)^{\frac{N-2s}{N}}}.$$

The classification of the solutions would provide the best constant in the inequality of the critical Sobolev imbedding from $H^{s}(\mathbb{R}^{N})$ to $L^{\frac{2N}{N-2s}}(\mathbb{R}^{N})$:

$$\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2s}} \mathrm{d}x\right)^{\frac{N-2s}{N}} \leq C \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}u\right|^2 \,\mathrm{d}x,$$

where the fractional Sobolev space $H^{s}(\mathbb{R}^{N})$ is defined by

$$H^{s}(\mathbb{R}^{N}) = \left\{ u \in L^{2}(\mathbb{R}^{N}) : \frac{u(x) - u(y)}{|x - y|^{\frac{N}{2} + s}} \in L^{2}(\mathbb{R}^{N} \times \mathbb{R}^{N}) \right\},\$$

endowed with the natural norm

$$||u||^{2} = \int_{\mathbb{R}^{N}} |u|^{2} dx + \int \int_{\mathbb{R}^{N} \times \mathbb{R}^{N}} \frac{|u(x) - u(y)|^{2}}{|x - y|^{N+2s}} dx dy.$$

From [24], we have

$$\|(-\Delta)^{\frac{s}{2}}u\|_{2}^{2} = \int_{\mathbb{R}^{N}} |\xi|^{2s} |\mathcal{F}(u)|^{2} d\xi = \frac{1}{2} C(N, s) \int_{\mathbb{R}^{N} \times \mathbb{R}^{N}} \frac{|u(x) - u(y)|^{2}}{|x - y|^{N + 2s}} dx dy.$$

And we also define the homogeneous fractional Sobolev space

$$D^{s,2}(\mathbb{R}^N) = \left\{ u \in L^{2^*_s}(\mathbb{R}^N) : \|u\| \coloneqq \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

Since the fractional Laplacian $(-\Delta)^s$ is a nonlocal operator, one cannot apply directly the usual techniques dealing with the classical Laplacian operator. By using the moving plane method of integral form, Chen et al. [25] proved that every positive regular solution of (1.3) is radially symmetric and monotone about some point, and therefore assumes the form

$$Q(x) = C(N,s) \left(\frac{\lambda}{\lambda^2 + |x - \xi|^2} \right)^{\frac{N-2s}{2}}, \quad C(N,s) \neq 0, \quad \lambda > 0, \ \xi \in \mathbb{R}^N.$$

Let us observe that

$$U_{\lambda,\xi}(x) = \lambda^{-\frac{N-2s}{2}} Q\left(\frac{x-\xi}{\lambda}\right), \quad Q(x) = C(N,s)\left(\frac{1}{1+|x|^2}\right)^{\frac{N-2s}{2}},$$

which actually reflects the invariance of the equation under the above scaling and translations. Then Dávila et al. [26] proved that the solution above is nondegenerate in the sense that all bounded solutions to the equation

$$(-\Delta)^{s}\phi = (2_{s}^{*} - 1)u^{2_{s}^{*}-2}\phi$$

are linear combinations of the functions $\frac{N-2s}{2}u + x \cdot \nabla u$ and $\partial_{x_i}u$, $1 \le i \le N$. We also refer to [27–33] for recent works for the nonlocal problems.

From the viewpoint of calculus of variation, the fractional Kirchhoff problem (1.1) is much more complex and difficult than the classical fractional Laplacian equation (1.3) as the appearance of the term $b\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 dx\right)(-\Delta)^s u$, which is of order four. So a fundamental task for the study of problem (1.1) is to make clear the effects of this nonlocal term. Recently, Rădulescu and Yang [34] established uniqueness and nondegeneracy for positive solutions to Kirchhoff equations with subcritical growth. More precisely, they proved that the following fractional Kirchhoff equation:

$$\left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)(-\Delta)^su+u=|u|^{p-2}u,\quad\text{in }\mathbb{R}^N,$$

where $\frac{N}{4} < s < 1$, 2 , has a unique nondegenerate positive radial solution. For the highdimensional case, Yang [35] proved that uniqueness breaks down for dimensions <math>N > 4s, i.e., there exist two nondegenerate positive solutions which seem to be completely different from the result of the fractional Schrödinger equation or the low-dimensional fractional Kirchhoff equation. As one application, combining this nondegeneracy result and Lyapunov-Schmidt reduction method, they also derive the existence of solutions to the singularly perturbation problems [36,37]. For the critical problem (1.1) with $\varepsilon = 0$,

$$\left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)(-\Delta)^s u=u^{2^*_s-1},\quad\text{in }\mathbb{R}^N,$$
(1.4)

Yang and Yu [1] established uniqueness and nondegeneracy for 1 < N < 4s. Then as a counterpart to these results, we consider the high-dimensional fractional Kirchhoff equations with critical growth. The first results of this article are collected in the following.

Theorem 1.1. Assume that *a*, *b* > 0. Then the following statements are true:

- (i) If 1 < N < 4s, then problem (1.4) has exactly one solution;
- (ii) If N = 4s, then problem (1.4) is solvable if and only if $b \left\| (-\Delta)^{\frac{s}{2}} Q \right\|_{2}^{2} < 1$, and in this case problem (1.1) has exactly one solution;
- (iii) If N > 4s, then problem (1.4) is solvable if and only if

$$b \left\| (-\Delta)^{\frac{s}{2}} Q \right\|_{2}^{2} \leq \frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}.$$

Furthermore, problem (1.4) *has exactly one solution when the equality holds and has exactly two solutions for the other case.*

Moreover, define the solution by U, which is of the form

$$U_{\lambda,\xi}(x) = \lambda^{-\frac{N-2s}{2}} Q\left(\frac{c^{-\frac{1}{2s}x}-\xi}{\lambda}\right),$$

where $\lambda > 0, \xi \in \mathbb{R}^N$ and positive constant $c = a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}Q|^2$. Furthermore, there exist some $x_0 \in \mathbb{R}^N$ such that $U(\cdot -x_0)$ is radial, positive, and strictly decreasing in $r = |x - x_0|$. Moreover, the function U belongs to $C^{\infty}(\mathbb{R}^N)$ and it satisfies

$$\frac{C_1}{1+|x|^{N+2s}} \leq U(x) \leq \frac{C_2}{1+|x|^{N+2s}}, \quad \forall x \in \mathbb{R}^N,$$

with some constants $C_2 \ge C_1 > 0$.

Theorem 1.2. Suppose that a, b > 0. Then any positive solution U(x) of problem (1.4) is nondegenerate in the sense that there holds

Ker
$$\mathcal{L}_{+} = \operatorname{span}\left\{\partial_{x_{1}}U, \partial_{x_{2}}U, \dots, \partial_{x_{N}}U, \frac{N-2s}{2}U + x \cdot \nabla U\right\},\$$

if one of the following conditions holds:

•
$$1 < N \le 4s$$
;
• $N > 4s$ and $b \left\| (-\Delta)^{\frac{s}{2}} Q \right\|_{2}^{2} \neq \frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}},$

where \mathcal{L}_+ is defined as

$$\mathcal{L}_{+}\varphi = \left(a + b \int_{\mathbb{R}^{N}} |(-\Delta)^{\frac{s}{2}} U|^{2} dx\right) (-\Delta)^{s} \varphi - (2^{*}_{s} - 1) U^{2^{*}_{s} - 2} \varphi + 2b \left(\int_{\mathbb{R}^{N}} (-\Delta)^{\frac{s}{2}} U(-\Delta)^{\frac{s}{2}} \varphi dx\right) (-\Delta)^{s} U$$
(1.5)

acting on $L^2(\mathbb{R}^N)$ with domain D.

By Theorem 1.2, it is now possible that we apply finite-dimensional reduction to study the perturbed fractional Kirchhoff equation (1.1). Our problem is motivated by an interesting work [38], in which the following local problem was studied

$$-\Delta u = (1 + \varepsilon K(x))u^{2^*_s - 1}, \quad u > 0, \quad u \in D^{1,2}(\mathbb{R}^N).$$
(1.6)

Equation (1.6) can be derived from the following scalar curvature equation:

$$-\frac{4(N-1)}{N-2}\Delta_{g_0}u + S_0u = (1 + \varepsilon K(x))u^{2^*_s - 1},$$
(1.7)

where Δ_{g_0} and S_0 denote the Laplace-Beltrami operator and the scalar curvature of the *N*-dimensional Riemann manifold (*M*, *g*₀), respectively. Equation (1.7) and its variants have been studied extensively by the mathematicians, and the reader can check [39–42] and references therein.

Note that if *u* is a (weak) solution to equation (1.1), then the following Pohozâev identity [19] holds:

$$\frac{(N-2s)a}{2} \int_{\mathbb{R}^{N}} |(-\Delta)^{\frac{s}{2}} u|^{2} + \frac{(N-2s)b}{2} \left(\int_{\mathbb{R}^{N}} |(-\Delta)^{\frac{s}{2}} u|^{2} \right)^{2} - \frac{N}{2^{*}_{s}} \int_{\mathbb{R}^{N}} (1+\varepsilon K) u^{2^{*}_{s}} - \frac{\varepsilon}{2^{*}_{s}} \int_{\mathbb{R}^{N}} (x \cdot \nabla K) u^{2^{*}_{s}} = 0.$$

Therefore, if u is a solution of equation (1.1), then

$$\int_{\mathbb{R}^N} (x \cdot \nabla K) u^{2^*_s} = 0$$

Obviously, equation (1.1) does not have any solution if $x \cdot \nabla K < 0$ or $x \cdot \nabla K > 0$. Thus, in order to ensure the existence of solutions of equation (1.1), it is natural to suppose that K has critical points. More precisely, we assume that

 (K_1) $K \in L^{\infty}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ has finitely many critical points and set $Crit(K) := \{x | \nabla K(x) = 0\}$.

 $(K_2) \ \forall \xi \in \operatorname{Crit}(K), \exists \beta \in (1, N), \text{ and } a_j(x) \in C(\mathbb{R}^N) \text{ with } \sum_{i=1}^N a_i(\xi) \neq 0, \text{ such that }$

$$K(x) = K(\xi) + \sum_{j=1}^{N} a_{j}(\xi) |x_{j} - \xi_{j}|^{\beta} + o(|x - \xi|^{\beta}), \quad x \in B_{\delta}(\xi),$$

where $\delta > 0$ is a small constant.

 (K_3) K satisfies

(i)
$$\exists \rho > 0$$
 such that $x \cdot \nabla K(x) < 0$, $\forall |x| \ge \rho$.

(ii)
$$x \cdot \nabla K(x) \in L^1(\mathbb{R}^N)$$
 and $\int_{\mathbb{R}^N} (x \cdot \nabla K(x)) dx < 0$.

 (K_4)

$$\sum_{a_{\xi}<0} \deg_{\text{loc}}(\nabla K, \xi) \neq (-1)^N,$$

where deg_{loc} denotes the local degree and $a_{\xi} = \sum_{j=1}^{N} a_j(\xi)$.

Now we state the existence result as follows.

Theorem 1.3. Let a, b > 0, K(x) satisfy $(K_1) - (K_4)$, 1 < N < 4s, or N = 4s with $b \left\| (-\Delta)^{\frac{s}{2}} Q \right\|_2^2 < 1$, then there exist constants ε_0 , $\lambda_0 > 0$, and $\xi_0 \in \mathbb{R}^N$ such that for all $|\varepsilon| < \varepsilon_0$, problem (1.1) has a solution u_{ε} and

$$u_{\varepsilon}(x) \to U_{\lambda_0,\xi_0}(x), \quad as \quad \varepsilon \to 0.$$

Theorem 1.4. Let a, b > 0, K(x) satisfy $(K_1) - (K_4)$, N > 4s with $b \left\| (-\Delta)^{\frac{s}{2}} Q \right\|_2^2 \le \frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}$, then there exist constants ε_0 , $\lambda_0 > 0$, and $\xi_0 \in \mathbb{R}^N$ such that for all $|\varepsilon| < \varepsilon_0$, problem (1.1) has two solutions $u_{\varepsilon}^i(i = 1, 2)$ (one solution when the equality holds) and

$$u_{\varepsilon}^{i}(x) \to U_{\lambda_{0},\xi_{0}}^{i}(x), \quad as \quad \varepsilon \to 0.$$

This article is organized as follows. We complete the proof of Theorem 1.1 in Section 2 and prove Theorem 1.2 in Section 3. In Section 4, we present some basic results and explain the strategy of the proof of Theorems 1.3 and 1.4.

Notation. Throughout this article, we make use of the following notations.

- For any R > 0 and for any $x \in \mathbb{R}^N$, $B_R(x)$ denotes the ball of radius R centered at x;
- $\|\cdot\|_q$ denotes the usual norm of the space $L^q(\mathbb{R}^N)$, $1 \le q \le \infty$;
- $o_n(1)$ denotes $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$;
- *C* or $C_i(i = 1, 2, ...)$ are some positive constants that may change from line to line.

2 Proof of Theorem 1.1

In this section we prove Theorem 1.1. Our methods depend on the following result for the well-known fractional critical problem

$$(-\Delta)^s u = u^{2^{s-1}_s}, \quad x \in \mathbb{R}^N.$$

By using the moving plan method of integral form, Chen et al. [25] proved that every positive regular solution of (2.1) is radially symmetric, monotone about some point, and therefore assumes the form

$$Q(x)=C(N,s)\left(\frac{\lambda}{\lambda^2+|x-\xi|^2}\right)^{\frac{N-2s}{2}},\quad C(N,s)\neq 0,\quad \lambda>0,\quad \xi\in\mathbb{R}^N,$$

which satisfies

$$\frac{C_1}{1+|x|^{N+2s}} \le Q(x) \le \frac{C_2}{1+|x|^{N+2s}}, \quad \forall x \in \mathbb{R}^N,$$

with some constants $C_2 \ge C_1 > 0$. Let

$$c = a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2.$$

It is not difficult to check that $v(x) = u(c^{\frac{1}{2s}}x)$ is a solution of equation (2.1). By the uniqueness of Q, we can deduce that

$$v(x) = \lambda^{-\frac{N-2s}{2}} Q\left(\frac{x-\xi}{\lambda}\right),$$

where $\xi \in \mathbb{R}^N$, $\lambda > 0$. Furthermore,

$$u(x) = \lambda^{-\frac{N-2s}{2}} Q\left(\frac{c^{-\frac{1}{2s}\chi} - \xi}{\lambda}\right).$$

Then, direct calculation shows that

$$f(\mathcal{E}) = \mathcal{E} - a - b \left\| (-\Delta)^{\frac{s}{2}} Q \right\|_{2}^{2} \mathcal{E}^{\frac{N-2s}{2s}} = 0, \quad \mathcal{E} \in (a, +\infty).$$

$$(2.2)$$

Therefore, to find solution U(x) of (1.4), it suffices to find positive solutions of the above algebraic equation (2.2), and \mathcal{E} is a constant, which only depends on a, b, and Q.

Case 1: 1 < N < 4s: In this case, we have $\frac{N-2s}{2s} < 1$, which implies that $\lim_{\mathcal{E} \to +\infty} f(\mathcal{E}) = +\infty$. Moreover, one has f(a) < 0. Consequently, there exists unique $\mathcal{E}_0 > a$ such that $f(\mathcal{E}_0) = 0$, which means that (1.4) has a unique solution.

Case 2: N = 4s: In this case, (2.2) becomes

$$\mathcal{E} - a - b \left\| (-\Delta)^{\frac{s}{2}} Q \right\|_{2}^{2} \mathcal{E} = 0, \qquad (2.3)$$

which means that this equation has a unique positive solution

$$\mathcal{E}_0 = \frac{a}{1 - b \left\| (-\Delta)^{\frac{s}{2}} Q \right\|_2^2} \tag{2.4}$$

if and only if $b < \frac{1}{\left\| (-\Delta)^{\frac{s}{2}}Q \right\|_{2}^{2}}$.

Case 3: *N* > 4*s*: A simple computation implies that

$$f'(\mathcal{E}) = 1 - \frac{N - 2s}{2s} b \left\| (-\Delta)^{\frac{s}{2}} Q \right\|_{2}^{2} \mathcal{E}^{\frac{N-4s}{2s}},$$
(2.5)

which means that $f(\mathcal{E})$ has a unique maximum point

$$\mathcal{E}_{0} = \left(\frac{2s}{(N-2s)\left\|\left(-\Delta\right)^{\frac{s}{2}}Q\right\|_{2}^{2}}b\right)^{\frac{2s}{N-4s}} > 0,$$
(2.6)

and the maximum of $f(\mathcal{E})$ is

$$f(\mathcal{E}_{0}) = \left(\frac{N-4s}{N-2s}\right) \left(\frac{2s}{(N-2s)\left\|(-\Delta)^{\frac{s}{2}}Q\right\|_{2}^{2}} b\right)^{\frac{2s}{N-4s}} - a.$$
(2.7)

It is easy to see that $f(\mathcal{E}_0) \ge 0$ implies

$$b \left\| (-\Delta)^{\frac{s}{2}} Q \right\|_{2}^{2} \le \frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}.$$
(2.8)

Since $f''(\mathcal{E}) < 0$ in $(0, +\infty)$ due to N > 4s, we know that $f(\mathcal{E})$ is concave in $(0, +\infty)$. Noting further that f(0) = -a < 0 and $\lim_{\mathcal{E} \to +\infty} f(\mathcal{E}) = -\infty$, a sufficient and necessary condition for the solvability of equation (2.2) in $(0, +\infty)$ is $f(\mathcal{E}_0) \ge 0$. Hence, equation (2.2) has a solution in $(0, +\infty)$ if and only if inequality (2.8) holds. Furthermore, we have

- (i) If $b \left\| (-\Delta)^{\frac{s}{2}} Q \right\|_{2}^{2} = \frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}$, then equation (2.2) has exactly one positive solution \mathcal{E}_{0} defined by (2.6);
- (ii) If $b \left\| (-\Delta)^{\frac{s}{2}} Q \right\|_{2}^{2} < \frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}$, then equation (2.2) has exactly two positive solutions \mathcal{E}_{1} and \mathcal{E}_{2} such that $\mathcal{E}_{1} \in (0, \mathcal{E}_{0})$ and $\mathcal{E}_{2} \in (\mathcal{E}_{0}, +\infty)$.

3 Nondegeneracy results

In this section, we prove the nondegeneracy results of Theorem 1.2. For positive constants a, b, we define the differential operator L as

$$L(u) = \left(a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x\right) (-\Delta)^s u - u^{2^*_s - 1}$$

for any $u \in H^{s}(\mathbb{R}^{N})$ in the weak sense. The linearized operator \mathcal{L}_{+} of L at U is defined as

$$\mathcal{L}_{+}(\varphi) = \frac{\mathrm{d}L(U+t\varphi)}{\mathrm{d}t}\Big|_{t=0}, \quad \forall \varphi \in H^{s}(\mathbb{R}^{N}).$$

It is easy to see that for any $\varphi \in H^{s}(\mathbb{R}^{N})$,

$$\mathcal{L}_{+}(\varphi) = \left(a + b \int_{\mathbb{R}^{N}} |(-\Delta)^{\frac{s}{2}} U|^{2} dx\right) (-\Delta)^{s} \varphi - (2^{*}_{s} - 1) U^{2^{*}_{s} - 2} \varphi + 2b \left(\int_{\mathbb{R}^{N}} (-\Delta)^{\frac{s}{2}} U(-\Delta)^{\frac{s}{2}} \varphi dx\right) (-\Delta)^{s} U$$
$$= T_{+}(\varphi) + L_{2}(\varphi) (-\Delta)^{s} U,$$

acting on $L^2(\mathbb{R}^N)$ with domain D(L), where

$$T_{+}(\varphi) = c(-\Delta)^{s}\varphi - (2_{s}^{*}-1)U^{2_{s}^{*}-2}\varphi,$$

with $c = a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} U|^2 dx$ and

$$L_2(\varphi) = 2b \left(\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U(-\Delta)^{\frac{s}{2}} \varphi dx \right).$$

We observe now that

$$U_{\lambda,\xi}(x) = \lambda^{-\frac{N-2s}{2}} Q\left(\frac{x-\xi}{\lambda}\right), \quad Q(x) = C(N,s)\left(\frac{1}{1+|x|^2}\right)^{\frac{N-2s}{2}}.$$

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Differentiating the equation

$$(-\Delta)^{s}U_{\lambda,\xi} = U_{\lambda,\xi}^{2^{*}_{s}-1}$$
 in \mathbb{R}^{N} ,

with respect of the parameters at $\lambda = 1$, $\xi = 0$, we see that the functions

$$\partial_{\lambda}U_{\lambda,\xi} = \frac{N-2s}{2}Q + x \cdot \nabla Q, \quad \partial_{\xi_i}U_{\lambda,\xi} = -\partial_{x_i}Q$$

annihilate the linearized operator around *Q*; namely, they satisfy the equation

 $(-\Delta)^{s}\phi = (2_{s}^{*} - 1)Q^{2_{s}^{*}-2}\phi$ in \mathbb{R}^{N} .

With no loss of generality, we assume that U(x) = U(|x|) is the unique positive radial energy solution to equation (2.1). Then we have

Ker
$$T_{+} = \operatorname{span}\left\{\frac{N-2s}{2}U + x \cdot \nabla U, U_{x_{1}}, U_{x_{2}}, \dots, U_{x_{N}}\right\}.$$
 (3.1)

Since $\frac{\partial U}{\partial x_i}$ is nonradially symmetric, we have the following corollary:

Corollary 3.1. T_+ is invertible on $L^2_{rad}(\mathbb{R}^N)$.

Lemma 3.1. Let U(x) be a positive solution to the equation L(u) = 0 in $H^s(\mathbb{R}^N)$. Then $L_2\left(\frac{\partial U}{\partial x_i}\right) = 0$ for $i \in \{1, 2, ..., N\}$.

Proof. From the definition of L_2 , and U is the solution of the equation

$$c(-\Delta)^s U = U^{2^*_s - 1}.$$

We have

$$L_2\left(\frac{\partial U}{\partial x_i}\right) = -2b \int_{\mathbb{R}^N} \frac{\partial U}{\partial x_i} (-\Delta)^s U \mathrm{d}x.$$

Therefore,

$$L_2\left(\frac{\partial U}{\partial x_i}\right) = \frac{-2b}{c} \int_{\mathbb{R}^N} U^{2^*_s - 1} \frac{\partial U}{\partial x_i} dx = \frac{-2b}{c} \int_{\mathbb{R}^N} \frac{\partial \left(\frac{1}{2^*_s} U^{2^*_s}\right)}{\partial x_i} dx.$$

Since, for any fixed *i*, up to a translation, the function $\frac{\partial \left(\frac{1}{2s}U^{2s}\right)}{\partial x_i}$ is odd in variable x_i , it is easy to see that

$$\int_{\mathbb{R}^N} \frac{\partial \left(\frac{1}{2_s^*} U^{2_s^*}\right)}{\partial x_i} \mathrm{d}x = 0.$$

Therefore, $L_2\left(\frac{\partial U}{\partial x_i}\right) = 0.$

Lemma 3.2. Let U(x) be a positive solution to the equation L(u) = 0 in $H^{s}(\mathbb{R}^{N})$. If N > 4s and

$$\frac{(N-2s)b\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}U\right|^2\,\mathrm{d}x}{2sc}=1,$$

then

$$b\int_{\mathbb{R}^{N}}\left|(-\Delta)^{\frac{s}{2}}Q\right|^{2} dx = \frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}},$$

where $Q \in H^{s}(\mathbb{R}^{N})$ is the unique positive solution to the equation (1.3).

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Proof. Noting that $c = a + b \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} U \right|^2 dx$, the assumption

$$\frac{(N-2s)b\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} U \right|^2 \, \mathrm{d}x}{2sc} = 1$$

implies

$$b\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} U \right|^2 \mathrm{d}x = \frac{2sa}{N-4s}$$
 and $c = \frac{(N-2s)a}{N-4s}.$

Since $U(x) \in H^{s}(\mathbb{R}^{N})$ is a positive solution to the equation L(u) = 0, we know that U(x) has the following form:

$$U(x) = Q\left(c^{-\frac{1}{2s}}x\right)$$

with $Q(x) \in H^{s}(\mathbb{R}^{N})$ being the unique positive solution to equation (1.3). Therefore,

$$\int_{\mathbb{R}^{N}} \left| (-\Delta)^{\frac{s}{2}} U \right|^{2} dx = c^{\frac{N-2s}{2s}} \int_{\mathbb{R}^{N}} \left| (-\Delta)^{\frac{s}{2}} Q \right|^{2} dx = \left(\frac{(N-2s)a}{N-4s} \right)^{\frac{N-2s}{2s}} \int_{\mathbb{R}^{N}} \left| (-\Delta)^{\frac{s}{2}} Q \right|^{2} dx.$$

Therefore, we have

$$b \int_{\mathbb{R}^{N}} \left| (-\Delta)^{\frac{s}{2}} Q \right|^{2} dx = \frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}.$$

This completes the proof.

Lemma 3.3. Let U(x) be a positive solution to the equation L(u) = 0 in $H^{s}(\mathbb{R}^{N})$. Suppose that

$$1 < N \leq 4s$$

or

$$N > 4s \quad and \quad b \int_{\mathbb{R}^{N}} \left| (-\Delta)^{\frac{s}{2}} Q \right|^{2} \, \mathrm{d}x \neq \frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}$$

Then

$$\operatorname{Ker} (\mathcal{L}_{+}) \cap L^{2}_{\operatorname{rad}}(\mathbb{R}^{N}) = \{0\}.$$

Proof. Direct computation shows that $\frac{N-2s}{2}U + x \cdot \nabla U = \frac{N-2s}{2}U + rU'(r)$ is indeed a radial solution to equation $\mathcal{L}_+\varphi = 0$. We have to prove that $\frac{N-2s}{2}U + x \cdot \nabla U$ is the unique radial solution to equation $\mathcal{L}_+\varphi = 0$ in D_{rad} up to a constant, where D_{rad} contains all the radial functions in $D^{s,2}(\mathbb{R}^N)$.

Let $c = a + b \left\| (-\Delta)^{\frac{s}{2}} U \right\|_2^2$. Recall that U is a ground state solution of (1.2). It follows from above that c is a constant independent of U under the assumptions of Theorem 1.1. Let $\varphi \in D_{\text{rad}}$ satisfy $\mathcal{L}_+ \varphi = 0$. It is equivalent to

$$T_{+}\varphi \doteq c(-\Delta)^{s}\varphi - (2_{s}^{*}-1)U^{2_{s}^{*}-1}\varphi = -2b\left(\int_{\mathbb{R}^{N}}(-\Delta)^{\frac{s}{2}}U(-\Delta)^{\frac{s}{2}}vdx\right)(-\Delta)^{s}U.$$

Write $e_0 = \frac{N-2s}{2}U + x \cdot \nabla U$ for simplicity. Since D_{rad} is a Hilbert space, denote by D_0 the orthogonal complement of $\mathbb{R}e_0$ in D_{rad} . Then $\varphi = \lambda e_0 + v$ for some $\lambda \in \mathbb{R}$ and $v \in D_0$. By a direct computation, we find that $\int (-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}} e_0 = 0$. This implies $u \in D_0$. Moreover, note that T_+ is invertible on D_0 . It follows from $T_+e_0 = 0$ and $\int (-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}} e_0 = 0$ that v satisfies

$$T_{+}v = -2b \left(\int_{\mathbb{R}^{N}} (-\Delta)^{\frac{s}{2}} U(-\Delta)^{\frac{s}{2}} v dx \right) (-\Delta)^{s} U = -\frac{2b\sigma_{v}}{c} \left(U^{2^{*}_{s}-1} \right),$$
(3.2)

where

$$\sigma_{\nu} = \int_{\mathbb{R}^{N}} (-\Delta)^{\frac{s}{2}} U(-\Delta)^{\frac{s}{2}} \nu dx.$$

By applying [43, Theorem 3.3], we conclude that

$$v = -\frac{2b\sigma_{v}}{c}T_{+}^{-1}\left(U^{2_{s}^{*}-1}\right) = -\frac{b\sigma_{v}}{sc}\psi,$$
(3.3)

where $\psi = x \cdot \nabla U$. Multiplying (3.3) by $(-\Delta)^{s}U$ and integrating over \mathbb{R}^{N} , we see that

$$\int_{\mathbb{R}^{N}} \nu(-\Delta)^{s} U dx = -\frac{b\sigma_{v}}{sc} \int_{\mathbb{R}^{N}} \psi(-\Delta)^{s} U dx.$$
(3.4)

Note that

$$\int_{\mathbb{R}^{N}} \nu(-\Delta)^{s} U dx = \int_{\mathbb{R}^{N}} (-\Delta)^{\frac{s}{2}} U(-\Delta)^{\frac{s}{2}} \nu dx$$
(3.5)

and

$$\int_{\mathbb{R}^{N}} \psi(-\Delta)^{s} U dx = \frac{2s - N}{2} \int_{\mathbb{R}^{N}} |(-\Delta)^{\frac{s}{2}} U|^{2} dx$$
(3.6)

(see, e.g., [44]). We then conclude from (3.4) to (3.6) that

$$\sigma_{\nu} = -\frac{b(2s-N)\sigma_{\nu}}{2sc}\int_{\mathbb{R}^{N}}|(-\Delta)^{\frac{s}{2}}U|^{2}\mathrm{d}x = -\frac{(c-a)(2s-N)}{2sc}\sigma_{\nu}.$$

It follows from Lemma 3.2 that

$$1 + \frac{(2s - N)b\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} U \right|^2 dx}{2sc} \neq 0$$
provided that $1 < N \le 4s$, or $N > 4s$ and $b\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} Q \right|^2 dx \neq \frac{2sa^{\frac{4s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}}}$. Therefore, under this assumption, we have $v \equiv 0$. This completes the proof.

Proof of Theorem 1.2. Since $(-\Delta)^{s}U = c^{-1}(U^{2^{*}_{s}-1})$ and U(x) = U(|x|) is radial function, the operator L_{+} commutes with rotations in \mathbb{R}^N (see, e.g., [45]). Therefore, we can decompose $L^2(\mathbb{R}^N)$ using spherical harmonics

$$L^2(\mathbb{R}^N) = \bigoplus_{l \ge 0} \mathcal{H}_l$$

so that L_+ acts invariantly on each subspace

$$\mathcal{H}_l = L^2(\mathbb{R}_+, r^{N-1} \mathrm{d} r) \otimes \mathcal{Y}_l.$$

Here $\mathcal{Y}_l = \operatorname{span}\{Y_{l,m}\}_{m \in M_l}$ denotes space of the spherical harmonics of degree l in space dimension N and M_l is an index set depending on l and N. Precisely, $M_l = \frac{(l+N-1)!}{(N-1)!l!}$ for $l \ge 0$ and $M_l = 0$ for l < 0. Moreover, denote by $\Delta_{Z^{N-1}}$ the Laplacian-Beltrami operator on the unit N - 1 dimensional sphere Z^{N-1} in \mathbb{R}^N and by $Y_{l,m}$, l = 0, 1, ... the spherical harmonics satisfy

$$-\Delta_{Z^{N-1}}Y_{l,m} = \lambda_l Y_{l,m}$$

for all l = 0, 1, ... and $1 \le m \le M_l - M_{l-2}$, where

$$\lambda_l = l(N+l-2) \quad \forall l \ge 0$$

is an eigenvalue of $-\Delta_{Z^{N-1}}$ with multiplicity $\leq M_l - M_{l-2}$ for all $k \in \mathbb{N}$. In particular, $\lambda_0 = 0$ is of multiplicity 1 with $Y_{0,1} = 1$, and $\lambda_1 = N - 1$ is of multiplicity N with $Y_{1,m} = x_m / |x|$ for $1 \leq m \leq N$ (see, e.g., Ambrosetti and Malchiodi [46, Chapter 2 and Chapter 4]).

We can describe the action of L_+ more precisely. For each l, the action of L_+ on the radial factor in H_l is given by

$$(L_{+,l}f)(r) = c((-\Delta_l)^s f)(r) - (2_s^* - 1)U^{2_s^* - 2}(r)f(r) + 2b(W_l f)(r)$$

with the nonlocal linear operator

$$(W_l f)(r) = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} ((-\Delta_l)^s U)(r) \int_0^{+\infty} ((-\Delta_l)^s U)(r) r^{N-1} f(r) dr$$

for $f \in C_0^{\infty}(\mathbb{R}^+) \subset L^2(\mathbb{R}_+, r^{N-1}dr)$. Here $(-\Delta_l)^s$ is given by spectral calculus and the known formula

$$-\Delta_l = -\partial_r^2 - \frac{N-1}{r}\partial_r + \frac{l(l+N-2)}{r^2}.$$

Applying arguments similar to that used in [43] and [45], one can verify that each $L_{+,l}$ enjoys a Perron-Frobenius property, that is, if $E = \inf \sigma(\mathcal{H}_l)$ is an eigenvalue, then E is simple and the corresponding eigenfunction can be chosen strictly positive. Moreover, we have $L_{+,l} > 0$ for $l \ge 2$ in the sense of quadratic forms (see, e.g., [45]).

Since $\partial_{x_i}U(x) = U'(r)\frac{x_i}{r} \in \mathcal{H}_1$, this shows that $L_{+,1}U' = 0$. Note that U'(r) < 0. It follows from the Perron-Frobenius property that 0 is the lowest eigenvalue of $L_{+,1}$, with -U'(r) being its corresponding eigenfunction. Therefore, for any $v \in \mathcal{H}_1$ satisfying $L_{+,1}v = 0$ must be some linear combination of $\{\partial_{x_i}U : i = 1, 2, ..., N\}$ and $\frac{N-2s}{2}U + x \cdot \nabla U$. Recall that $L_{+,l} > 0$ for $l \ge 2$. Applying the Perron-Frobenius property again, 0 cannot be an eigenvalue of $L_{+,l} > 0$ for $l \ge 2$. Finally, Lemma 3.3 implies that $L_{+,0} = \{0\}$. Consequently, for any $v \in \ker L_+$, we conclude that $v \in \mathcal{H}_1$ and hence $\ker L_+ = \ker L_{+,1} = \operatorname{span}\{\partial_{x_i}U, \partial_{x_2}U, \ldots, \partial_{x_N}U, \frac{N-2s}{2}U + x \cdot \nabla U\}$. The proof is completed.

4 Singularly perturbation problem

In this section, we are concerned with the singularly perturbation fractional Kirchhoff equation

$$\left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)(-\Delta)^s u=(1+\varepsilon K(x))u^{2^*_s-1},\quad\text{in }\mathbb{R}^N.$$
(4.1)

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It is known that every solution to (4.1) is a critical point of the energy functional I_{ε} : $D^{s,2}(\mathbb{R}^N) \to \mathbb{R}$, given by

$$I_{\varepsilon}(u) = \frac{a}{2} \int_{\mathbb{R}^{N}} |(-\Delta)^{\frac{s}{2}} u|^{2} + \frac{b}{4} \left(\int_{\mathbb{R}^{N}} |(-\Delta)^{\frac{s}{2}} u|^{2} dx \right)^{2} - \frac{1}{2^{*}_{s}} \int_{\mathbb{R}^{N}} (1 + \varepsilon K(x)) u^{2^{*}_{s}} dx$$

for $u \in D^{s,2}(\mathbb{R}^N)$, which is the fractional Sobolev space equipped with the inner product and norm given by $(u, v) = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} v$ and $||u||^2 = (u, u)$, respectively. It is standard to verify that $I_{\varepsilon} \in C^2(D^{s,2}(\mathbb{R}^N))$. So we are left to find a critical point of I_{ε} . However, because of the noncompactness of the injection of $D^{s,2}(\mathbb{R}^N)$ into $L^{2^s}(\mathbb{R}^N)$, it is impossible to prove that functional I_{ε} satisfies the Palais-Smale condition. To prove main results, based on the nondegeneracy of solution to (1.3), and the perturbation method [38], we use the

finite-dimensional reduction arguments. Moreover, due to the presence of the double nonlocal terms $(-\Delta)^s$ and $\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\right)$, it requires more careful estimates in the procedure, which is more complicated than the case of the fractional Schrödinger equation.

4.1 The abstract perturbation method

In this subsection, we state the abstract results we will use in the rest of the article. They are reported below for the reader's convenience.

Let *E* be a Hilbert space and let I_0 , $G \in C^2(E, \mathbb{R})$ be given. Consider the perturbed functional

$$I_{\varepsilon}(u) = I_0(u) - \varepsilon G(u).$$

Suppose that I_0 satisfies

(1) I_0 has a finite dimensional manifold of critical points *Z*; let $b = I_0(z)$, for all $z \in Z$;

(2) for all $z \in Z$, $D^2I_0(z)$ is a Fredholm operator with index zero;

(3) for all $z \in Z$ there results $T_z Z = \text{Ker } D^2 I_0(z)$.

Hereafter, we denote by Γ the functional $G \mid_Z$.

Theorem 4.1. [38, Theorem 2.1] Let I_0 satisfy (1)–(3) and suppose that there exists a critical point $\overline{z} \in Z$ of Γ such that one of the following conditions hold:

- (i) \bar{z} is nondegenerated;
- (ii) \bar{z} is a proper local minimum or maximum;
- (iii) \bar{z} is isolated and the local topological degree of Γ' at \bar{z} , deg_{loc}(Γ' , 0) is different from zero.

Then for $|\varepsilon|$ small enough, the functional I_{ε} has a critical point u_{ε} such that $u_{\varepsilon} \to \bar{z}$ as $\varepsilon \to 0$.

To apply Theorem 4.1, we set $E = D^{s,2}(\mathbb{R}^N)$,

$$I_{0}(u) = \frac{a}{2} \int_{\mathbb{R}^{N}} |(-\Delta)^{\frac{s}{2}} u|^{2} dx + \frac{b}{4} \left(\int_{\mathbb{R}^{N}} |(-\Delta)^{\frac{s}{2}} u|^{2} dx \right)^{2} - \frac{1}{2_{s}^{*}} \int_{\mathbb{R}^{N}} u^{2_{s}^{*}} dx$$

and

$$G(u) = \frac{1}{2_s^*} \int_{\mathbb{R}^N} K(x) u^{2_s^*} \mathrm{d}x.$$

Letting

$$Z = \left\{ U_{\lambda,\xi} = \lambda^{-\frac{N-2\varepsilon}{2}} Q\left(\frac{c^{-\frac{1}{2\varepsilon}\chi} - \xi}{\lambda}\right) : \lambda > 0, \, \xi \in \mathbb{R}^N \right\}.$$

Then *Z* is an N + 1 dimensional manifold of critical points for the function I_0 corresponding to (1.3). In order to apply the abstract setting we will check the assumptions on I_0 introduced above. The nondegeneracy condition comes from Theorem 1.2, so we only need to prove the following result.

Lemma 4.1. Let

$$I_0''(U_{\lambda,\xi})\varphi = \left(a + b\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} U_{\lambda,\xi}\right|^2 dx\right)(-\Delta)^s \varphi + 2b \left(\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U_{\lambda,\xi}(-\Delta)^{\frac{s}{2}} \varphi dx\right)(-\Delta)^s U_{\lambda,\xi} - (2^*_s - 1)U_{\lambda,\xi}^{2^*_s - 2} \varphi.$$

Then $I_0''(U_{\lambda,\xi})$ is a Fredholm operator with zero index.

Proof. Recalling that

$$c = a + b \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} U_{\lambda,\xi} \right|^2 \, \mathrm{d}x,$$

then for any $\psi \in D^{s,2}(\mathbb{R}^N)$, we have

$$\langle I_0''(U_{\lambda,\xi})\varphi,\psi\rangle = c \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \varphi(-\Delta)^{\frac{s}{2}} \psi dx + 2b \left(\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U_{\lambda,\xi}(-\Delta)^{\frac{s}{2}} \varphi dx \right) \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U_{\lambda,\xi}(-\Delta)^{\frac{s}{2}} \psi dx$$
$$- (2^*_s - 1) \int_{\mathbb{R}^N} U_{\lambda,\xi}^{2^*_s - 2} \varphi \psi dx.$$

Suppose that $\{\varphi_n\}$ is bounded in $D^{s,2}(\mathbb{R}^N)$, then there exists a subsequence of $\{\varphi_n\}$ (we still denote $\{\varphi_n\}$), such that

$$\begin{split} \varphi_n &\to \varphi \; \text{ weakly in } D^{s,2}(\mathbb{R}^N), \\ \varphi_n &\to \varphi \; \text{ strongly in } L^q_{\text{loc}}(\mathbb{R}^N), \quad 2 \leq q < 2^*_s, \\ \varphi_n &\to \varphi \; \text{ a.e. on } \mathbb{R}^N. \end{split}$$

Let

$$\langle \mathcal{L}'\varphi,\psi\rangle \coloneqq 2b \left(\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U_{\lambda,\xi}(-\Delta)^{\frac{s}{2}} \varphi dx \right) \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U_{\lambda,\xi}(-\Delta)^{\frac{s}{2}} \psi dx - (2^*_s - 1) \int_{\mathbb{R}^N} U_{\lambda,\xi}^{2^*_s - 2} \varphi \psi dx.$$

Then, using Hölder inequality we obtain

$$\begin{split} |\langle \mathcal{L}'\varphi_{n} - \mathcal{L}'\varphi, \psi\rangle| &\leq 2b \left| \int_{\mathbb{R}^{N}} (-\Delta)^{\frac{s}{2}} U_{\lambda,\xi}(-\Delta)^{\frac{s}{2}}(\varphi_{n} - \varphi) dx \right| \|U\| \|\psi\| + (2^{*}_{s} - 1) \left(\int_{\mathbb{R}^{N}} (U^{2^{*}_{s}-2}_{\lambda,\xi} |\varphi_{n} - \varphi|)^{\frac{2^{*}_{s}}{2^{*}_{s}-1}} dx \right)^{\frac{2^{*}_{s}-1}{2^{*}_{s}}} \|\psi\|_{2^{*}_{s}} \\ &\leq C \left(\left| \int_{\mathbb{R}^{N}} (-\Delta)^{\frac{s}{2}} U_{\lambda,\xi}(-\Delta)^{\frac{s}{2}}(\varphi_{n} - \varphi) dx \right| + \left(\int_{\mathbb{R}^{N}} (U^{2^{*}_{s}-2}_{\lambda,\xi} |\varphi_{n} - \varphi|)^{\frac{2^{*}_{s}}{2^{*}_{s}-1}} dx \right)^{\frac{2^{*}_{s}-1}{2^{*}_{s}}} \right) \|\psi\|. \end{split}$$

Thus, we can obtain

$$\|\mathcal{L}'\varphi_n - \mathcal{L}'\varphi\| \leq C \left(\left| \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U_{\lambda,\xi}(-\Delta)^{\frac{s}{2}} (\varphi_n - \varphi) \mathrm{d}x \right| + \left(\int_{\mathbb{R}^N} U_{\lambda,\xi}^{\frac{2^*_s(2^*_s - 2)}{2^*_s - 1}} |\varphi_n - \varphi|^{\frac{2^*_s}{2^*_s - 1}} \mathrm{d}x \right)^{\frac{2^*_s - 1}{2^*_s}} \right).$$

On one hand, since $\varphi_n \to \varphi$ in $D^{s,2}(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U_{\lambda,\xi}(-\Delta)^{\frac{s}{2}} (\varphi_n - \varphi) \mathrm{d}x \to 0 \text{ as } n \to \infty.$$

On the other hand, by Vitali convergence theorem, we have

$$\int_{\mathbb{R}^N} \frac{2^{*}_{s(2^*_s-2)}}{2^{*}_{s-1}} |\varphi_n - \varphi|^{\frac{2^*}_{2^*_s-1}} dx \to 0 \text{ as } n \to \infty.$$

Therefore, we have $\|\mathcal{L}'\varphi_n - \mathcal{L}'\varphi\| \to 0$ as $n \to \infty$. Hence, \mathcal{L}' is a compact operator.

Up to now, we have proved that I_0 satisfies assumptions (1)–(3). As described above, one has

$$\Gamma(\lambda,\xi) = G(U_{\lambda,\xi}) = \frac{1}{2_s^*} \int_{\mathbb{R}^N} K(c^{\frac{1}{2s}}(\lambda x + \xi)) Q^{2_s^*}(x) dx,$$

which is nothing but a Poincaré-Melnikov-type function. Our next goal is to show that, locally near any $z \in Z$, there exists a manifold Z_{ε} , diffeomorphic to Z which is a natural constraint for I_{ε} . By this we mean that $u \in Z_{\varepsilon}$ and $I'_{\varepsilon}|_{Z_{\varepsilon}}$ implies $I'_{\varepsilon}(u) = 0$. In this way, the search of critical points of I_{ε} on E is reduced to the search of critical points of $I'_{\varepsilon}|_{Z_{\varepsilon}}$.

In the sequel, we define $T_{\lambda,\xi}Z = \text{span}\{q_1, q_2, \dots, q_{N+1}\}$ by the tangent space to *Z* at $U_{\lambda,\xi}$, where

$$q_j = \frac{\partial U_{\lambda,\xi}}{\partial \xi_j}, \quad j = 1, 2, \dots, N, \ q_{N+1} = \frac{\partial U_{\lambda,\xi}}{\partial \lambda}.$$

Lemma 4.2. For given R > 0, there exists constant $\varepsilon_0 > 0$ and a C^1 function

$$w = w(U_{\lambda,\xi},\varepsilon) : B_R^{N+1}(0) \times (-\varepsilon_0,\varepsilon_0) \to D^{s,2}(\mathbb{R}^N)$$

such that for any $(\lambda, \xi) \in B_R^{N+1}(0)$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the following properties hold:

- (*i*) $w(U_{\lambda,\xi}, \varepsilon)$ is orthogonal to $T_{\lambda,\xi}Z$,
- (*ii*) $I'_{\varepsilon}(U_{\lambda,\xi} + w(U_{\lambda,\xi}, \varepsilon)) \in T_{\lambda,\xi}Z$,
- (iii) $w(U_{\lambda,\xi}, 0) = 0$, where $B_R^N(0)$ denotes the N dimension ball centering at zero with radius R.

Proof. We will find $w(U_{\lambda,\xi}, \varepsilon)$ by means of the local inversion theorem applied to the map

$$H: B^{N+1}_R(0) \times D^{s,2}(\mathbb{R}^N) \times \mathbb{R} \times \mathbb{R}^{N+1} \to D^{s,2}(\mathbb{R}^N) \times \mathbb{R}^{N+1}$$

with components $H_1 \in E$ and $H_2 \in \mathbb{R}^{N+1}$ given by

$$H_{1}(Q_{\lambda,\xi}, w, \varepsilon, \sigma) = I_{\varepsilon}'(Q_{\lambda,\xi} + w) - \sum_{j=1}^{N+1} \sigma_{j}q_{j},$$
$$H_{2}(Q_{\lambda,\xi}, w, \varepsilon, \sigma) = (\langle w, q_{1} \rangle, \dots, \langle w, q_{N+1} \rangle),$$

where

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}} v dx.$$

Let us remark that $H_1 = 0$ means that $I'_{\varepsilon}(U_{\lambda,\xi} + w(U_{\lambda,\xi}, \varepsilon)) \in T_{\lambda,\xi}Z$ namely that (*ii*) holds, while $H_2 = 0$ means that *w* is orthogonal to $T_{\lambda,\xi}Z$, namely that (*i*) holds.

Note that for each $U_{\lambda,\xi} \in \mathbb{S}$,

$$H(U_{\lambda,\xi}, 0, 0, 0) = (H_1(U_{\lambda,\xi}, 0, 0, 0), H_2(U_{\lambda,\xi}, 0, 0, 0)) = (I_0'(U_{\lambda,\xi}), 0) = 0$$

and the derivative of *H* at $Q_{\lambda,\xi} \in S$, $\varepsilon = 0$, $\sigma = 0$ and w = 0 can be stated as follows:

$$\frac{\partial H}{\partial(w,\sigma)}(Q_{\lambda,\xi},0,0,0)[\phi,d] = \left(I_0''(Q_{\lambda,\xi})[\phi] - \sum_{j=1}^{N+1} d_j q_j, \langle \phi, q_1 \rangle, \dots, \langle \phi, q_{N+1} \rangle\right),$$

where $\phi \in D^{1,2}(\mathbb{R}^N)$ and $d = (d_1, ..., d_{N+1})$.

From Lemma 4.1, we can prove that $\frac{\partial H}{\partial(w,\sigma)}(Q_{\lambda,\xi}, 0, 0, 0)$ is a Fredholm operator of index 0, so it is enough to prove that it is injective. Then let us assume that $\frac{\partial H}{\partial(w,\sigma)}(Q_{\lambda,\xi}, 0, 0, 0)[d, \phi] = (0, 0)$. From

$$I_0''(Q_{\lambda,\xi})[\phi] = \sum_{j=1}^{N+1} d_j q_j$$

taking the inner product with q_i , we infer that $d_j = 0$ and

$$I_0''(Q_{\lambda,\xi})[\phi]=0.$$

Using again (3), we deduce that $\phi \in T_{\lambda,\xi}Z$. On the other side, $\frac{\partial H_1}{\partial(w,\sigma)} = 0$ implies that ϕ is orthogonal to $T_{\lambda,\xi}Z$ and thus $\phi = 0$. This shows that $\frac{\partial H}{\partial(w,\sigma)}(Q_{\lambda,\xi}, 0, 0, 0)$ is invertible and an application of the implicit function theorem yields the results.

Based on Lemma 4.2, it is natural to introduce the perturbed manifold

$$Z_{\varepsilon} = \{U_{\lambda,\xi} + w(U_{\lambda,\xi},\varepsilon) | (\lambda,\xi) \in B_R^{N+1}(0) \}.$$

It is easy to prove that Z_{ε} is a natural constraint for I_{ε} , that is to say the critical point of I_{ε} on Z_{ε} is also a critical point of I_{ε} on $D^{s,2}(\mathbb{R}^N)$. In fact, if $u = U_{\lambda,\xi} + w(U_{\lambda,\xi}, \varepsilon) \in Z_{\varepsilon}$ is a critical point of I_{ε} on Z_{ε} , then $I'_{\varepsilon}(u)$ is orthogonal to $T_u Z_{\varepsilon}$, where $T_u Z_{\varepsilon}$ is the tangent space to Z_{ε} at u. On the other hand, from Lemma 4.2(ii) we can know that $I'_{\varepsilon}(u) \in T_{\lambda,\xi}Z$ and $T_u Z_{\varepsilon}$ is near $T_{\lambda,\xi}Z$ provided ε is small enough. Thus, $I'_{\varepsilon}(u) = 0$. Moreover, we have that

$$I_{\varepsilon}(U_{\lambda,\xi} + w(U_{\lambda,\xi},\varepsilon)) = I_0(U_{\lambda,\xi}) - \varepsilon \Gamma(\lambda,\xi) + o(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

Consequently, if (λ, ξ) is a critical point of the restriction $\Gamma(\lambda, \xi)$, then $U_{\lambda,\xi} + w(U_{\lambda,\xi}, \varepsilon)$ will turn out to be a critical point of I_{ε} . From the analysis above, then we turn to solve the finite-dimensional functional $\Gamma(\lambda, \xi)$.

4.2 Behavior of $\Gamma(\lambda, \xi)$

We begin by proving some general properties of $\Gamma(\lambda, \xi)$. First of all, it is convenient to extend $\Gamma(\lambda, \xi)$ by continuity to $\lambda = 0$ for all fixed $\xi \in \mathbb{R}^N$ by setting

$$\Gamma(0,\,\xi)=\frac{1}{2_s^*}\int_{\mathbb{R}^N} K(c^{\frac{1}{2s}}\xi)Q^{2_s^*}(x)\mathrm{d}x:=e_0K(c^{\frac{1}{2s}}\xi),$$

where

$$e_0=\frac{1}{2_s^*}\int_{\mathbb{R}^N}Q^{2_s^*}(x)\mathrm{d}x.$$

Moreover, we have

$$\frac{\partial\Gamma(\lambda,\xi)}{\partial\lambda} = \frac{1}{2_s^*} \int_{\mathbb{R}^N} (\nabla K(c^{\frac{1}{2s}}(\lambda x + \xi)) \cdot c^{\frac{1}{2s}}x) Q^{2_s^*}(x) \mathrm{d}x.$$

Since

 $\int_{\mathbb{R}^N} x_i Q^{2^*_s}(x) \mathrm{d}x = 0, \quad \text{for all } i = 1, 2, \dots, N,$

then

$$\frac{\partial\Gamma(0,\xi)}{\partial\lambda} = 0. \tag{4.2}$$

As a consequence, we can further extend Γ by symmetry to \mathbb{R}^{N+1} as a C^1 function. We will use the same symbol Γ for such a function. Moreover, from (4.2), we can find that

$$\boldsymbol{\xi} \in \operatorname{Crit}(K) \Leftrightarrow \left(0, \, \boldsymbol{\xi} c^{-\frac{1}{2s}}\right) \in \operatorname{Crit}(\Gamma).$$
(4.3)

In the sequel, we are going to introduce some lemmas which is crucial in studying the finite dimensional functional $\Gamma(\lambda, \xi)$.

Lemma 4.3. Suppose that (K_3) holds. Then there exists R > 0 such that

 $(\nabla\Gamma(\lambda, \xi) \cdot (\lambda, \xi)) < 0$ for any $|\lambda| + |\xi| \ge R$.

Proof. Let $z = (\lambda, \xi)$, and then

$$2_{\mathcal{S}}^{*}(\nabla_{z}\Gamma(z)\cdot z) = \int_{\mathbb{R}^{N}} (\nabla K(c^{\frac{1}{2s}}(\lambda y + \xi))\cdot c^{\frac{1}{2s}}(\lambda y + \xi))Q^{2*}(y)dy$$
$$= c^{-\frac{N}{2s}}\lambda^{-N} \int_{\mathbb{R}^{N}} (\nabla K(x)\cdot x)Q^{2*}\left(\frac{c^{-\frac{1}{2s}}x - \xi}{\lambda}\right)dx$$
$$= c^{-\frac{N}{2s}}\lambda^{-N}(J_{1,R} + J_{2,R}),$$

where

$$J_{1,R} = \int_{B_R(0)} (\nabla K(x) \cdot x) Q^{2^*_s} \left(\frac{C^{-\frac{1}{2s}} x - \xi}{\lambda}\right) dx,$$
$$J_{2,R} = \int_{\mathbb{R}^N \setminus B_R(0)} (\nabla K(x) \cdot x) Q^{2^*_s} \left(\frac{C^{-\frac{1}{2s}} x - \xi}{\lambda}\right) dx.$$

From $(K_3)(ii)$, we can deduce that there exists R_0 such that for each $R \ge R_0$,

$$I(R) \coloneqq \int_{B_R(0)} (\nabla K(x) \cdot x) \mathrm{d}x < 0.$$
(4.4)

Let $f(x) = (\nabla K(x) \cdot x)$. We define $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$,

$$\max(\lambda, \xi) \coloneqq \max_{x \in B_R(0)} Q^{2^*_s} \left(\frac{C^{-\frac{1}{2s}\chi} - \xi}{\lambda} \right),$$
$$\min(\lambda, \xi) \coloneqq \min_{x \in B_R(0)} Q^{2^*_s} \left(\frac{C^{-\frac{1}{2s}\chi} - \xi}{\lambda} \right).$$

By a direct calculation, we can prove that

$$J_{1,R} = \int_{B_{R}(0)} f^{+}(x) Q^{2_{s}^{*}} \left(\frac{c^{-\frac{1}{2s}} x - \xi}{\lambda} \right) dx - \int_{B_{R}(0)} f^{-}(x) Q^{2_{s}^{*}} \left(\frac{c^{-\frac{1}{2s}} x - \xi}{\lambda} \right) dx$$

$$\leq \max(\lambda, \xi) \cdot \int_{B_{R}(0)} f^{+}(x) dx - \min(\lambda, \xi) \cdot \int_{B_{R}(0)} f^{-}(x) dx.$$
(4.5)

If $|\lambda| + |\xi|$ is large enough, then we have

$$\max(\lambda, \xi) \simeq \frac{C_N \lambda^{2N}}{\left(\lambda^2 + \left(\frac{R}{C^{\frac{1}{2s}}} - |\xi|\right)^2\right)^N},$$
$$\min(\lambda, \xi) \simeq \frac{C_N \lambda^{2N}}{\left(\lambda^2 + \left(\frac{R}{C^{\frac{1}{2s}}} + |\xi|\right)^2\right)^N},$$

thus

$$\lim_{|\lambda|+|\xi|\to\infty}\frac{\max(\lambda,\xi)}{\min(\lambda,\xi)} = 1.$$
(4.6)

Furthermore, by (4.4)–(4.6), we know that $J_{1,R} < 0$ provided $|\lambda| + |\xi|$ is large enough. Same as the proof above, from (K_3)(i), we can choose sufficient large $R > \rho$ such that $J_{2,R} < 0$.

Lemma 4.4. Assume that (K_2) holds. Then

$$\lim_{\lambda \to 0^+} \frac{\Gamma\left(\lambda, \, \xi c^{-\frac{1}{2s}}\right) - \Gamma\left(0, \, \xi c^{-\frac{1}{2s}}\right)}{\lambda^{\beta}} = A_{\xi},\tag{4.7}$$

where

$$A_{\xi} = \frac{c^{\frac{\beta}{2}}}{2_{s}^{*}} \sum_{j=1}^{N} a_{j}(\xi) \int_{\mathbb{R}^{N}} |x_{1}|^{\beta} Q^{2_{s}^{*}}(x) dx.$$

Proof. Note that if $\beta < N$, then $|x|^{\beta}Q^{2^*_s}(x) \in L^1(\mathbb{R}^N)$. Thus, from (K_2), we have

$$\lim_{\lambda \to 0^{+}} \frac{\Gamma\left(\lambda, \xi c^{-\frac{1}{2s}}\right) - \Gamma\left(0, \xi c^{-\frac{1}{2s}}\right)}{\lambda^{\beta}} = \lim_{\lambda \to 0^{+}} \frac{1}{2_{s}^{*}} \int_{\mathbb{R}^{N}} \frac{(K(c^{\frac{1}{2s}}\lambda x + \xi) - K(\xi))Q^{2*}(x)}{\lambda^{\beta}} dx$$
$$= \frac{1}{2_{s}^{*}} \int_{\mathbb{R}^{N}} \sum_{j=1}^{N} a_{j}(\xi) \left| c^{\frac{1}{2s}}x_{j} \right|^{\beta} Q^{2*}(x) dx$$
$$= \frac{c^{\frac{\beta}{2}}}{2_{s}^{*}} \sum_{j=1}^{N} a_{j}(\xi) \int_{\mathbb{R}^{N}} |x_{j}|^{\beta} Q^{2*}(x) dx$$
$$= \frac{c^{\frac{\beta}{2}}}{2_{s}^{*}} \sum_{j=1}^{N} a_{j}(\xi) \int_{\mathbb{R}^{N}} |x_{j}|^{\beta} Q^{2*}(x) dx.$$

Lemma 4.5. Let $\xi \in \text{Crit}(K)$ be isolated and suppose (K_2) holds. Then $z = (0, \xi c^{-\frac{1}{2s}})$ is an isolated critical point of Γ and the following properties hold:

$$\deg_{\text{loc}}(\nabla\Gamma, z) = \deg_{\text{loc}}(\nabla K, \xi), \quad if \sum_{j=1}^{N} a_j(\xi) > 0,$$
$$\deg_{\text{loc}}(\nabla\Gamma, z) = -\deg_{\text{loc}}(\nabla K, \xi), \quad if \sum_{j=1}^{N} a_j(\xi) < 0.$$

Proof. By (4.3), we know that *z* is a critical point of Γ . Since $a_j(x) \in C(\mathbb{R}^N)$ and $\sum_{j=1}^N a_j(\xi) \neq 0$, then there exist $\delta > 0$ such that for any $y \in B_{\delta}(\xi)$, we have $\sum_{j=1}^N a(y) \neq 0$. From (4.7), we can conclude $\Gamma(\lambda, yc^{-\frac{1}{2s}}) \sim \Gamma(0, yc^{-\frac{1}{2s}}) + A_v \lambda^{\varepsilon}$ for $y \in B_{\delta}(\xi)$, which together with isolated property of ξ imply that *z* is an isolated critical point of Γ .

Let $L_{\delta} = [-\delta, \delta] \times B_{\delta}(\xi c^{-\frac{1}{2s}})$. For $\delta > 0$ small the degree deg $(\nabla \Gamma, L_{\delta}, 0)$ is well defined, then by property of multiplicative yields to

$$\deg(\nabla\Gamma, L_{\delta}, 0) = \deg_{loc}(\nabla K, \xi) \cdot \deg_{loc}\left(\frac{\partial\Gamma}{\partial\lambda}, 0\right).$$
(4.8)

By (4.7) again, then we conclude that

$$deg_{loc}\left(\frac{\partial\Gamma}{\partial\lambda}, 0\right) = 1, \quad \text{if } \sum_{j=1}^{N} a_j(\xi) > 0,$$

$$deg_{loc}\left(\frac{\partial\Gamma}{\partial\lambda}, 0\right) = -1, \quad \text{if } \sum_{j=1}^{N} a_j(\xi) < 0.$$
(4.9)

Thus, by (4.8) and (4.9), we can prove this conclusion.

4.3 Proof of Theorems 1.3 and 1.4

Let

$$C^+ = \{(\lambda, \xi) | (\lambda, \xi) \in \operatorname{Crit}(\Gamma), \lambda > 0\}.$$

According to Lemmas 4.3 and 4.4, C^+ is a (possibly empty) compact set. Since the extended Γ is even in λ , then

$$C^{-} = \{(-\lambda, \xi) | (\lambda, \xi) \in C^+\}$$

is also a critical point set of Γ . Denote $C = C^+ \cup C^-$. We claim that there is a bounded open set $\Omega \in (0, \infty) \times \mathbb{R}^N$ with $C^+ \in \Omega$ such that

$$\deg(\nabla\Gamma, \Omega, 0) \neq 0.$$

Argue by contradiction, we may assume that there is an open bounded set Ω with $C^+ \subset \Omega$ such that

$$\deg(\nabla\Gamma, \Omega, 0) = 0.$$

Using Lemma 4.3, we can prove that

$$\deg(\nabla\Gamma, B_R^{N+1}(0), 0) = (-1)^{N+1}.$$

Let $\Omega^- = \{(-\lambda, \xi) : (\lambda, \xi) \in \Omega\}$ and set $\Omega' = \Omega \cup \Omega^-$. Of course, one has that

$$\deg(\nabla\Gamma, \Omega^-, 0) = 0$$

and hence

$$\deg(\nabla\Gamma, B_R^{N+1} \setminus \overline{\Omega}', 0) = (-1)^{N+1}$$

where $\overline{\Omega}'$ denotes the closure of Ω' . According to (4.3) any $z \in \operatorname{Crit}(\Gamma) \setminus C$ must be of the form $z = (0, \xi c^{-\frac{1}{2s}})$ with $\xi \in \operatorname{Crit}(K)$. Using Lemma 4.5, then we can deduce that

$$\deg(\nabla\Gamma, B_R^{N+1} \setminus \overline{\Omega}', 0) = \sum \deg_{loc}(\nabla\Gamma, z) = \sum_{a_{\xi} > 0} \deg_{loc}(\nabla K, \xi) - \sum_{a_{\xi} < 0} \deg_{loc}(\nabla K, \xi).$$

Thus, one has

$$\sum_{a_{\xi}>0} \deg_{\operatorname{loc}}(\nabla K, \xi) - \sum_{a_{\xi}<0} \deg_{\operatorname{loc}}(\nabla K, \xi) = (-1)^{N+1}.$$

Let $R \ge \rho$, where ρ is defined as (K_3). Since ($\nabla K \cdot x$) < 0 for all $|x| = R(R \ge \rho)$, then we have

$$\deg(\nabla K, B_R^N(0), 0) = (-1)^N$$

which combining with the properties of topological degree yield to

$$\sum_{\xi \in \operatorname{Crit}(K)} \deg_{\operatorname{loc}}(\nabla K, \xi) = (-1)^N.$$

Moreover, since $a_{\xi} \coloneqq \sum_{i=1}^{N} a_i(\xi) \neq 0$ for all $\xi \in \text{Crit}(K)$, then

$$\sum_{a_{\xi}>0} \deg_{\mathrm{loc}}(\nabla K, \xi) + \sum_{a_{\xi}<0} \deg_{\mathrm{loc}}(\nabla K, \xi) = (-1)^{N}.$$

Therefore, we obtain

$$\sum_{a_{\xi}<0} \deg_{\mathrm{loc}}(\nabla K,\,\boldsymbol{\xi}) = (-1)^N,$$

which contradicts to the assumption (K_4). Then the claim is true and we can prove Theorems 1.3 and 1.4 by Theorem 4.1.

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