On fractional Schrödinger-Poisson system

Zhipeng Yang

Georg-August Universität Göttingen
Mathematisches Institut

Janurary 11, 2020

GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

Recall RTG lecture

For any $\alpha \in (0,1)$, the fractional Sobolev space $H^{\alpha}(\mathbb{R}^3) = W^{\alpha,2}(\mathbb{R}^3)$ is defined as follows:

$$H^{\alpha}(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\xi|^{2\alpha}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2) d\xi < \infty\},$$

whose norm is defined as

$$\|u\|^2_{H^{\alpha}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \left(|\xi|^{2\alpha}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2\right) d\xi.$$

We also define the homogeneous fractional Sobolev space $\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ as the completion of $\mathcal{C}_0^{\infty}(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} := \left(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2\alpha}} dxdy \right)^{\frac{1}{2}}.$$

Recall RTG lecture

The fractional Laplacian $(-\Delta)^\alpha$ is defined by

$$\mathcal{F}((-\Delta)^\alpha u)(\xi) = |\xi|^{2\alpha}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^3,$$

for functions ϕ in the Schwartz class. $(-\Delta)^\alpha u$ can be presented as

$$(-\Delta)^\alpha u(x) = -\frac{1}{2}C(\alpha)\int_{\mathbb{R}^3} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{3+2\alpha}}dy, \ \forall x \in \mathbb{R}^3.$$

My previous works

By the Plancherel Theorem, the norms on $H^\alpha(\mathbb{R}^3)$ defined above

$$u \longmapsto \Big(\int_{\mathbb{R}^3} |u|^2 dx + \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}} dxdy\Big)^{\frac{1}{2}};$$

$$u \longmapsto \Big(\int_{\mathbb{R}^3} (|\xi|^{2\alpha}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2) d\xi\Big)^{\frac{1}{2}};$$

$$u \longmapsto \Big(\int_{\mathbb{R}^3} |u|^2 dx + \|(-\Delta)^{\frac{\alpha}{2}} u\|_2^2\Big)^{\frac{1}{2}}$$

are equivalent.

My previous works

$$\lim_{s \to 1^-} (-\Delta)^s u = -\Delta u \qquad \lim_{s \to 0^+} (-\Delta)^s u = u.$$

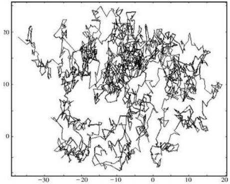


Fig. 2. An illustration of Brownian motion which corresponds to normal diffusion. It is obtained by iterating any random walk with identically independently distributed elementary steps having finite variance.

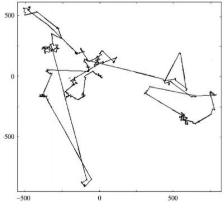


Fig. 3. This is the pattern of Lévy flight shown for the Lévy index $\alpha = 1.5$. Contrary to Brownian motion, the variance and any moment of order μ, $\mu \geq \alpha$ are infinite. As a consequence, any Lévy flight path is almost surely not continuous, and the figure indeed displays jumps.

Brownian motion: 2nd order PDE Lèvy processes: fractional PDE

px

Fractional Schrödinger-Poisson system

An interesting class of Schrödinger equations arises when it describes quantum (nonrelativistic) particles interacting with the electromagnetic field generated by the motion. That is a nonlinear fractional Schrödinger- Poisson system (also called fractional Schrödinger-Maxwell system)

$$\begin{cases} i\varepsilon \frac{\partial \Psi}{\partial t} = \hbar^{2s}(-\Delta)^s \Psi + V(x)\Psi + \mu\phi\Psi - f(|\Psi|)\Psi & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ \varepsilon^{2t}(-\Delta)^t \phi = |\Psi|^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where i is the imaginary unit, $\varepsilon > 0$ is associate to the Planck constant.

px

Fractional Schrödinger-Poisson system

$(e^{-iEt/\hbar}u(x), \phi(x))$ is a standing wave solution of (1) iff $(u(x), \phi(x))$ satisfies the following fractional Schrödinger-Poisson system:

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ \varepsilon^{2t}(-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{2}$$

When $s = t = 1$, this is the famous Schrödinger-Poisson system

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\varepsilon^2 \Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{3}$$

Many previous known results: P.L. Lions, CMP. 1984 (Hartree-Fock equation).

Fractional Schrödinger-Poisson system

In the following, we study the fractional Schrödinger-Poisson system with critical nonlinearity :

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = \lambda |u|^{p-2}u + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ \varepsilon^{2t}(-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{4}$$

where $2_s^* = \frac{6}{3-2s}$, $V \in C$ satisfies the following conditions

(V_1) $0 < \inf\limits_{x \in \mathbb{R}^3} V(x)$.

(V_2) There is a bounded domain Ω such that

$$V_0 := \inf_{\Omega} V(x) < \min_{\partial\Omega} V.$$

Work space

Making the change of variable $x \mapsto \varepsilon x$, we can rewrite the system (4) as the following equivalent system

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x)u + \phi u = \lambda |u|^{p-2}u + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{5}$$

If u is a solution of the system (5), then $v(x) := u(\frac{x}{\varepsilon})$ is a solution of the system (4).

In view of the presence of potential $V(x)$, we introduce the subspace

$$H_\varepsilon = \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx < +\infty \right\},$$

which is a Hilbert space equipped with the norm

$$\|u\|_{H_\varepsilon}^2 = (u, u) = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx.$$

Energy function

System (5) is the Euler-Lagrange equations of the functional $J : H_\varepsilon \times \mathcal{D}^{t,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$J(u,\phi) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\phi|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi u^2 dx \\ - \frac{\lambda}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx,$$

Evidently, the action functional $J \in C^1(H_\varepsilon \times \mathcal{D}^{t,2}(\mathbb{R}^3), \mathbb{R})$ and its critical points are the solutions of (5). The first difficulty is that J exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinitely dimensional subspaces.

The reduction method

For a fixed $u \in H_\varepsilon$, there exists a unique $\phi_u^t \in \mathcal{D}^{t,2}(\mathbb{R}^3)$ which is the solution of

$$(-\Delta)^t \phi = u^2 \ \text{ in } \mathbb{R}^3.$$

We can write an integral expression for ϕ_u^t in the form

$$\phi_u^t(x) = C_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} dy, \quad \forall\, x \in \mathbb{R}^3,$$

which is called t-Riesz potential.

px

Fractional Schrödinger-Poisson system

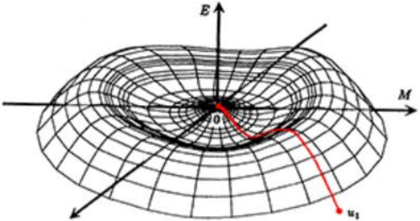
Putting $\phi = \phi_u^t$ into the first equation of (5), we obtain a semi-linear elliptic equation

$$(-\Delta)^s u + V(\varepsilon x)u + \phi_u^t u = \lambda |u|^{p-2}u + |u|^{2_s^*-2}u \ \text{ in } \mathbb{R}^3.$$

The corresponding functional $I : H_\varepsilon \to \mathbb{R}$ is defined by

$$\begin{aligned} I(u) = & \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx \\ & + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx. \end{aligned}$$

Mountain pass theorem



A. Ambrosetti

P.H. Rabinowitz

Pohožaev-Nehari manifold

If $u \in H^s(\mathbb{R}^3)$ is a weak solution to problem (5), then we have the following Pohožaev identity:

$$P(u) = \frac{3-2s}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} Vu^2 dx + \frac{2t+3}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{3\lambda}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{3}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx = 0.$$

We define $G : H^s(\mathbb{R}^3) \to \mathbb{R}$ as

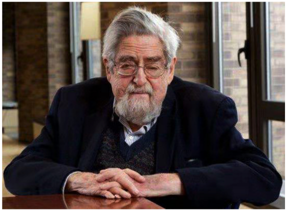
$$G(u) = (s+t)\langle I'(u), u\rangle - P(u)$$

Next we study the functional I restricted on the manifold M defined as

$$M := \{u \in H^s(\mathbb{R}^3) \backslash \{0\} : G(u) = 0\}.$$

Critical problem: $H^s(\mathbb{R}^n) \hookrightarrow L^{2_s^*}(\mathbb{R}^n)$ is not compact.

Haim Brezis



Louis Nirenberg

Positive Solutions of Nonlinear Elliptic Equations Involving Critical Sobolev Exponents

HAÏM BREZIS
Paris VI

AND

LOUIS NIRENBERG
Courant Institute

0. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geqq 3$. We are concerned with the problem of existence of a function u satisfying the nonlinear elliptic equation

$$\begin{aligned} -\Delta u &= u^p + f(x,u) && \text{on } \Omega, \\ u &> 0 && \text{on } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{0.1}$$

where $p = (n+2)/(n-2)$, $f(x,0) = 0$ and $f(x,u)$ is a lower-order perturbation of u^p in the sense that $\lim_{u\to+\infty} f(x,u)/u^p = 0$. A typical example is $f(x,u) = \lambda u$, where λ is a real constant. The exponent $p = (n+2)/(n-2)$ is critical from the viewpoint of Sobolev embedding. Indeed solutions of (0.1) correspond to critical points of the functional

$$\Phi(u) = \tfrac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int |u|^{p+1} - \int F(x,u),$$

where $F(x,u) = \int_0^u f(x,t)\,dt$. Note that $p+1 = 2n/(n-2)$ is the limiting Sobolev exponent for the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$. Since this embedding is *not compact*, the functional Φ *does not satisfy the* (PS) *condition*. Hence there are serious difficulties when trying to find critical points by standard variational methods. In fact, there is a sharp contrast between the case $p < (n+2)/(n-2)$ for which the Sobolev embedding is compact, and the case $p = (n+2)/(n-2)$. Many *existence* results for problem (0.1) are known when $p < (n+2)/(n-2)$ (see the review article by P. L. Lions [20] and the abundant list of references in [20]). On the other hand, a well-known *nonexistence* result of Pohozaev [24]

Communications on Pure and Applied Mathematics, Vol. XXXVI, 437–477 (1983)

CCC 0010-3640/83/040437-41$05.10

Energy classes

For the Mountain-Pass level c for I, the following inequality holds

$$0 < c < \frac{s}{3} S_s^{\frac{3}{2s}},$$

if one of the following conditions is satisfied

(i) $s > \frac{3}{4}$: $p \in \left(\frac{4s}{3-2s}, 2_s^*\right)$ and any $\lambda > 0$;

(ii) $s > \frac{3}{4}$: $p \in \left(\frac{4s+2t}{s+t}, \frac{4s}{3-2s}\right]$ and any $\lambda > 0$ large enough;

(iii) $\frac{1}{2} < s \leq \frac{3}{4}$: $p \in \left(\frac{4s+2t}{s+t}, 2_s^*\right)$ and any $\lambda > 0$,

where

$$S_s := \inf_{u \in \mathcal{D}^{s,2}(\mathbb{R}^3)} \frac{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)}}{\|u\|^2_{2_s^*(\mathbb{R}^3)}}.$$

Our results

Theorem 3.2 (Yang-Yu-Zhao, Commun. Contemp. Math., 2019)

Assume that V satisfies (V_1) and (V_2),

(i) If $p \in (\frac{4s+2t}{s+t}, \frac{4s}{3-2s}]$, then there exist $\varepsilon^* > 0$ and $\lambda^* > 0$ such that for each $\lambda \in [\lambda^*, \infty)$ and $\varepsilon \in (0, \varepsilon^*)$, system (1) possesses a positive ground state solution $(u_\varepsilon, \phi_\varepsilon) \in H^s(\mathbb{R}^3) \times \mathcal{D}^{t,2}(\mathbb{R}^3)$.

(ii) If $p \in (\frac{4s}{3-2s}, 2_s^*)$, then system (1) possesses a positive ground state solution for any $\lambda > 0$.

(iii) If $x_\varepsilon \in \Omega$ is a maximum point of u_ε, then

$$\lim_{\varepsilon \to 0} V(x_\varepsilon) = V_0,$$

and there exists a constant $C > 0$ (independent of ε) such that

$$u_\varepsilon(x) \leq \frac{C\varepsilon^{3+2s}}{\varepsilon^{3+2s} + |x - x_\varepsilon|^{3+2s}}, \ \forall x \in \mathbb{R}^3.$$

px

Thanks for your attention!