# Existence and concentration behavior of solutions for some fractional Schrödinger-Poisson systems

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# Introduction

The Schrödinger equation is a fundamental model in quantum physics and nonlinear optics. The semilinear Schrödinger equation is typically of the form

$$i\varepsilon\frac{\partial\Psi}{\partial t} = -\frac{\varepsilon^2}{2m}\Delta\Psi + V(x)\Psi - |\Psi|^{p-1}\Psi \quad \text{in } \mathbb{R}\times\mathbb{R}^N,$$

where $\varepsilon$ denotes the Plank constant, $i$ is the imaginary unit, $1 < p \leq \frac{N+2}{N-2}$ if $N \geq 2$ and $1 < p < \infty$ if $N = 1, 2$.

In physical problems, $p = 3$ leads to the Gross-Pitaevskii equation.

# Introduction

Laskin (Laskin, 2000, Phys. Lett. A) introduced the following equation

$$i\varepsilon\frac{\partial\Psi}{\partial t}=\varepsilon^{2s}(-\Delta)^s\Psi+V(x)\Psi-f(|\Psi|)\Psi \quad \text{in } \mathbb{R}\times\mathbb{R}^N,$$

which appeares in fractional Quantum Mechanics and comes from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. The space derivative in this equation is of fractional (noninteger) order $s$. So the equation is called the fractional Schrödinger equation. See Fractional quantum mechanics, 2018, World Scientific Publishing.

The standing wave solution of the nonlinear fractional Schrödinger equation is of the form $\Psi(t,x)=e^{iEt/\varepsilon}u(x)$, where $u(x)$ satisfies

$$\varepsilon^{2s}(-\Delta)^s u+V(x)u=f(x,u), \quad x\in\mathbb{R}^N.$$

# Introduction

From other physical point of view, nonlocal operators play a crucial rule in describing several different physical phenomena.

- the anomalous diffusion (R. Metzler and J. Klafter, Phys. Rep.,2000; S. Abe and S. Thurner, Physica A., 2005);
- the dynamics of the dislocation of atoms in crystals (S. Dipierro, G. Palatucci and E. Valdinoci, Comm. Math. Phys., 2015);
- Vázquez (Book, Nonlinear Partial Differential Equations, Abel Symp. 2012 ) describes two models of flow in porous media, including nonlocal diffusion effects.

# Fractional Schrödinger-Poisson system

An interesting Schrödinger equation class is when it describes non-relativistic particles interacting with the electromagnetic field generated by the motion. That is a nonlinear fractional Schrödinger-Poisson system (also called fractional Schrödinger-Maxwell system)

$$\begin{cases} i\varepsilon\frac{\partial\Psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s\Psi + V(x)\Psi + \phi\Psi - f(|\Psi|)\Psi & \text{in } \mathbb{R}^3\times\mathbb{R}, \\ \varepsilon^{2t}(-\Delta)^t\phi = |\Psi|^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.1}$$

$(e^{-iEt/\varepsilon}u(x), \phi(x))$ is a standing wave solution of (2.1) iff $(u(x), \phi(x))$ satisfies the following fractional Schrödinger-Poisson system:

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ \varepsilon^{2t}(-\Delta)^t\phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.2}$$

# Introduction

- J. Zhang, J. M. do ó and M. Squassina, Adv. Nonlinear Stud. 2016 considered the following problem

$$\begin{cases} (-\Delta)^s u + \lambda\phi u = f(u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = \lambda u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

  where $\lambda > 0$, $s, t \in [0, 1]$ with $2t + 4s \geq 3$. For the sub-crtical and critical case, under the Berestycki-Lions type conditions, the existence of a positive radial solution $(u_\lambda, \phi_\lambda)$ was obtained for $\lambda > 0$ small and $(u_\lambda, \phi_\lambda)$ converges to $(u, 0)$ in $H^s \times D^{s,2}$, where $u$ is a ground state solution of

$$(-\Delta)^s u = f(u) \text{ in } \mathbb{R}^3.$$

# Introduction

- K. Teng, JDE 2016 considered the following problem

$$\begin{cases}(-\Delta)^s u + V(x)u + \phi u = \mu|u|^{q-1}u + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3,\\ (-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3.\end{cases}$$

where $\mu > 0$, $s, t \in (0,1)$ with $2t + 2s > 3$, $1 < q < 2_s^* - 1$. Under certain assumptions on $V(x)$, using the method of Nehari-Pohozaev manifold and the arguments of Brezis-Nirenberg, the monotonic trick and global compactness Lemma, Teng obtained a nontrivial ground state solution. This generalized the results in L. Zhao and Zhao, NA, 2009.

# Introduction

- Z. Liu and J. Zhang, ESAIM-COCV, 2018 considered

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = f(u) + |u|^{2_s^*-2}u, \text{ in } \mathbb{R}^3, \\ \varepsilon^{2t}(-\Delta)^t \phi = u^2, \text{ in } \mathbb{R}^3, \end{cases}$$

where $f, V \in C^1$, $f$ is subcritical and $V$ satisfies a global condition due to Rabinowitz, 1992, ZAMP

$$(GC)\quad 0 < \inf_{x\in\mathbb{R}^3} V(x) < \liminf_{|x|\to\infty} V(x).$$

They obtained a positive ground state solution for $\varepsilon$ small and a multiplicity result related to the categary of the minima set of $V$, and showed that these solutions concentrate around a global minimum point of $V$.

# Introduction

In Y. Yu, F. Zhao and L. Zhao, CVPDE, 2017, they consider

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = K(x)|u|^{p-2}u, \text{ in } \mathbb{R}^3, \\ \varepsilon^{2s}(-\Delta)^s \phi = u^2, \text{ in } \mathbb{R}^3, \end{cases}$$

where $\frac{3}{4} < s < 1$, $4 < p < 2_s^* := \frac{6}{3-2s}$, $V(x) \in C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ has positive global minimum, and $K(x) \in C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is positive and has global maximum.

- Existence of a a positive ground state solution.
- Concentration set of ground state solutions as $\varepsilon \to 0$.
- Convergence and decay estimate of ground state solutions as $\varepsilon \to 0$.

# Fractional Schrödinger-Poisson system

In the following, we study the fractional Schrödinger-Poisson system with critical nonlinearity :

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = \lambda |u|^{p-2}u + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ \varepsilon^{2t}(-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{2.3}$$

where $2_s^* = \frac{6}{3-2s}$, $V \in C$ satisfies the following conditions

$(V_1)$ $0 < \inf_{x \in \mathbb{R}^3} V(x)$.

$(V_2)$ There is a bounded domain $\Omega$ such that

$$V_0 := \inf_{\Omega} V(x) < \min_{\partial\Omega} V.$$

# Fractional Schrödinger-Poisson system

Making the change of variable $x \mapsto \varepsilon x$, we can rewrite the system (2.3) as the following equivalent system

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x)u + \phi u = \lambda |u|^{p-2}u + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.4}$$

If $u$ is a solution of the system (2.4), then $v(x) := u(\frac{x}{\varepsilon})$ is a solution of the system (2.3).

In view of the presence of potential $V(x)$, we introduce the subspace

$$H_\varepsilon = \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx < +\infty \right\},$$

which is a Hilbert space equipped with the norm

$$\|u\|_{H_\varepsilon}^2 = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx.$$

# Fractional Schrödinger-Poisson system

System (2.4) is the Euler-Lagrange equations of the functional $J : H_\varepsilon \times \mathcal{D}^{t,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$J(u,\phi) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\phi|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi u^2 dx$$
$$- \frac{\lambda}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx,$$

where $\mathcal{D}^{t,2}(\mathbb{R}^3)$ is the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{t,2}(\mathbb{R}^3)} := \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}} dxdy\right)^{\frac{1}{2}}.$$

Evidently, the action functional $J \in C^1(H_\varepsilon \times \mathcal{D}^{t,2}(\mathbb{R}^3), \mathbb{R})$ and the solutions of (2.4) are its critical points. The first difficulty is that $J$ exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinitely dimensional subspaces.

# Fractional Schrödinger-Poisson system

For a fixed $u \in H_\varepsilon$ , there exists a unique $\phi_u^t \in \mathcal{D}^{t,2}(\mathbb{R}^3)$ which is the solution of

$$(-\Delta)^t \phi = u^2 \ \text{ in } \mathbb{R}^3.$$

We can write an integral expression for $\phi_u^t$ in the form of a $t$-Riesz potential

$$\phi_u^t(x) = C_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} dy, \quad \forall\, x \in \mathbb{R}^3.$$

# Fractional Schrödinger-Poisson system

Putting $\phi = \phi_u^t$ into the first equation of (2.4), we obtain a semilinear elliptic equation

$$(-\Delta)^s u + V(\varepsilon x)u + \phi_u^t u = \lambda |u|^{p-2}u + |u|^{2_s^*-2}u \ \text{ in } \mathbb{R}^3,$$

with a nonlocal term. The corresponding functional $I : H_\varepsilon \to \mathbb{R}$ is defined by

$$\begin{aligned} I(u) = & \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx \\ & + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx. \end{aligned}$$

# Difficulties

- Non-compactness due to the unboundedness of the domain and $2_s^*$ ,
- If $p \in (\frac{4s+2t}{s+t}, 2_s^*)$, the nonlinearity $\lambda|u|^{p-2}u+|u|^{2_s^*-2}u$ does not satisfy Ambrosetti-Rabinowitz condition,
- If $p \in (\frac{4s+2t}{s+t}, 2_s^*)$, the function $\frac{\lambda|u|^{p-1}+|u|^{2_s^*-1}}{u^3}$ is not increasing for $u > 0$.

# Fractional Schrödinger-Poisson system

If $u \in H^s(\mathbb{R}^3)$ is a weak solution to problem (2.4), then we have the following Pohožaev identity:

$$P(u) = \frac{3-2s}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} Vu^2 dx + \frac{2t+3}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{3\lambda}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{3}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx = 0.$$

We define $G : H^s(\mathbb{R}^3) \to \mathbb{R}$ as

$$G(u) = (s+t)\langle I'(u), u\rangle - P(u).$$

Next we study the functional $I$ restricted to the manifold $M$ defined as

$$M := \{u \in H^s(\mathbb{R}^3)\backslash\{0\} : G(u) = 0\}.$$

# Our results

Define the Mountain-Pass level of $I$:

$$c := \inf_{\gamma\in\Gamma} \sup_{h\in[0,1]} I(\gamma(h)), c^* = \inf_{u\in H^s(\mathbb{R}^3)\setminus\{0\}} \max_{\theta>0} I(u_\theta), \ c^{**} = \inf_{u\in M} I(u).$$

where the set of paths is defined as

$$\Gamma := \{\gamma \in C([0,1], H^s(\mathbb{R}^3)) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}.$$

## Lemma

- There exists a sequence $\{u_n\} \in H^s(\mathbb{R}^3)$ such that

$$I(u_n) \to c, \quad I'(u_n) \to 0, \quad G(u_n) \to 0, \quad as\ n \to \infty.$$

- The following equalities hold

$$c = c^* = c^{**} > 0.$$

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# Our results

## Lemma (Yang-Yu-Zhao, Commun. Contemp. Math., 2019)

For the Mountain-Pass level $c$ for $I$, the following inequality holds

$$0 < c < \frac{s}{3} S_s^{\frac{3}{2s}},$$

if one of the following conditions is satisfied

(1) $s > \frac{3}{4} : p \in (\frac{4s}{3-2s}, 2_s^*)$ and any $\lambda > 0$,

(2) $s > \frac{3}{4} : p \in (\frac{4s+2t}{s+t}, \frac{4s}{3-2s}]$ and any $\lambda > 0$ large enough,

(3) $\frac{1}{2} < s \leq \frac{3}{4} : p \in (\frac{4s+2t}{s+t}, 2_s^*)$ and any $\lambda > 0$,

where

$$S_s := \inf_{u \in \mathcal{D}^{s,2}(\mathbb{R}^3)} \frac{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)}}{\|u\|^2_{2_s^*(\mathbb{R}^3)}}.$$

# Our results

## Theorem (Yang-Yu-Zhao, Commun. Contemp. Math., 2019)

Assume that $V$ satisfies $(V_1)$ and $(V_2)$,
$(1)$ If $p \in (\frac{4s+2t}{s+t}, \frac{4s}{3-2s}]$, then there exist $\varepsilon^* > 0$ and $\lambda^* > 0$ such that for each $\lambda \in [\lambda^*, \infty)$ and $\varepsilon \in (0, \varepsilon^*)$, system (2.3) possesses a positive ground state solution $(u_\varepsilon, \phi_\varepsilon) \in H^s(\mathbb{R}^3) \times \mathcal{D}^{t,2}(\mathbb{R}^3)$.
$(2)$ If $p \in (\frac{4s}{3-2s}, 2_s^*)$ , then system (2.3) possesses a positive ground state solution for any $\lambda > 0$.
$(3)$ If $x_\varepsilon \in \Omega$ is a maximum point of $u_\varepsilon$, then

$$\lim_{\varepsilon \to 0} V(x_\varepsilon) = V_0,$$

and there exists a constant $C > 0$ (independent of $\varepsilon$) such that

$$u_\varepsilon(x) \leq \frac{C\varepsilon^{3+2s}}{\varepsilon^{3+2s} + |x - x_\varepsilon|^{3+2s}}, \ \forall x \in \mathbb{R}^3.$$

*Thanks for your attention!*