Harmonic analysis on 2-step stratified Lie groups without the Moore-Wolf condition

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Nilpotent Lie groups

Graded Lie groups

- A Lie algebra $\mathfrak{g}$ of step r is called graded if it is endowed with a vector space decomposition

$$\mathfrak{g} = \oplus_{j=1}^{r} V_j \quad \text{such that} \quad [V_i, V_j] \subseteq V_{i+j}.$$

- A Lie group is called graded if it is a connected simply-connected Lie group whose Lie algebra is graded.

Nilpotent Lie groups

Stratified Lie groups

- A graded Lie algebra $\mathfrak{g}$ of step r is called stratified if V_1 generates $\mathfrak{g}$ as an algebra. In this case, we have

$$\mathfrak{g} = \oplus_{j=1}^{r} V_j, \quad [V_j, V_1] = V_{j+1}.$$

  The natural dilations of $\mathfrak{g}$ are given by

$$\delta_\lambda \left(\sum_{k=1}^{r} X_k \right) = \sum_{k=1}^{r} \lambda^k X_k, \quad (X_k \in V_k).$$

- A Lie group is called stratified if it is a connected simply-connected Lie group whose Lie algebra is stratified.

Nilpotent Lie groups

The sub-Laplacians on stratified Lie groups

Let $X_1, \ldots, X_n$ be a basis of V_1. Then the second-order differential operator

$$\mathcal{L} = \sum_{j=1}^{n} X_j^2$$

is called a sub-Laplacian on $\mathbb{G}$. The vector-valued operator $\nabla_{\mathcal{L}} = (X_1, \ldots, X_n)$ is the $\mathcal{L}$-gradient (or horizontal $\mathcal{L}$-gradient).

Some properties

- $\mathcal{L}$ is hypoelliptic.
- $\mathcal{L}$ is invariant with respect to left translations on $\mathbb{G}$.
- $\mathcal{L}$ is homogeneous of degree two.

The Heisenberg Group

Let us consider in $\mathbb{C}^n \times \mathbb{R} = \mathbb{R}^{2n+1}$:

$$(z,t) \equiv (x,y,t) = (x_1,\ldots,x_n,y_1,\ldots,y_n,t)$$

with $z = (z_1,\ldots,z_n)\,, z_j = x_j + iy_j$ and $x_j, y_j, t \in \mathbb{R}$. Then, the composition law $\circ$ can be explicitly written as

$$(x,y,t)\circ\left(x',y',t'\right) = \left(x+x', y+y', t+t'+2\left\langle y,x'\right\rangle - 2\left\langle x,y'\right\rangle\right),$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^n$. Let us now consider the dilations

$$\delta_\lambda : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}, \quad \delta_\lambda(z,t) = \left(\lambda z, \lambda^2 t\right).$$

Then $\mathbb{H}^n = \left(\mathbb{R}^{2n+1}, 0, \delta_\lambda\right)$ is called the Heisenberg group in $\mathbb{R}^{2n+1}$.

Heisenberg-type group

Consider the homogeneous Lie group

$$\mathbb{H} = \left(\mathbb{R}^{n+m}, \circ, \delta_\lambda\right)$$

with composition law as

$$(x,t)\circ(\xi,\tau) = \left(x+\xi, t_1+\tau_1+\frac{1}{2}\left\langle B^{(1)}x,\xi\right\rangle, \ldots, t_m+\tau_m+\frac{1}{2}\left\langle B^{(m)}x,\xi\right\rangle\right.$$

where $B^{(1)}, \ldots, B^{(m)}$ are fixed $n\times n$ matrices with the following properties:

(1) $B^{(j)}$ is skew-symmetric and orthogonal for every $j \leq m$;

(2) $B^{(i)}B^{(j)} = -B^{(j)}B^{(i)}$ for every $i, j \in \{1, \ldots, m\}$ with $i \neq j$.

If all these conditions are satisfied, $\mathbb{H}$ is called a group of Heisenberg-type, in short, a H-type group.

Métivier group

Definition (G. Métivier, Duke Math.J., 1980)

Let $\mathfrak{g}$ be a (finite-dimensional) Lie algebra, and let us denote by $\mathfrak{z}$ its center. We say that $\mathfrak{g}$ is a Métivier Lie algebra if it admits a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad ([\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2 \subseteq \mathfrak{z}, \quad [\mathfrak{g}_1, \mathfrak{g}_2] = \{0\})$$

with the following additional property: for every $\eta \in \mathfrak{g}_2^*$, the skew-symmetric bilinear form on $\mathfrak{g}_1$ defined by

$$B_\eta : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{R}, \quad B_\eta\left(X, X'\right) := \eta\left(\left[X, X'\right]\right)$$

is non-degenerate.

Following D. Müller and F. Ricci. Ann. of Math. 1996, we call this 2-step nilpotent Lie algebra, Moore-Wolf algebra (MW in short).

H-type group $\subsetneq$ Métivier group

For example, consider the group on $\mathbb{R}^5$ (points are denoted by $(x,t), x \in \mathbb{R}^4$, $t \in \mathbb{R}$) with the composition law

$$(x,t) \circ (\xi,\tau) = \left(x + \xi, t + \tau + \frac{1}{2}\langle Bx, \xi\rangle\right),$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}.$$

Then $\mathbb{G}$ is a Métivier group, for B is a non-singular skew-symmetric matrix. But $\mathbb{G}$ is not a H-type group, for B is not orthogonal.

graded $\supset$ stratified $\supset$ Métivier $\supset$ H-type $\supset$ Heisenberg

2-step stratified Lie group without MW condition

We will study a special 2-step stratified Lie groups, we are going to assume that the Lie algebra $\mathfrak{g}$ decomposes into subspaces

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

with $\dim \mathfrak{g}_1 = n, \dim \mathfrak{g}_2 = m$ and

$$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_2 \subseteq \mathfrak{z} = \text{the center of } \mathfrak{g}.$$

Then, there exists a bilinear, antisymmetric map

$$\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$$

such that, for $Z, Z' \in \mathbb{R}^n$ and $t, t' \in \mathbb{R}^m$,

$$[(Z,t),(Z',t')] = (0, \sigma(Z,Z')).$$

It follows that

$$(Z,t)\cdot(Z',t') = \left(Z + Z', t + t' + \frac{1}{2}\sigma\left(Z,Z'\right)\right).$$

2-step stratified Lie group without MW condition

Fix a basis $\mathcal{B} = \{X_1, X_2 \cdots, X_n, X_{n+1}, \cdots, X_{n+m}\}$ so that $\mathfrak{g}_1 = \mathrm{span}_{\mathbb{R}}\{X_1, X_2 \cdots, X_n\}$ and $\mathfrak{g}_2 = \mathrm{span}_{\mathbb{R}}\{X_{n+1}, \cdots, X_m\}$. Since $\mathfrak{g}$ is nilpotent the exponential map is an analytic diffeomorphism. We can identify $\mathbb{G}$ with $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and write (X, T) for $\exp(X + T)$, where $X \in \mathfrak{v}$ and $T \in \mathfrak{z}$.

For any $\lambda \in \mathfrak{z}^*$, we define a skew-symmetric bilinear form on $\mathfrak{g}_1$ by

$$B^{(\lambda)}(X, Y) := \lambda([X, Y]) \quad \text{for all } X, Y \in \mathfrak{g}_1.$$

One can find a Zariski-open subset Λ of $\mathfrak{z}^\star$ such that the number of distinct eigenvalues of $B^{(\lambda)}$ is maximal. We denote by k the dimension of the radical $\mathfrak{r}_\lambda$ of $B^{(\lambda)}$. Since $B^{(\lambda)}$ is skew-symmetric, the dimension of the orthogonal complement of $\mathfrak{r}_\lambda$ in $\mathfrak{g}_1$ is even, which we shall denote by $2d$.

2-step stratified Lie group without MW condition

Therefore, there exists an orthonormal basis

$$(X_1(\lambda), \dots, X_d(\lambda), Y_1(\lambda), \dots, Y_d(\lambda), R_1(\lambda), \dots, R_k(\lambda))$$

and d continuous functions

$$\eta_j : \mathbb{R}^m \to \mathbb{R}_+, \quad 1 \leq j \leq d$$

such that $B^{(\lambda)}$ reduces to the form

$$\begin{pmatrix} 0 & \eta(\lambda) & 0 \\ -\eta(\lambda) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}),$$

where

$$\eta(\lambda) := \operatorname{diag}\left(\eta_1(\lambda), \dots, \eta_d(\lambda)\right) \in \mathcal{M}_d(\mathbb{R}),$$

and each $\eta_j(\lambda) > 0$ is smooth and homogeneous of degree 1 in $\lambda = (\lambda_1, \cdots, \lambda_m)$, hence the basis vectors are chosen to depend smoothly on λ in Λ.

2-step stratified Lie group without MW condition

Decompose $\mathfrak{g}_1$ as

$$\mathfrak{g}_1 = \mathfrak{p}_\lambda \oplus \mathfrak{q}_\lambda \oplus \mathfrak{r}_\lambda$$

with

$$\begin{aligned} \mathfrak{p}_\lambda &:= \mathrm{span}_{\mathbb{R}} \left(X_1(\lambda), \ldots, X_d(\lambda)\right), \\ \mathfrak{q}_\lambda &:= \mathrm{span}_{\mathbb{R}} \left(Y_1(\lambda), \ldots, Y_d(\lambda)\right), \\ \mathfrak{r}_\lambda &:= \mathrm{span}_{\mathbb{R}} \left(R_1(\lambda), \ldots, R_k(\lambda)\right). \end{aligned}$$

Then we have the decomposition $\mathfrak{g} = \mathfrak{p}_\lambda \oplus \mathfrak{q}_\lambda \oplus \mathfrak{r}_\lambda \oplus \mathfrak{g}_2$. Further we can write

$$(X, Y, R, T) = \sum_{j=1}^{d} x_j(\lambda) X_j(\lambda) + \sum_{j=1}^{d} y_j(\lambda) Y_j(\lambda) + \sum_{j=1}^{k} r_j(\lambda) R_j(\lambda) + \sum_{j=1}^{m} t_j T_j$$

and denote it by (x, y, r, t) suppressing the dependence of λ which will be understood from the context.

2-step stratified Lie group without MW condition

For (λ, ν, w) in $\Lambda \times \mathbb{R}^k \times \mathbb{R}^N$ with

$$w = (x, y, r, t) \in \mathbb{R}^d \oplus \mathbb{R}^d \oplus \mathbb{R}^k \oplus \mathbb{R}^m = \mathbb{R}^N,$$

we define the irreducible unitary representations of $\mathbb{R}^N$, equipped with the group law defined above, on $L^2\left(\mathbb{R}^d\right)$

$$\begin{aligned}(\pi_{\lambda,\nu}(w)\phi)(\xi) &:= \exp\left(i\sum_{j=1}^{m}\lambda_j t_j + i\sum_{j=1}^{k}\nu_j r_j + i\sum_{j=1}^{d}\eta_j(\lambda)\left(y_j\xi_j + \frac{1}{2}x_j y_j\right)\right. \\ &= e^{i\langle\nu,r\rangle}e^{i\langle\lambda,t\rangle}e^{i\sum_{j=1}^{d}\eta_j(\lambda)\left(y_j\xi_j+\frac{1}{2}x_j y_j\right)}\phi(\xi+x) \\ &= e^{i\langle\nu,r\rangle}e^{i\langle\lambda,t\rangle}e^{i\langle\eta(\lambda)\cdot(\xi+\frac{1}{2}x),y\rangle}\phi(\xi+x).\end{aligned}$$

2-step stratified Lie group without MW condition

- G. B. Folland. Harmonic analysis in phase space. Annals of Mathematics Studies. 1989.
- M. W. Wong. Weyl transforms. Springer-Verlag, New York, 1998.

We develop the (λ,ν)-Weyl transform $W^{\lambda,\nu}$ and (λ,ν)-Wigner transform $W_{\lambda,\nu}(f,g)$ on 2-step stratified Lie groups $\mathbb{G}$ to prove the classic theorem of Stone and von Neumann for the 2-step stratified Lie group, which says in effect that any irreducible unitary representation of $\mathbb{G}$ that is nontrivial on the center is equivalent to some $\pi_{\lambda,\nu}$.

Theorem (Stone and von Neumann)

Let π be any unitary representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$, such that for some $\lambda \in \Lambda$, $\pi(0,0,0,t) = e^{i\lambda t} I$. Then $\mathcal{H} = \bigoplus \mathcal{H}_\alpha$ where the $\mathcal{H}_\alpha$ are mutually orthogonal subspaces of $\mathcal{H}$, each invariant under π, such that $\pi|_{\mathcal{H}_\alpha}$ is unitarily equivalent to $\pi_{\lambda,\nu}$ for each α and some $\nu \in \mathbb{R}^k$. In particular, if π is irreducible then π is equivalent to $\pi_{\lambda,\nu}$.

2-step stratified Lie group without MW condition

For (λ, ν, w) in $\Lambda \times \mathbb{R}^k \times \mathbb{R}^N$ with

$$w = (x, y, r, t) \in \mathbb{R}^d \oplus \mathbb{R}^d \oplus \mathbb{R}^k \oplus \mathbb{R}^m = \mathbb{R}^N.$$

Definition (Fourier transform)

The Fourier transform of the function $f \in L^1(\mathbb{G})$ at the point

$$(\lambda, \nu) \in \Lambda \times \mathbb{R}^k$$

is a unitary operator acting on $L^2(\mathbb{G})$ with

$$\mathcal{F}(f)(\lambda, \nu) = (\hat{f}(\lambda, \nu) := \int_{\mathbb{G}} f(w)\pi_{\lambda,\nu}(w^{-1})dw.$$

2-step stratified Lie group without MW condition

Further, the Fourier transform can be extended to an isometry from $L^2(\mathbb{G})$ onto the Hilbert space of two-parameter families $A = \{A(\lambda, v)\}$ of operators on $L^2\left(\mathbb{R}^d\right)$ which are Hilbert-Schmidt for almost every $(\lambda, v) \in \Lambda \times \mathbb{R}^k$, with $\|A(\lambda, v)\|_{\mathrm{HS}\left(L^2\left(\mathbb{R}^d\right)\right)}$ measurable and with norm

$$\|A\| := \left(\iint_{\Lambda \times \mathbb{R}^k} \|A(\lambda, v)\|^2_{\mathrm{HS}\left(L^2\left(\mathbb{R}^d\right)\right)} \mathrm{Pf}(\lambda) d\nu d\lambda \right)^{\frac{1}{2}} < \infty,$$

where $\mathrm{Pf}(\lambda) := \prod_{j=1}^{d} \eta_j(\lambda)$ is the Pfaffian of $B^{(\lambda)}$.

2-step stratified Lie group without MW condition

Lemma (Fourier-Plancherel formula)

There exists some constant $\kappa > 0$ depending only on the choice of the group such that, for any $f \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, there holds

$$\int_{\mathbb{G}} |f(w)|^2 dw = \kappa \iint_{\Lambda \times \mathbb{R}^k} \|\mathcal{F}(f)(\lambda, \nu)\|^2_{HS(L^2(\mathbb{R}^d))} \operatorname{Pf}(\lambda) d\lambda d\nu.$$

Lemma (Inversion formula)

For $f \in L^1\left(\mathbb{R}^N\right)$ and almost every $w \in \mathbb{R}^N$, the following inversion formula holds:

$$f(w) = \kappa \iint_{\Lambda \times \mathbb{R}^k} \operatorname{tr}\left(\left(\pi_{\lambda,\nu}(w)\right)^* \mathcal{F}(f)(\lambda, \nu)\right) \operatorname{Pf}(\lambda) d\lambda d\nu$$

with the same constant $\kappa > 0$.

2-step stratified Lie group without MW condition

We can now define the sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$ by

$$\mathcal{L} = -\Delta_x - \Delta_y - \Delta_r - \frac{1}{4}\left(|x|^2 + |y|^2\right)\Delta_t$$
$$+ \sum_{s=1}^{m}\sum_{j=1}^{d}\left\{-\left(B_s y, e_j\right)\frac{\partial}{\partial x_j} + \left(x, B_s e_j\right)\frac{\partial}{\partial y_j}\right\}\frac{\partial}{\partial t_s}.$$

By taking the Fourier transform of the sub-Laplacian $\mathcal{L}$ with respect to t, we get parametrized λ-twisted sub-Laplacian $\mathcal{L}^\lambda, \lambda \in \mathbb{R}^m$, given by

$$\mathcal{L}^\lambda = -\Delta_x - \Delta_y - \Delta_r + \frac{1}{4}\left(|x|^2 + |y|^2\right)|\lambda|^2$$
$$-i\sum_{j=1}^{d}\left\{-\left(B^{(\lambda)} y, e_j\right)\frac{\partial}{\partial x_j} + \left(x, B^{(\lambda)} e_j\right)\frac{\partial}{\partial y_j}\right\},$$

where we use $B^{(\lambda)} = \sum_{s=1}^{m}\lambda_s B_s$.

2-step stratified Lie group without MW condition

If $\eta = (\eta_1, \ldots, \eta_d) \in \left(\mathbb{R}_+^*\right)^d$ and $\alpha \in \mathbb{N}^d$, we define the rescaled Hermite function Φ_α^λ by

$$\Phi_\alpha^\lambda := |\eta|^{\frac{d}{4}} \Phi_\alpha \left(|\eta|^{\frac{1}{2}} \cdot \right),$$

where Φ_α is the usual Hermite function and define the special Hermite function

$$\Phi_{\alpha,\beta}^\lambda(x) = \mathrm{Pf}(\lambda)^{\frac{1}{2}} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\eta(\lambda)\cdot px} \Phi_\alpha^\lambda \left(x + \frac{q}{2} \right) \overline{\Phi_\beta^\lambda \left(x - \frac{q}{2} \right)} dx.$$

In particular, they form an orthonormal basis of $L^2\left(\mathbb{R}^d\right)$ and we have the rescaled harmonic oscillator

$$\mathcal{H}(\lambda)\Phi_\alpha^\lambda := (-\Delta + |\eta \cdot x|^2)\Phi_\alpha^\lambda = \sum_{j=1}^d \eta_j(\lambda)(2\alpha_j + 1)\Phi_\alpha^\lambda.$$

2-step stratified Lie group without MW condition

We therefore can write

$$\mathcal{F}\left(\mathcal{L}f\right)(\lambda,\nu)(u)=\mathcal{F}(f)(\lambda,\nu)\left(\mathcal{H}(\lambda)+|\nu|^2\right)(u).$$

Theorem

For $\lambda\in\Lambda,\nu\in\mathbb{R}^k$, one has the formula

$$\mathcal{L}^\lambda(\Phi^\lambda_{\alpha,\beta})=\left(\sum_{j=1}^d\eta_j(\lambda)(2\alpha_j+1)+\sum_{j=1}^k\nu_j^2\right)\Phi^\lambda_{\alpha,\beta}.$$

Applications

Heat kernels of $\mathcal{H}(\lambda)$

The associated heat kernel of the rescaled harmonic oscillator $\mathcal{H}(\lambda)$ is

$$G_\tau(x) = \prod_{j=1}^{d} \frac{1}{2\sinh(\eta_j(\lambda)\tau)} \exp\left\{ -\sum_{j=1}^{d} \frac{\eta_j(\lambda)\left|x_j\right|^2}{2} \coth\left(\eta_j(\lambda)\tau\right) \right\},$$

i.e., $G_\tau(x)$ satisfies the heat equation

$$\frac{\partial G_\tau}{\partial \tau} + \sum_{j=1}^{d} \left(\eta_j^2(\lambda) x_j^2 - \frac{\partial^2}{\partial x_j^2} \right) G_\tau(x) = 0,$$

$$\lim_{\tau \to 0} \int_{\mathbb{R}^d} G_\tau(x) f(x) dx = f(0).$$

Applications

Now, we consider the initial-value problem given by

$$\begin{cases} \partial_\tau u(\omega,t,\tau) + (\mathcal{L}u)(w,t,\tau) = 0, \\ u(\omega,t,0) = f(\omega,t), \\ \omega = (z,r) \in \mathbb{R}^{2d+k}, t \in \mathbb{R}^m, \tau > 0. \end{cases}$$

By taking the Fourier transform with respect to t and evaluated at λ, we get an initial-value problem for the heat equation governed by the λ-twisted sub-Laplacian $\mathcal{L}^\lambda$, i.e.

$$\begin{cases} \partial_\tau u_\lambda(\omega,\tau) + (\mathcal{L}^\lambda u_\lambda)(\omega,\tau) = 0, \\ u_\lambda(\omega,0) = f_\lambda(\omega), \end{cases}$$

for all $\omega = (z,r) \in \mathbb{R}^{2d+k}, \tau > 0$ and $\lambda \in \mathbb{R}^m \backslash \{0\}$.

Applications

With the heat kernels of $\mathcal{H}(\lambda)$, we can obtain the heat kernel of $\mathcal{L}$.

Heat kernel of $\mathcal{L}$

For all f in $L^2(\mathbb{G}), e^{-\tau\mathcal{L}} f = f *_{\mathbb{G}} K_\tau$, where

$$K_\tau(\omega, t) = (2\pi)^{-(d+m)} \int_{\mathbb{R}^m} e^{-it\cdot\lambda} e^{-\tau|\nu|^2} \prod_{j=1}^{d} \frac{\eta_j(\lambda)}{2\sinh(\eta_j(\lambda)\tau)} \times \exp\left\{ -\frac{\eta_j(\lambda)\omega_j^2}{4} \coth(\eta_j(\lambda)\tau) \right\} d\lambda$$

for all $(\omega, t) \in \mathbb{G}$.

Applications

Why calculate heat kernels ?

- Heat kernel estimates;
- Li-Yau/gradient estimates;
- Long/short time asymptotics;
- Restriction theorems;
- ......

Thanks for your attention!

Recent works on 2-step nilpotent Lie groups

- Ruzhansky et. al: Wave equations and Gevrey spaces;
- Bahouri et. al: Dispersive estimates and Fourier restriction theorems
- Kumar et. al: Trace class and Hilbert-Schmidt operators
  Chang et. al: Laguerre calculus;
- Thangavelu et. al: Hardy/Beurling's theorem;
- Garofalo et. al: Variational problems;
- Müller et. al: Singular integral and multipliers;
- ......